12-2015

Towards Selecting the Best Abstraction for a Patrolling Game

Anjon Basak
The University of Texas at El Paso, abasak@miners.utep.edu

Chris Kiekintveld
The University of Texas at El Paso, cdkiekintveld@utep.edu

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

Comments:

Technical Report: UTEP-CS-15-93

To appear in *Journal of Uncertain Systems*, 2017, Vol. 11.

Towards Selecting the Best Abstraction for a Patrolling Game

Anjon Basak, Christopher Kiekintveld,
and Vladik Kreinovich
Department of Computer Science
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
abasak@miners.utep.edu,
cdkiekintveld@utep.edu,
vladik@utep.edu

Abstract

When the number of possible strategies is large, it is not computationally feasible to compute the optimal strategy for the original game. Instead, we select our strategy based on an approximate approximate description of the original game. The quality of the resulting strategy depends on which approximation we select. In this paper, on an example of a simple game, we show how to find the optimal approximation, the approximation whose use results in the best strategy.

1 Formulation of the Problem

Need for abstraction. Ideally, in a conflict situation, we should select a strategy which is optimal in some reasonable sense. For example, in a zero-sum game, it makes sense to select a strategy that minimizes the worst-case loss.

The more strategies we need to consider, the more computations we need to perform to find the optimal strategy. When the number of strategies becomes large, it is often not feasible to exactly compute the optimal strategy. In such situations, to produce a reasonable strategy, it makes sense to approximate the actual difficult-to-analyze game with a simpler-to-analyze game that has fewer strategies. One of the natural ways to come up with such an approximation is *abstraction*, when we divide all possible strategies into a small number of groups, so that strategies within each group lead to approximately the same consequences. Then, in the "abstracted" (simplified) game, we just select a group (instead of selecting an actual strategy); see, e.g., [2, 3].

Which abstraction is the best? Of course, since we are using only an

approximate description of the actual game situation to come up with a strategy, the resulting strategy is only approximately optimal. The quality of the resulting strategy depends on the choice of the abstraction. It is therefore desirable to find an abstraction that results in the best possible performance.

What we do in this paper. In this paper, we solve the problem of selecting the best abstraction on the example of a simple patrolling game.

2 1-D Patrolling Game: A Brief Description

Description of the game. Let us assume that have a 1-D area that we want to protect against an adversary: e.g., a border segment. Points along this area will be marked by a parameter x that takes values between $\underline{x}$ and $\overline{x}$. For example, for border protection, x is the distance along the border.

We also assume that for each point x, we know the utility gain $u(x)$ that the adversary will get if his penetration at this point succeeds. This value may differ from one location to another. For example, in case of border protection, a successive smuggling in an area with a smooth terrain will enable the adversary to smuggle a larger amount than in an area with a harsher terrain – and thus, gain more utility.

To prevent the adversary's attacks, we place agents along the 1-D area. The closer the agent to the adversary, the less probable is the adversary's success. For simplicity, let us assume that there is a threshold distance r such that the attack is prevented if and only if the distance between the adversary and the agent does not exceed this threshold h.

We randomly allocate agents to different points x. Let $\rho_a(x)$ be the probability density that describes the corresponding agents' distribution. This means that for a small distance Δx, the probability of finding an agent at locations between x and $x + \Delta x$ is equal to $\rho_a(x) \cdot \Delta x$. Overall, we have n_a agents, so we must have

$$\int_{\underline{x}}^{\overline{x}} \rho_a(x)\, dx = n_a. \tag{1}$$

We assume that the adversary knows our strategy, i.e., knows the values $\rho_a(x)$. This is a reasonable assumption, since the adversary can observe agents at different locations and thus, determine the frequencies with which agents are posted to different locations. Such games in which the adversary knows the agent's strategy are known as *Stackelberg games*; see, e.g., [6].

Based on his knowledge, the adversary selects an attack location x at which his expected utility gain is the largest possible. We assumed that we know the adversary's gain in the situation when the adversary's attack at a point x is successful: this gain is equal to $u(x)$. Under our assumptions, the probability of being unsuccessful is equal to the probability to find an agent within a distance h from the point x, i.e., in the interval $[x-h, x+h]$ of width $\Delta x = 2h$. Based on the probability density $\rho_a(x)$ describing the agents' appearance, we can therefore estimate the probability of thwarting an attack as the product $2h \cdot \rho_a(x)$. Thus,

the probability of a successful attack is $1 - 2h \cdot \rho_a(x)$, and so, the expected gain of the adversary is equal to the product $(1 - 2h \cdot \rho_a(x)) \cdot u(x)$.

For each selection of the agent distribution $\rho_a(x)$, the adversary selects an attack location at which his expected utility gain $(1 - 2h \cdot \rho_a(x)) \cdot u(x)$ is the largest possible. Thus, the adversary's gain is equal to

$$\max_x \, (1 - 2h \cdot \rho_a(x)) \cdot u(x). \tag{2}$$

We select the agent distribution $\rho_a(x)$ for which this gain is as small as possible:

$$\max_x \, (1 - 2h \cdot \rho_a(x)) \cdot u(x) \to \min_{\rho_a(x)} \tag{3}$$

under the constraint that

$$\int \rho_a(x)\, dx = 1. \tag{4}$$

Known optimal strategy for the patrolling game. Let u_0 denote the desired maximum $\max\limits_x (1 - 2h \cdot \rho_a(x)) \cdot u(x)$. Then, we have

$$(1 - 2h \cdot \rho_a(x)) \cdot u(x) \le u_0 \tag{5}$$

for all u.

For all the points x at which $\rho(x) = 0$, the gain is equal to $u(x)$, so we have $u(x) \le u_0$.

