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On Geometry of Finsler Causality: For Convex Cones, There Is No Affine-Invariant Linear Order (Similar to Comparing Volumes)

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Abstract

Some physicists suggest that to more adequately describe the causal structure of space-time, it is necessary to go beyond the usual pseudo-Riemannian causality, to a more general Finsler causality. In this general case, the set of all the events which can be influenced by a given event is, locally, a generic convex cone, and not necessarily a pseudo-Reimannian-style quadratic cone. Since all current observations support pseudo-Riemannian causality, Finsler causality cones should be close to quadratic ones. It is therefore desirable to approximate a general convex cone by a quadratic one. This cane be done if we select a hyperplane, and approximate intersections of cones and this hyperplane. In the hyperplane, we need to approximate a convex body by an ellipsoid. This can be done in an affine-invariant way, e.g., by selecting, among all ellipsoids containing the body, the one with the smallest volume; since volume is affine-covariant, this selection is affine-invariant. However, this selection may depend on the choice of the hyperplane. It is therefore desirable to directly approximate the convex cone describing Finsler causality with the quadratic cone, ideally in an affine-invariant way. We prove, however, that on the set of convex cones, there is no affine-covariant characteristic like volume. So, any approximation is necessarily not affine-invariant.

1 Formulation of the Corresponding Physical Problem: Analysis of Finsler Causality Relations

Geometric description of physical causality: a brief reminder (for details, see, e.g., [5]). In Newton's mechanics, a space-time event (t, x) can causally influence an event (t', x') if and only if $t \le t'$. In Special Relativity Theory, a

space-time event (t,x), with $x=(x_1,\ldots,x_m)$, can causally influence an event (t',x') with $x'=(x'_1,\ldots,x'_m)$ if the difference vector $(\Delta t,\Delta x_1,\ldots,\Delta x_m)\stackrel{\rm def}{=}(t'-t,x'_1-x_1,\ldots,x'_m-x_m)$ belongs to the quadratic cone

$$C=\{(\Delta t,\Delta x_1,\ldots,\Delta x_m):\Delta t\ge 0\,\&\,(\Delta t)^2-(\Delta x_1)^2-\ldots-(\Delta x_m)^2\ge 0\}.$$

According to the General Relativity Theory, similar cones describe *local* causality: a space-time event (t,x), with $x=(x_1,\ldots,x_m)$, can causally influence an event $(t+dt,x+dx)$ with $dx=(dx_1,\ldots,dx_m)$ if the difference vector $(dt,dx_1,\ldots,dx_m)$ belongs to the cone

$$C=\{(dt,dx_1,\ldots,dx_m):dt\ge 0\,\&\,(dt)^2-(dx_1)^2-\ldots-(dx_m)^2\ge 0\}.$$

Finsler causality. Some physical theories use more general – not necessarily quadratic – convex cones to describe causality; see, e.g., [1, 4, 6]. This generalization of the usual causality is known as *Finsler causality.*

It is desirable to approximate Finsler causality with the usual one. As of now, all observations are consistent with the usual causality relation, namely with the locally quadratic causality cones of General Relativity theory. Thus, even if the actual causality relation is a Finsler one, it is close to the usual quadratic one. So, to be able to efficiently deal with Finsler causality relations, it is is therefore desirable to be able to approximate Finsler causality cones with quadratic ones. This way, we will be able to use the formulas corresponding to the usual quadratic causality as the first approximation, and thus to concentrate our analysis on the – empirically small – differences between these causality relations.

How can we find such an approximation?

A possible approach to the desired approximation. A convex cone is uniquely by its intersection with a hyperplane – which is a convex body. For the quadratic cone that corresponds to the usual causality relation, this intersection is a bounded quadratic body, i.e., an ellipsoid – and vice versa, for each ellipsoid, the corresponding cone is a quadratic cone. For general convex cones, the intersection is a generic convex body.

So, to approximate a generic convex cone by a quadratic one, it is sufficient to approximate a general convex body by an ellipsoid. This is indeed possible. For example, to every bounded convex body $B\subseteq \mathbb{R}^n$ with a non-empty interior, we can associate a unique ellipsoid E enclosing this body: namely, out of all ellipsoids E that contain the set B, we can select the ellipsoid E_0 with the smallest possible volume $V(E)$ (see, e.g., [2, 3]):

$$V(E_0)=\min\{V(E):E \text{ is an ellipsoid and } B\subseteq E\}.$$

This definition makes perfect sense, since for ellipsoids, $E\subseteq E'$ and $E\ne E'$ imply that $V(E)<V(E')$. Thus, the fact that the ellipsoid E_0 has the minimal

possible volume implies that there is no sub-ellipsoid $E' \subseteq E_0$ that contains the original body B.