One can easily prove that in the optimal agent distribution,

$$(1 - 2h \cdot \rho_a(x)) \cdot u(x) = u_0 \tag{6}$$

for all the points $x \in [\underline{x}, \overline{x}]$ for which $\rho_a(x) > 0$. Indeed, if at some of such points x_0, we have

$$(1 - 2h \cdot \rho_a(x_0)) \cdot u(x_0) < \max_x \, (1 - 2h \cdot \rho_a(x)) \cdot u(x), \tag{7}$$

then we can decrease the density in the vicinity of this point x_0 without changing the maximum, and distribute these probabilities among all the points where the product $(1 - 2h \cdot \rho_a(x)) \cdot u(x)$ attains its maximum. This will guarantee that the maximum adversary's gain decreases.

So, the optimal solution is to select some threshold u_0 and take $\rho(x) = 0$ for all values x for which $u(x) < u_0$ and $(1 - 2h \cdot \rho_a(x)) \cdot u(x) = u_0$ for all other x. This equality implies that

$$1 - 2h \cdot \rho_a(x) = \frac{u_0}{u(x)}, \tag{8}$$

hence

$$2h \cdot \rho_a(x) = 1 - \frac{u_0}{u(x)} \tag{9}$$

and

$$\rho_a(x) = \frac{1}{2h} \cdot \left(1 - \frac{u_0}{u(x)}\right). \tag{10}$$

The threshold u_0 should be determined by the condition that $\int \rho_a(x)\,dx = n_a$, i.e., that

$$\int_{x:u(x)\geq u_0} \frac{1}{2h} \cdot \left(1 - \frac{u_0}{u(x)}\right) = n_a. \tag{11}$$

The case that we will consider in this paper. In general, we may have points which are not worth defending, i.e., for which the possible loss $u(x)$ is so low, that we do not even bother sending agents there.

In many realistic important patrolling problems, however, we cannot ignore any of the points, so we have $\rho_a(x) > 0$ for all x. This is the case that we consider in this paper. In this realistic case, we have $u(x) \geq u_0$ for all x, hence the equality (11) takes the form

$$\int_{\underline{x}}^{\overline{x}} \frac{1}{2h} \cdot \left(1 - \frac{u_0}{u(x)}\right) = \frac{\overline{x} - \underline{x}}{2h} - u_0 \cdot \int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)} = n_a. \tag{12}$$

Thus, in this case, we have

$$u_0 = \frac{(\overline{x} - \underline{x}) - 2n_a \cdot h}{h \cdot \int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx}. \tag{13}$$

Thus, the optimal strategy is to select the agent density (10), where the parameter u_0 is determined by the formula (13).

3 Approximations to the 1-D Patrolling Game: A Description

How to describe an approximation? As we have mentioned earlier, an approximation means that instead of considering all possible values x, we divide the interval $[\underline{x}, \overline{x}]$ into a small number of sub-zones, and, when planning the agent distribution, only decide on the sub-zone – so that agents are uniformly distributed within each sub-zone.

At some important locations, we may have a denser concentration of these sub-zones, in other locations, we may have fewer sub-zones. Let us describe the distribution of sub-zones by the sub-zone density $\rho_s(x)$ that describes how many sub-zones per unit length we have at a location x. The overall number of sub-zones should be equal to the the total number of sub-zones n_s that our computational abilities allows us:

$$\int \rho_s(x)\,dx = n_s. \tag{14}$$

How to select a solution based on the approximate game? In the original game, for each location x, we take into account the adversary's gain $u(x)$. In the approximate game, instead of a single value x, we need to consider the whole sub-zone. By definition of the sub-zone density, a unit interval into $\rho_s(x)$ sub-zones, thus the width w of each sub-zone is equal to

$$w = \frac{1}{\rho_s(x)}. \tag{15}$$

So, the distance from the midpoint to each of the endpoints of the zone is equal to half of this width, and thus, instead of a single point x, we have a sub-zone

$$S = \left[\widetilde{x} - \frac{1}{2\rho_s(x)}, \widetilde{x} + \frac{1}{2\rho_s(x)}\right], \tag{16}$$

where $\widetilde{x}$ is the midpoint of this sub-zone.

For different points x within this sub-zone, we have different values of adversary's gain $u(x)$ (and thus, different values of our loss $-u(x)$). So, instead of a single value $u(x)$, we now only know that the actual (unknown) value $u(x)$ belongs to the interval

$$[\underline{u}, \overline{u}] = \{u(x) : x \in S\} \tag{17}$$

of possible values x from this sub-zone.

How can we make a decision in situations with such interval uncertainty? The way to make such decisions was developed by a Nobelist Leo Hurwicz in the early 1950s [1, 4, 5]: he argued that in such situations, we should optimize a weighted combination of the best-case gain and the worst-case gain, with weights adding up to 1 and with the weight α at the best-case gain determined by the decision maker's optimism-pessimism.

In our case, the best-case situation is when we have the smallest possible loss $\underline{u}$ and the worst-case situation is when we have the largest possible loss $\overline{u}$. So, Hurwicz's recommendation means that in our decision making, we should use the function

$$u^*(x) = \alpha \cdot \underline{u} + (1 - \alpha) \cdot \overline{u} = \alpha \cdot \min_{x \in S} u(x) + (1 - \alpha) \cdot \max_{x \in S} u(x) \tag{18}$$

which is constant on each sub-zone.

Formulation of the problem. By definition of an approximate game, we select the agent distribution $\rho_a^*(x)$ based not on the actual utility function $u(x)$, bur rather based on the piece-wise constant approximation $u^*(x)$.

The adversary knows this selection $\rho_a^*(x)$, so he selects an attack point for which his expected gain $(1 - 2h \cdot \rho_a^*(x)) \cdot u(x)$ is the largest possible. As a result, for each approximation density, the adversary's gain is equal to

$$\max_x \, (1 - 2h \cdot \rho_a^*(x)) \cdot u(x). \tag{19}$$

Since we used an approximation, the resulting gain is, in general, larger, than in the ideal situation when we have the optimal agent density. It is therefore desirable to select the sub-zone distribution $\rho_s(x)$ for which the adversary's gain is the smallest possible:

$$\max_x \, (1 - 2h \cdot \rho_a^*(x)) \cdot u(x) \to \min_{\rho_s(x)} . \tag{20}$$

4 Analysis of the Problem: General Case

Let us first come up with explicit formulas for $u^*(x)$. In this paper, we assume that the approximation is reasonable, i.e., that the sub-zones are narrow. When the sub-zone is narrow, we can expand the value of $u(x)$ on each zone $[\widetilde{x}-w, \widetilde{x}+w]$ into Taylor series and only keep linear terms in this expansion (and safely ignore terms proportional to terms which are quadratic or of higher order with respect to w:

$$u(x) = u(\widetilde{x}) + u'(\widetilde{x}) \cdot \Delta x,$$

where $\Delta x \stackrel{\text{def}}{=} x - \widetilde{x}$. In the sub-zone, the difference Δx takes all possible values between $-\dfrac{w}{2}$ and $\dfrac{w}{2}$.

In this approximation, the function $u(x)$ is linear on each sub-zone, and it is easy to find its smallest and largest values:

$$\underline{u} = u - |u'| \cdot \frac{w}{2} \tag{21}$$

and

$$\overline{u} = u + |u'| \cdot \frac{w}{2}. \tag{22}$$

Indeed, when the derivative u' is positive, the maximum is attained when Δx attains its largest possible value

$$\Delta x = \frac{w}{2}, \tag{23}$$

and the minimum is attained when Δx attains its smallest possible value

$$\Delta x = -\frac{w}{2}. \tag{24}$$

When the derivative is negative, locations of the maximum and minimum switch places. In both cases, we get the desired formula.

So, we get

$$u^* = \alpha \cdot \left(u - |u'| \cdot \frac{w}{2}\right) + (1 - \alpha) \cdot \left(u + |u'| \cdot \frac{w}{2}\right) =$$

$$u - (2\alpha - 1) \cdot |u'| \cdot \frac{w}{2}. \tag{25}$$

Our agent allocation is based on this approximate piece-wise constant utility function $u^*(x)$.

What is the agent allocation $\rho_a^*(x)$ based on this approximate utility function $u^*(x)$. According to formulas (8) and (13), we have

$$1 - 2h \cdot \rho_a^*(x) = \frac{u_0^*}{u^*(x)}, \tag{26}$$

where

$$u_0^* = \frac{(\overline{x} - \underline{x}) - 2n_a \cdot h}{h \cdot \displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u^*(x)}\,dx}. \tag{27}$$

How much the adversary gains. On each zone, the adversary selects a point at which the expected gain $(1 - 2h \cdot \rho_a^*(x)) \cdot u(x)$ is the largest. The value $(1 - 2h \cdot \rho_a^*(x))$ is constant within the zone, so within each zone, the adversary selects the point at which $u(x)$ is the largest, i.e., the point x at which $u(x) = \overline{u} = u(x) + |u'(x)| \cdot \dfrac{w}{2}$. For this point, the adversary's gain is equal to $(1 - 2h \cdot \rho_a^*(x)) \cdot \overline{u}$. From (26), we conclude that

$$(1 - 2h \cdot \rho_a^*(x)) = \frac{u_0^*}{u^*(x)}, \tag{28}$$

thus the adversary's gain is equal to

$$u_0^* \cdot \frac{\overline{u}(x)}{u^*(x)}. \tag{29}$$

Here,

$$\overline{u}(x) = u(x) + |u'(x)| \cdot \frac{w}{2} \tag{30}$$

and

$$u^*(x) = u(x) - (2\alpha - 1) \cdot |u'(x)| \cdot \frac{w}{2} \qquad (31),$$

thus

$$\frac{\overline{u}(x)}{u^*(x)} = \frac{u(x) + |u'(x)| \cdot \dfrac{w}{2}}{u(x) - (2\alpha - 1) \cdot |u'(x)| \cdot \dfrac{w}{2}} =$$

$$\frac{1 + \dfrac{|u'(x)|}{u(x)} \cdot \dfrac{w}{2}}{1 - (2\alpha - 1) \cdot \dfrac{|u'(x)|}{u(x)} \cdot \dfrac{w}{2}}. \tag{32}$$

For narrow w, when we can ignore terms which are quadratic and higher order in terms of w, this ratio is equal to

$$\frac{\overline{u}(x)}{u^*(x)} = 1 + 2 \cdot (1 - \alpha) \cdot \frac{|u'(x)|}{u(x)} \cdot \frac{w}{2}. \tag{33}$$

Since $w = \dfrac{1}{\rho_s(x)}$, the gain is thus equal to

$$u_0^* \cdot \left(1 + (1-\alpha) \cdot \frac{|u'(x)|}{u(x)} \cdot \frac{1}{\rho_s(x)}\right). \tag{34}$$

This value characterizes the adversary's gain for each sub-zone. Thus, the adversary will select a sub-zone at which this value is the largest possible. The resulting gain is equal to

$$u_0^* \cdot \left(1 + (1-\alpha) \cdot \max_x \frac{|u'(x)|}{u(x) \cdot \rho_s(x)}\right). \tag{35}$$

Here, due to (13) and (27), we have

$$u_0^* = u_0 \cdot \frac{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx}{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u^*(x)}\,dx}. \tag{36}$$

Due to (25), we have

$$u^*(x) = u(x) \cdot \left(1 - (2\alpha - 1) \cdot \frac{|u'(x)|}{u(x)} \cdot \frac{w}{2}\right) =$$

$$u(x) \cdot \left(1 - \frac{2\alpha - 1}{2} \cdot \frac{|u'(x)|}{u(x) \cdot \rho_s(x)}\right). \tag{37}$$