Strictly speaking, to describe the volume, we need to fix (affine) coordinates in the corresponding hyperplane. However, while the actual values of the volume depend on the coordinates, the resulting ellipsoid is the same no matter what coordinates we use. The reason for this is that the volume is *affine-covariant*: for any affine transformation T that describes the transition to new affine coordinates, and for every ellipsoid E, we have $V(TE) = c(T) \cdot V(E)$ for some constant $c(T)$. As a result, for every two ellipsoids E and E', we have $V(E) < V(E') \Leftrightarrow V(TE) < V(TE')$, i.e., which ellipsoid has larger volume and which has smaller volume does not depend on the choice of the coordinates.

Limitations of the usual approximation, and the resulting problem. The problem with the above approximation idea is that to follow this idea, we must first choose a hyperplane, and there is no guarantee that for a different hyperplane, we will not get a different approximation.

To make the approximation more physically meaningful, it is therefore desirable to avoid such an arbitrary choice, and instead of approximating a body by an ellipsoid, to directly approximate a general convex cone C by a quadratic cone Q. Our of all possible quadratic cones Q that contain C, it is desirable to select the "smallest" one in the sense of an appropriate total order $\le$ on the class of all quadratic ellipsoids. This selection should not depend on the choice of coordinates. To guarantee that, it is desirable to make sure that the corresponding order is affine-covariant, i.e., that $Q < Q'$ if and only if $TQ < TQ'$.

What we do in this paper. In this paper, we show that for quadratic cones, there are no affine-invariant volume-like characteristics. Moreover, we show that there is no affine-invariant way ordering relation between quadratic cones.

The physical meaning of this result is that it is not possible to approximate Finsler causality by quadratic causality in an affine-invariant way.

2 Definitions and the Main Result

Definition 1. *Let $n \ge 2$.*

- *By an* affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, *we mean a reversible mapping of the type* $x_i \to \sum\limits_{j=1}^{n} t_{ij} \cdot x_j$.
- *By a* quadratic cone, *we mean an image TQ_0 of the standard quadratic cone*
$$Q_0 = \{(x_1, \ldots, x_n) : x_1 \ge 0 \,\&\, x_1^2 \ge x_2^2 + \ldots + x_n^2\}$$
under an affine transformation.

Notation. For each $n \ge 2$, let $\mathcal{Q}_n$ denote the set of all n-dimensional quadratic cones.

Definition 2. *By a* linear (total) pre-order *we mean a transitive relation $\le$ for which, for every two objects a and b, we have $a \le b$ or $b \le a$.*

Comment. For some objects a and b, it is possible to have both $a \le b$ and $b \le a$. For example, for the corresponding relation on the ellipsoids $E \le E' \Leftrightarrow V(E) \le V(E')$, when we have two ellipsoids E and E' of the same volume, then we have both $E \le E'$ and $E' \le E$.

Notation. We will denote:

- the situation when $a \le b$ and $b \not\le a$ by $a < b$, and
- the situation when $a \le b$ and $b \le a$ by $a \sim b$.

Definition 3. *Let $n \le 2$.*

- *A linear pre-order on the set $\mathcal{Q}_n$ is called* $\subseteq$-consistent *if $Q \subseteq Q'$ and $Q \ne Q'$ imply that $Q < Q'$.*
- *A linear pre-order on the set $\mathcal{Q}_n$ is called* affine-invariant *if for each affine transformation T, $Q \le Q'$ implies $TQ < TQ'$.*

Proposition. *For each $n \ge 2$, no linear order on the set $\mathcal{Q}_n$ of all n-dimensional quadratic cones is both $\subseteq$-consistent and affine-invariant.*

Proof. We will prove this result by contradiction. Let us assume that a linear order $\le$ is a $\subseteq$-consistent and affine-invariant linear order on the set $\mathcal{Q}_n$, and let us deduce a contradiction from this assumption.

1°. Let us first consider the case $n = 2$.

1.1°. Let us first consider the following quadratic cone

$$Q_1 \stackrel{\text{def}}{=} \{(x_1, x_2) : x_1 \ge 0 \,\&\, x_2 \ge 0\}.$$

An affine transformation $T_1 : (x_1, x_2) \to (-x_2, x_1)$ maps this cone into $T_1 Q_1 = \{(x_1, x_2) : x_1 \le 0 \,\&\, x_2 \ge 0\}$. If we apply the same transformation T_1 again, we get $T_1^2 Q_1 = \{(x_1, x_2) : x_1 \le 0 \,\&\, x_2 \le 0\}$,

$$T_1^3 Q_1 = \{(x_1, x_2) : x_1 \ge 0 \,\&\, x_2 \le 0\},$$

and $T_1^4 Q_1 = Q_1$.

1.2°. Since $\le$ is a linear order, we have either $Q_1 \le T_1 Q_1$ or $T_1 Q_1 \le Q_1$. Let us consider these two possibilities one by one.

$1.2.1^\circ$. If $Q_1 \le T_1 Q_1$, then, since the order $\le$ is affine-invariant, we get $T_1 Q_1 \le T_1^2 Q_1$, $T_1^2 Q_1 \le T_1^3 Q_1$, and $T_1^3 Q_1 \le Q_1$. Thus, by transitivity, we get $T_1 Q_1 \le Q_1$ and hence, $Q_1 \sim T_1 Q_1$.