Thus, since the width $w = \dfrac{1}{\rho_s(x)}$ is small, we have

$$\frac{1}{u^*(x)} = \frac{1}{u(x)} \cdot \left(1 + \frac{2\alpha - 1}{2} \cdot \frac{|u'(x)|}{u(x) \cdot \rho_s(x)}\right) =$$

$$\frac{1}{u(x)} + \frac{2\alpha - 1}{2} \cdot \frac{|u'(x)|}{u^2(x) \cdot \rho_s(x)}. \tag{38}$$

Hence,

$$\int_{\underline{x}}^{\overline{x}} \frac{1}{u^*(x)}\,dx = \int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx + \frac{2\alpha - 1}{2} \cdot \int_{\underline{x}}^{\overline{x}} \frac{|u'(x)|}{u^2(x) \cdot \rho_s(x)}\,dx, \tag{39}$$

and

$$\int_{\underline{x}}^{\overline{x}} \frac{1}{u^*(x)}\,dx = \int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx \cdot \left(1 + \frac{2\alpha - 1}{2} \cdot \frac{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{|u'(x)|}{u^2(x) \cdot \rho_s(x)}\,dx}{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx}\right). \tag{40}$$

Thus,

$$\frac{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u^*(x)}\,dx}{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx} = 1 + \frac{2\alpha-1}{2} \cdot \frac{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{|u'(x)|}{u^2(x)\cdot\rho_s(x)}\,dx}{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx}, \tag{41}$$

hence

$$\frac{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx}{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u^*(x)}\,dx} = 1 - \frac{2\alpha-1}{2} \cdot \frac{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{|u'(x)|}{u^2(x)\cdot\rho_s(x)}\,dx}{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx}. \tag{42}$$

So, due to (36), we have

$$u_0^* = u_0 \cdot \left(1 - \frac{2\alpha-1}{2} \cdot \frac{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{|u'(x)|}{u^2(x)\cdot\rho_s(x)}\,dx}{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx}\right). \tag{43}$$

Substituting the expression (43) into the formula (35) and taking into account that terms proportional to $\dfrac{1}{\rho_s(x)}$ (i.e., to width of sub-zones) are small, we conclude that the adversary's gain is equal to

$$u_0 \cdot (1 + G) \tag{44}$$

where

$$G \stackrel{\text{def}}{=} (1-\alpha) \cdot \max_x \frac{|u'(x)|}{u(x)\cdot\rho_s(x)} - \frac{2\alpha-1}{2} \cdot \frac{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{|u'(x)|}{u^2(x)\cdot\rho_s(x)}\,dx}{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx}. \tag{45}$$

Resulting constraint optimization problem. To minimize the adversary's gain, we need to find the sub-zone distribution function $\rho_s(x)$ that minimizes the expression (45) under the constraint (14), that expresses the fact that the overall number of sub-zones is equal to n_s: $\displaystyle\int_{\underline{x}}^{\overline{x}} \rho_s(x)\,dx = n_s$.

How we will solve this problem. The minimized function (45) is the sum of two terms: the maximum term and the integral terms. To solve the corresponding minimization problem, we first consider the cases when one of these terms is equal to 0. In each of the corresponding two cases, the minimized function is easier-to-describe and thus, the corresponding minimization problem is easier to solve.

After that, we will show how the solutions to these two simplified cases can be combined to produce a solution to the general case.

5 First Special Case: $\alpha = 0.5$

Let us first consider the case when $\alpha = 0.5$, i.e., when we consider a midpoint of an interval in case of uncertainty.

In this case, the integral term in the formula (45) is equal to 0, and thus, the minimized function (45) takes the simplified form

$$G = 0.5 \cdot \max_x \frac{|u'(x)|}{u(x) \cdot \rho_s(x)}. \tag{46}$$

Similarly to Section 2, we can conclude that the optimal sub-zone distribution corresponds to the case when the maximized ratio

$$\frac{|u'(x)|}{u(x) \cdot \rho_s(x)} \tag{47}$$

is constant, i.e., when

$$\rho_s(x) = c \cdot \frac{|u'(x)|}{u(x)}. \tag{48}$$

This makes sense:

- When the derivative $u'(x)$ is small, i.e., when the gain $u(x)$ practically does not change, we can have fewer sub-zones of larger size, since for all x from this sub-zone we will have, in effect, the same gain value $u(x)$.
- On the other hand, when the derivative $u'(x)$ is large, we need narrower sub-zones if we want to accurately represent the utility function $u(x)$ – and thus, we need more frequent sub-zones.

Since

$$\frac{u'(x)}{u(x)} = \frac{d}{dx} \ln(u(x)) \tag{49}$$

and

$$\frac{1}{\rho_s(x)} = w, \tag{50}$$

the product

$$\frac{|u'(x)|}{u(x) \cdot \rho_s(x)} \tag{51}$$

is equal to

$$\frac{d}{dx} \ln(u(x)) \cdot w, \tag{52}$$

i.e., in the first approximation, to the absolute value of the difference between the values of

$$\ln\left(u\left(x - \frac{w}{2}\right)\right) \tag{53}$$

and

$$\ln\left(u\left(x + \frac{w}{2}\right)\right) \tag{54}$$

on the two sides of the sub-zone. For each sub-zone, this difference should be the same. This difference is the logarithm of the ratio

$$\frac{u\left(x+\frac{w}{2}\right)}{u\left(x-\frac{w}{2}\right)}. \tag{55}$$

Thus, we should select zones in which the ratio of utilities at the endpoints is a constant $C > 1$.

So, for the case when $\alpha = 0.5$, we arrive at the following optimal location of sub-zones (i.e., optimal approximation):

- We start with the leftmost location $x_1 = \underline{x}$. This location is the left-end point of the first zone.
- Its right-hand point is the value x_2 at which $u(x_2) = C \cdot u(x_1)$.
- Then, we take x_3 at which $u(x_3) = C \cdot u(x_2)$, etc.
- When we reach the maximum, we start decreasing the utility value by the factor of C.

In short, the endpoints of the sub-zones are points at which the utility $u(x)$ take values from a geometric progression $u(\underline{x})$, $C \cdot u(\underline{x})$, $C^2 \cdot u(\underline{x})$, etc.

6 Second Special Case: $\alpha = 1$

In the case when $\alpha = 1$, i.e., when we only consider the worst-case scenarios when making a decision, the maximum term in the formula (45) disappears, and thus, the minimized function (45) takes the simplified form

$$G = -\frac{1}{2} \cdot \frac{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{|u'(x)|}{u^2(x) \cdot \rho_s(x)}\,dx}{\displaystyle\int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx}. \tag{56}$$

In this expression, the denominator

$$\int_{\underline{x}}^{\overline{x}} \frac{1}{u(x)}\,dx \tag{57}$$

does not depend on our choice of the sub-zone distribution function $\rho_s(x)$, so minimizing the expression (56) is equivalent to maximizing the integral

$$\int_{\underline{x}}^{\overline{x}} \frac{|u'(x)|}{u^2(x) \cdot \rho_s(x)}\,dx \tag{58}$$

under the constraint

$$\int_{\underline{x}}^{\overline{x}} \rho_s(x)\,dx = n_s. \tag{59}$$

To solve this constraint maximization problem, we can use the Lagrange multiplier technique and optimize the expression

$$\int_{\underline{x}}^{\overline{x}} \frac{|u'(x)|}{u^2(x)\cdot\rho_s(x)}\,dx + \lambda\cdot\left(\int_{\underline{x}}^{\overline{x}} \rho_s(x)\,dx - n_s\right) \tag{60}$$

for an appropriate Lagrange multiplier λ. Differentiating the unconstrained objective function (60) with respect to $\rho_s(x)$ and equating derivatives to 0, we conclude that

$$-\frac{|u'(x)|}{u^2(x)\cdot\rho_s^2(x)} + \lambda = 0, \tag{61}$$

hence

$$\rho_s(x) = c\cdot\frac{\sqrt{|u'(x)|}}{u(x)} \tag{62}$$

for an appropriate constant c. This constant can be easily determined from the condition (59).

It is worth mentioning that the resulting formula (62) is similar to the formula (48) corresponding to the case $\alpha = 0.5$:

- in both cases, the density $\rho_s(x)$ of sub-zones is inverse proportional to the utility $u(x)$, and
- in both case, the density $\rho_s(x)$ increases as the utility's non-homogeneity $|u'(x)|$ increases.

The difference between these two formulas is that:

- when $\alpha = 0.5$, the density of sub-zones is proportional to the non-homogeneity itself: $\rho_s(x) \sim |u'(x)|$, while
- when $\alpha = 1$, the density of sub-zones grows slower, proportionally to the square root of non-homogeneity: $\rho_s(x) \sim \sqrt{|u'(x)|}$.

Let us now show how these two special cases can be combined into a generic case.

7 General Cases: $0 \le \alpha \le 1$

Analysis of the problem. We would like to find the sub-zone density $\rho_s(x)$ that minimizes the loss function (45). This loss function is the sum of two terms: the maximum term and the integral term.

Let us denote the optimal value of the maximum term by $c_{\max}$:

$$c_{\max} \stackrel{\text{def}}{=} \max_x \frac{|u'(x)|}{u(x)\cdot\rho_s(x)}. \tag{63}$$

For the locations x at which this maximum is attained, i.e., at which

$$\frac{|u'(x)|}{u(x)\cdot\rho_s(x)} = c_{\max}, \tag{64}$$

we thus have

$$\rho_s(x) = C_{\max}\cdot\frac{|u'(x)|}{u(x)}, \tag{65}$$

where $C_{\max} \stackrel{\text{def}}{=} \dfrac{1}{c_{\max}}$.

For the locations x at which the maximum is not attained, i.e., at which

$$\frac{|u'(x)|}{u(x)\cdot\rho_s(x)} < c_{\max} \tag{66}$$

and where small changes in $\rho_s(x)$ do not affect the maximum part of the objective function, we need to select the values of $\rho_s(x)$ at these locations so as to minimize the integral term

$$-\frac{2\alpha-1}{2}\cdot\frac{\displaystyle\int_{\underline{x}}^{\overline{x}}\frac{|u'(x)|}{u^2(x)\cdot\rho_s(x)}\,dx}{\displaystyle\int_{\underline{x}}^{\overline{x}}\frac{1}{u(x)}\,dx} \tag{67}$$

under the constraint $\displaystyle\int_{\underline{x}}^{\overline{x}}\rho_s(x) = n_s$. Let us denote the set of all locations x that satisfy the inequality (66) by X. Since we are only looking for the values $\rho_s(x)$ for $x \in X$, this means that we need to minimize the corresponding part of the integral term, i.e., the term

$$-\frac{2\alpha-1}{2}\cdot\frac{\displaystyle\int_X\frac{|u'(x)|}{u^2(x)\cdot\rho_s(x)}\,dx}{\displaystyle\int_{\underline{x}}^{\overline{x}}\frac{1}{u(x)}\,dx} \tag{68}$$

under the constraint

$$\int_X \rho_s(x) = \text{const.} \tag{69}$$

Similarly to the case $\alpha = 1$, we can use the Lagrange multiplier method to solve this constraint optimization problem and thus conclude that for $x \in X$, we have

$$\rho_s(x) = C_{\text{int}}\cdot\frac{\sqrt{|u'(x)|}}{u(x)} \tag{70}$$

for an appropriate constant C_{int}.

These values must satisfy the inequality (66), so we must have

$$\rho_s(x) > C_{\max} \cdot \frac{|u'(x)|}{u(x)} \tag{71}$$

and thus,

$$C_{\text{int}} \cdot \frac{\sqrt{|u'(x)|}}{u(x)} > C_{\max} \cdot \frac{|u'(x)|}{u(x)}. \tag{72}$$

Multiplying both sides of this inequality by $u(x)$ and dividing by $\sqrt{|u'(x)|}$, we get an equivalent inequality

$$C_{\text{int}} > C_{\max} \cdot \sqrt{|u'(x)|}, \tag{73}$$

squaring which we get

$$|u'(x)| < \left(\frac{C_{\text{int}}}{C_{\max}}\right)^2. \tag{74}$$

This is thus the threshold for allocating locations x either to the set X or to the set of points where the maximum is attained:

- when the absolute value $|u'(x)|$ of the derivative exceeds the threshold, the maximum is attained and thus, we get the formula (65);
- on the other hand, when the absolute value of the derivative is smaller than the threshold, we use the formula (70).