$1.2.2^\circ$. If $T_1 Q_1 \le Q_1$, then we similarly get $Q_1 \le T_1 Q_1$ and thus, $Q_1 \sim T_1 Q_1$. So, in both cases, we have $Q_1 \sim T_1 Q_1$.

1.3°. Let us now consider a different affine transformation

$$T_2(x_1, x_2) = (x_1 + x_2, x_2).$$

For this transformation, $T_2Q_1 = \{(x_1, x_2) : 0 \le x_2 \le x_1\}$ and $T_2T_1Q_1 = \{(x_1, x_2) : x_2 \ge 0 \,\&\, x_1 \le x_2\}$. Since $Q_1 \sim T_1Q_1$, we have $T_2Q_1 \sim T_2T_1Q_1$.

On the other hand, since $T_2Q_1 \subseteq Q_1$ and $T_2Q_1 \ne Q_1$, we have $T_2Q_1 < Q_1$. Similarly, since $T_1Q_1 \subseteq T_2T_1Q_1$ and $T_1Q_1 \ne T_2T_1Q_1$, we have $T_1Q_1 < T_2T_1Q_1$. From $T_2Q_1 < Q_1$, $Q_1 \sim T_1Q_1$, and $T_1Q_1 < T_2T_1Q_1$, we conclude that $T_2Q_1 < T_2T_1Q_1$, which contradicts to the previous conclusion that $T_2Q_1 \sim T_2T_1Q_1$. This contradiction shows that for $n = 2$, indeed, no $\subseteq$-consistent and affine-invariant is possible on the class $\mathcal{Q}_n$.

2°. Let us now consider the case $n > 2$. In this case, we consider a cone

$$Q_1 = \{(x_1, \ldots, x_n) : x_1 \ge 0 \,\&\, x_1 \cdot x_2 \ge x_3^2 + \ldots + x_n^2\},$$

and transformations

$$T_1(x_1, x_2, x_3, \ldots, x_n) = (-x_2, x_1, x_3, \ldots, x_n)$$

and

$$T_2(x_1, x_2, x_3, \ldots, x_n) = (x_1 + x_2, x_2, x_3, \ldots, x_n).$$

Thus, the inverse transformations take the form

$$T_1^{-1}(x'_1, x'_2, x'_3, \ldots, x'_n) = (x'_2, -x'_1, x'_3, \ldots, x'_n)$$

and

$$T_2^{-1}(x'_1, x'_2, x'_3, \ldots, x'_n) = (x'_1 - x'_2, x'_2, x'_3, \ldots, x'_n).$$

Here too, $T_1^4Q_1 = Q_1$, thus $Q_1 \sim T_1Q_1$. The set $T_2Q_1 = \{x' : T_2^{-1}x' \in Q_1\}$ has the form

$$T_2Q_1 = \{(x_1, \ldots, x_n) : x_1 - x_2 \ge 0 \,\&\, (x_1 - x_2) \cdot x_2 \ge x_3^2 + \ldots + x_n^2\}.$$

The corresponding inequality

$$(x_1 - x_2) \cdot x_2 \ge x_3^2 + \ldots + x_n^2$$

is equivalent to

$$x_1 \cdot x_2 - x_2^2 \ge x_3^2 + \ldots + x_n^2$$

and thus, implies that

$$x_1 \cdot x_1 \ge x_1 \cdot x_2 - x_2^2 \ge x_3^2 + \ldots + x_n^2,$$

so $T_2Q_1 \subseteq Q_1$ and $T_2Q_1 \ne Q_1$ and hence, $T_2Q_1 < Q_1$.

Similarly, the set $T_2T_1Q_1$ consists of all the points $(x_1, \ldots, x_n)$ for which

$$-x_2 \cdot (x_1 - x_2) \ge x_3^2 + \ldots + x_n^2,$$

i.e., for which
$$-x_2 \cdot x_1 + x_2^2 \geq x_3^2 + \ldots + x_n^2.$$
So, if $(x_1, \ldots, x_n) \in T_1Q_1$, i.e., if
$$-x_2 \cdot x_1 \geq x_3^2 + \ldots + x_n^2,$$
then
$$-x_2 \cdot x_1 + x_2^2 \geq x_3^2 + \ldots + x_n^2$$
and thus, $x \in T_2T_1Q_1$. So here too, $T_1Q_1 \subseteq T_2T_1Q_1$ and $T_1Q_1 \neq T_2T_1Q_1$, hence $T_1Q_1 < T_2T_1Q_1$.

Similarly to Part 1 of this proof, from $T_2Q_1 < Q_1$, $Q_1 \sim T_1Q_1$, and $T_1Q_1 < T_2T_1Q_1$, we conclude that $T_2Q_1 < T_2T_1Q_1$, which contradicts to the previous conclusion that $T_2Q_1 \sim T_2T_1Q_1$. This contradiction shows that for any n, no $\subseteq$-consistent and affine-invariant is possible on the class $\mathcal{Q}_n$.

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