So, we arrive at the following solution to our optimization problem.

Main result for the 1-D case: optimal abstraction. Let us assume that we know the gain function $u(x)$ on the interval $[\underline{x}, \overline{x}]$. Instead of using the original gain function $u(x)$, we consider an abstraction, i.e.:

- we divide the interval $[\underline{x}, \overline{x}]$ into n_s sub-zones, and
- we only consider sub-zones when deciding on the best patrolling strategy.

In this situation, the optimal sub-zone density $\rho_s(x)$ – i.e., the sub-zone density for which the expected losses are the smallest – can be obtained if we select two parameters C_{int} and $C_{\max}$ and take

$$\rho_s(x) = C_{\text{int}} \cdot \frac{\sqrt{|u'(x)|}}{u(x)} \text{ when } |u'(x)| < \left(\frac{C_{\text{int}}}{C_{\max}}\right)^2; \tag{75}$$

$$\rho_s(x) = C_{\max} \cdot \frac{|u'(x)|}{u(x)} \text{ when } |u'(x)| \geq \left(\frac{C_{\text{int}}}{C_{\max}}\right)^2. \tag{76}$$

The parameters C_{int} and $C_{\max}$ should be selected in such a way that

$$\int_{\underline{x}}^{\overline{x}} \rho_s(x)\, dx = n_s. \tag{77}$$

Under this constraint, we must select the values C_{int} and $C_{\max}$ that minimizes the functional (45), in which the optimism-pessimism parameter α describes the decision maker's degree of optimism.

8 Multi-D Patrolling Game

Description of the game. Let us now consider a more general case, when the agents need to patrol a multi-D area A. For each location $x = (x_1, \ldots, x_d)$ within this area, we know the value $u(x)$ that the adversary gains if he succeeds in attacking at this point. Similarly to the 1-D case, we assume that the adversary succeeds if there is no agent within a certain distance h from the attack point.

Our objective is to find the distribution of the agents $\rho_a(x)$ that will minimize our expected loss, under the constraint that the overall number of agents is n_a:

$$\int_A \rho_a(x)\,dx = n_a. \tag{78}$$

For each distribution, the adversary's gain is equal to

$$\max_x \,(1 - V_d \cdot h^d \cdot \rho_a(x)) \cdot u(x), \tag{79}$$

where V_d is the d-dimensional volume of a unit ball $B = \{x : \|x\| \le 1\}$:

- for $d = 1$, we have $V_1 = 2$, so the length of an interval of radius h is equal to $2h$;
- for $d = 2$, we have $V_1 = \pi$, so the area of a circle of radius h is equal to $\pi \cdot h^2$;
- for $d = 3$, we have $V_3 = \dfrac{4\pi}{3}$, so the volume of a ball of radius h is equal to $\dfrac{4\pi}{3} \cdot h^3$; etc.

Optimal strategy for the multi-D patrolling game. Similarly to the 1-D case, in situations in which there are no not-worth-defending locations, the optimal patrolling strategy can be determined from the condition that

$$1 - V_d \cdot h^d \cdot \rho_a(x) = \frac{u_0}{u(x)}, \tag{80}$$

i.e.,

$$\rho_a(x) = \frac{1}{V_d \cdot h^d} \cdot \left(1 - \frac{u_0}{u(x)}\right), \tag{81}$$

where the threshold u_0 is determined by the condition (78), as

$$u_0 = \frac{V_A - 2n_a \cdot V_d \cdot h^d}{V_d \cdot h^d \cdot \displaystyle\int_A \frac{1}{u(x)}\,dx}, \tag{82}$$

where V_A is the volume of the area A.

How to describe approximations. Instead of considering individual locations, we divide the area A into rectangular boxes

$$[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_d, \overline{x}_d]. \tag{83}$$

To describe the distribution of such boxes, we now need to have d different density functions:

- the function $\rho_1(x)$ describes how many intervals $[\underline{x}_1, \overline{x}_1]$ we have per unit length at location x;
- the function $\rho_2(x)$ describes how many intervals $[\underline{x}_2, \overline{x}_2]$ we have per unit length at location x; etc.

The x_1-width of each box is thus equal to $\dfrac{1}{\rho_1(x)}$, the x_2-width of each box is equal to $\dfrac{1}{\rho_2(x)}$, etc. Thus, the volume $V(x)$ of each box (83) is equal to the product of these widths:

$$V(x) = \prod_{i=1}^{d} \frac{1}{\rho_i(x)}. \tag{84}$$

In a unit volume, we can thus place

$$\frac{1}{V(x)} = \prod_{i=1}^{d} \rho_i(x) \tag{85}$$

such boxes. The overall number of boxes can be obtained by adding up these numbers for all locations, i.e., as an integral of this product over the area A. This overall number should be equal to the largest number of n_a of boxes that we can computationally handle when selecting a strategy, so we must have

$$\int_A \prod_{i=1}^{d} \rho_i(x)\,dx = n_b. \tag{86}$$

Analysis of the problem. Similarly to the 1-D case, we can represent the i-th coordinate x_i of each point x in a box as $x_i = \widetilde{x}_i + \Delta x_i$, where $\widetilde{x}_i$ is the midpoint of the i-th interval $[\underline{x}_i, \overline{x}_i]$ and $|\Delta x_i| \le \dfrac{w_i}{2}$, where $w_i = \dfrac{1}{\rho_i(x)}$ is the width of this interval.

In the linear approximation, we have

$$u(x_1, \ldots, x_n) = u(\widetilde{x}_1, \ldots, \widetilde{x}_n) + \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} \cdot \Delta x_i. \tag{87}$$

The largest possible value $\overline{u}$ of the gain function on this box and its smallest possible value $\underline{u}$ thus have the form

$$\overline{u} = u + D(\widetilde{x}), \quad \underline{u} = u - D(\widetilde{x}), \tag{88}$$

where

$$D(\widetilde{x}) \stackrel{\text{def}}{=} \frac{1}{2} \cdot \sum_{i=1}^{d} \left| \frac{\partial u}{\partial x_i} \right| \cdot \frac{1}{\rho_i(x)}. \tag{89}$$

So, when we select a patrolling strategy, we use the following Hurwicz optimism-pessimism combination:

$$u^* = \alpha \cdot \underline{u} + (1-\alpha) \cdot \overline{u} = u(x) - (2\alpha - 1) \cdot D(x). \tag{90}$$

Based on this approximate gain function $u^*(x)$, we select the agent allocation $\rho_a^*(x)$ for which

$$1 - V_d \cdot h^d \cdot \rho_a^*(x) = \frac{u_0^*}{u^*(x)}, \tag{91}$$

where

$$u_0^* = \frac{V_A - n_a \cdot V_d \cdot h^d}{V_d \cdot h^d \cdot \displaystyle\int_A \frac{1}{u^*(x)}\,dx}. \tag{92}$$

On each zone, the adversary selects a point x at which the expected gain

$$(1 - V_d \cdot h^d \cdot \rho_a^*(x)) \cdot u(x) \tag{93}$$

is the largest. The value $(1 - V_d \cdot h^d \cdot \rho_a^*(x))$ is constant within the zone, so within each zone, the adversary selects the point at which $u(x)$ is the largest, i.e., the point x at which $u(x) = \overline{u}(x) = u(x) + D(x)$.

Similarly to the 1-D case, we thus conclude that the adversary's gain is equal to

$$u_0 \cdot (1 + G), \tag{94}$$

where

$$G = (1-\alpha) \cdot \max_x \frac{\displaystyle\sum_{i=1}^{d} \left|\frac{\partial u}{\partial x_i}\right| \cdot \frac{1}{\rho_i(x)}}{u(x)} -$$

$$\frac{2\alpha - 1}{2} \cdot \frac{\displaystyle\int_A \frac{1}{u^2(x)} \cdot \sum_{i=1}^{d} \left|\frac{\partial u}{\partial x_i}\right| \cdot \frac{1}{\rho_i(x)}\,dx}{\displaystyle\int_A \frac{1}{u(x)}\,dx}. \tag{95}$$

We need to minimize this gain based on the condition that

$$\int_A \left(\prod_{i=1}^{d} \rho_i(x)\right) dx = n_b. \tag{96}$$

For each location x, we need to select the values of d functions $\rho_1(x), \ldots, \rho_d(x)$. The only constraint that comes from (96) is the constraint on their product:

$$\prod_{i=1}^{d} \rho_i(x) = c \tag{97}$$

for some constant c. The only effect of all these d values $\rho_1(x), \ldots, \rho_d(x)$ on the objective function G is via the combination

$$\sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}\right| \cdot \frac{1}{\rho_i(x)} \tag{98}$$

Thus, for each x, we need to optimize the combination (98) under the constraint (97). For this constraint optimization problem, the Lagrange multiplier method leads to the need to optimize the following objective function:

$$\sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}\right| \cdot \frac{1}{\rho_i(x)} + \lambda \cdot \left(\prod_{i=1}^d \rho_i(x) - c\right). \tag{99}$$

Differentiating this expression with respect to $\rho_i(x)$ and equating the derivative to 0, we conclude that

$$-\left|\frac{\partial u}{\partial x_i}\right| \cdot \frac{1}{\rho_i^2(x)} + \lambda \cdot \frac{1}{\rho_i(x)} \cdot \prod_{j=1}^d \rho_j(x) = 0, \tag{100}$$

hence

$$\rho_i(x) = c(x) \cdot \left|\frac{\partial u}{\partial x_i}\right|, \tag{101}$$

where

$$c(x) \stackrel{\text{def}}{=} \frac{1}{\lambda \cdot \prod\limits_{j=1}^d \rho_j(x)} \tag{102}$$

does not depend on i. For the functions (101), we have

$$\sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}\right| \cdot \frac{1}{\rho_i(x)} = \frac{d}{c(x)}, \tag{103}$$

thus the objective function G takes the form

$$G = (1-\alpha) \cdot d \cdot \max_x \frac{1}{c(x) \cdot u(x)} - \frac{2\alpha - 1}{2} \cdot d \cdot \frac{\displaystyle\int_A \frac{1}{c(x) \cdot u^2(x)}\,dx}{\displaystyle\int_A \frac{1}{u(x)}\,dx}. \tag{104}$$

In terms of the new unknown $c(x)$, the constraint (96) takes the form

$$\int_A c^d(x) \cdot \left(\prod_{i=1}^d \left|\frac{\partial u}{\partial x_i}\right|\right) dx = n_b. \tag{105}$$

For the locations x at which the maximum is attained, i.e., at which

$$\frac{1}{c(x) \cdot u(x)} = c_{\max} \stackrel{\text{def}}{=} \max_x \frac{1}{c(x) \cdot u(x)}, \tag{106}$$

we get

$$c(x) = \frac{1}{c_{\max} \cdot u(x)}, \tag{107}$$

i.e.,

$$c(x) = \frac{C_{\max}}{u(x)}, \tag{108}$$

where $C_{\max} \stackrel{\text{def}}{=} \dfrac{1}{c_{\max}}$. For the set X of locations x at which the maximum is not attained, Lagrange multiplier method leads to optimizing the combination

$$-\frac{2\alpha - 1}{2} \cdot d \cdot \frac{\displaystyle\int_X \frac{1}{c(x) \cdot u^2(x)}\, dx}{\displaystyle\int_A \frac{1}{u(x)}\, dx} + \lambda \cdot \left(\int_A c^d(x) \cdot \left(\prod_{i=1}^d \left| \frac{\partial u}{\partial x_i} \right| \right) dx - n_b \right). \tag{109}$$

Differentiating this expression with respect to $c(x)$ and equating the derivative to 0, we conclude that

$$\text{const} \cdot \frac{1}{c^2(x) \cdot u^2(x)} + \lambda \cdot d \cdot c^{d-1}(x) \cdot \prod_{i=1}^d \left| \frac{\partial u}{\partial x_i} \right| = 0, \tag{110}$$

hence

$$c^{d+1}(x) = \frac{\text{const}}{u^2(x) \cdot \prod\limits_{i=1}^d \left| \dfrac{\partial u}{\partial x_i} \right|}, \tag{111}$$

and

$$c(x) = \frac{C_{\text{int}}}{(u(x))^{2/(d+1)} \cdot \left(\prod\limits_{i=1}^d \left| \dfrac{\partial u}{\partial x_i} \right| \right)^{1/(d+1)}} \tag{112}$$

for some constant C_{int}.

For this value, the fact that the maximum is not attained means that $c(x) < \dfrac{C_{\max}}{u(x)}$, i.e., that

$$\frac{C_{\text{int}}}{(u(x))^{2/(d+1)} \cdot \left(\prod\limits_{i=1}^d \left| \dfrac{\partial u}{\partial x_i} \right| \right)^{1/(d+1)}} < \frac{C_{\max}}{u(x)}. \tag{113}$$

Raising both sides by the power $d+1$, we get an equivalent inequality

$$\frac{(C_{\text{int}})^{d+1}}{u^2(x) \cdot \prod\limits_{i=1}^d \left| \dfrac{\partial u}{\partial x_i} \right|} < \frac{(C_{\max})^{d+1}}{(u(x))^{d+1}}, \tag{114}$$

i.e., equivalently,

$$\frac{\prod_{i=1}^{d} \left|\frac{\partial u}{\partial x_i}\right|}{(u(x))^{d-1}} < \left(\frac{C_{\rm int}}{C_{\max}}\right)^{d+1}. \tag{115}$$

Thus, we arrive at the following formulas for the optimal abstraction.

Main result for the multi-D case: optimal abstraction. Let us assume that we know the gain function $u(x)$ on the d-dimensional area A. Instead of using the original gain function $u(x)$, we consider an abstraction, i.e.:

- we divide the area A into boxes

$$[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_d, \overline{x}_d], \tag{116}$$

  and

- we only consider boxes when deciding on the best patrolling strategy.

To describe the boxes, we use d density functions $\rho_1(x)$, ..., $\rho_d(x)$:

- the function $\rho_1(x)$ describes how many intervals $[\underline{x}_1, \overline{x}_1]$ we take per unit intervals in the vicinity of a given location x,
- the function $\rho_2(x)$ describes how many intervals $[\underline{x}_2, \overline{x}_2]$ we take per unit intervals in the vicinity of a given location x, etc.

In this situation, the optimal densities $\rho_i(x)$ have the form

$$\rho_i(x) = c(x) \cdot \left|\frac{\partial u}{\partial x_i}\right|. \tag{117}$$

To find the function $c(x)$, we need to select two parameters $C_{\rm int}$ and $C_{\max}$ and take

$$c(x) = \frac{C_{\max}}{u(x)} \text{ when } \frac{\prod_{i=1}^{d} \left|\frac{\partial u}{\partial x_i}\right|}{(u(x))^{d-1}} \ge \left(\frac{C_{\rm int}}{C_{\max}}\right)^{d+1}; \tag{118}$$

$$c(x) = \frac{C_{\rm int}}{(u(x))^{2/(d+1)} \cdot \left(\prod_{i=1}^{d} \left|\frac{\partial u}{\partial x_i}\right|\right)^{1/(d+1)}} \text{ otherwise.} \tag{119}$$

The parameters $C_{\rm int}$ and $C_{\max}$ should be selected in such a way that

$$\int_A \left(\prod_{i=1}^{d} \rho_i(x)\right) dx = \int_A (c(x))^d \cdot \left(\prod_{i=1}^{d} \left|\frac{\partial u}{\partial x_i}\right|\right) dx = n_b. \tag{120}$$

Under this constraint, we must select the values $C_{\rm int}$ and $C_{\max}$ that minimizes the functional (104), in which the optimism-pessimism parameter α describes the decision maker's degree of optimism.

Acknowledgments

This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721.

References

[1] L. Hurwicz, *Optimality Criteria for Decision Making under Ignorance*, Cowles Commission Discussion Paper, Statistics, No. 370, 1951.

[2] C. Kiekintveld, J. Marecki, and M. Tambe, "Approximation methods for infinite Bayesian Stackelberg games: modeling distributional payoff uncertainty", *Proceedings of the Tenth International Conference on Autonomous Agents and Multiagent Systems AAMAS'2011*, Taipei, Taiwan, May 2–6, 2011, Vol. 3, pp. 1005–1012.

[3] C. Kiekintveld, J. Marecki, and M. Tambe, "Approximation methods for infinite Bayesian Stackelberg games: modeling distributional payoff uncertainty", in [6], pp. 213–232.

[4] V. Kreinovich, "Decision making under interval uncertainty (and beyond)", In: P. Guo and W. Pedrycz (eds.), *Human-Centric Decision-Making Models for Social Sciences*, Springer Verlag, 2014, pp. 163–193.

[5] R. D. Luce and R. Raiffa, *Games and Decisions: Introduction and Critical Survey*, Dover, New York, 1989.

[6] M. Tambe (ed.), *Security and Game Theory: Algorithms, Deployed Systems, Lessons Learned*, Cambridge University Press, 2012.