8-2015

Invariance Explains Multiplicative and Exponential Skedactic Functions

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

Olga Kosheleva
The University of Texas at El Paso, olgak@utep.edu

Hung T. Nguyen
New Mexico State University - Main Campus, hunguyen@nmsu.edu

Songsak Sriboonchitta
Chiang Mai University, songsakecon@gmail.com

Comments:

Technical Report: UTEP-CS-15-63

Published in: Van Nam Huynh, Vladik Kreinovich, and Songsak Sriboonchitta (eds.), *Causal Inference in Econometrics*, Springer Verlag, Cham, Switzerland, 2016, pp. 119-131.

Invariance Explains Multiplicative and Exponential Skedactic Functions

Vladik Kreinovich, Olga Kosheleva, Hung T. Nguyen, and Songsak Sriboonchitta

Abstract In many situation, we have an (approximately) linear dependence between several quantities: $y \approx c + \sum\limits_{i=1}^{n} a_i \cdot x_i$. The variance $v = \sigma^2$ of the corresponding approximation error $\varepsilon = y - \left(c + \sum\limits_{i=1}^{n} a_i \cdot x_i\right)$ often depends on the values of the quantities $x_1, \ldots, x_n$: $v = v(x_1, \ldots, x_n)$; the function describing this dependence is known as the *skedactic function*. Empirically, two classes of skedactic functions are most successful: multiplicative functions $v = c \cdot \prod\limits_{i=1}^{n} |x_i|^{\gamma_i}$ and exponential functions $v = \exp\left(\alpha + \sum\limits_{i=1}^{n} \gamma_i \cdot x_i\right)$. In this paper, we use natural invariance ideas to provide a possible theoretical explanation for this empirical success; we explain why in some situations multiplicative skedactic functions work better and in some exponential ones. We also come up with a general class of invariant skedactic function that includes both multiplicative and exponential functions as particular cases.

Vladik Kreinovich
Department of Computer Science, University of Texas at El Paso,
500 W. University, El Paso, Texas 79968, USA,
e-mail: vladik@utep.edu

Olga Kosheleva
University of Texas at El Paso, 500 W. University, El Paso, Texas 79968, USA,
e-mail: olgak@utep.edu

Hung T. Nguyen
Department of Mathematical Sciences, New Mexico State University,
Las Cruces, New Mexico 88003, USA, and
Faculty of Economics, Chiang Mai University, Chiang Mai, Thailand,
e-mail: hunguyen@nmsu.edu

Songsak Sriboonchitta
Faculty of Economics, Chiang Mai University, Chiang Mai, Thailand,
e-mail: songsak@econ.chiangmai.ac.th

1 Why Are Multiplicative and Exponential Skedactic Functions Empirically Successful: Formulation of the Problem

Linear dependencies are ubiquitous. In many practical situations, a quantity y depends on several other quantities $x_1,\ldots,x_n$: $y=f(x_1,\ldots,x_n)$. Often, the ranges of x_i are narrow: $x_i \approx x_i^{(0)}$ for some $x_i^{(0)}$, so the differences $\Delta x_i \stackrel{\text{def}}{=} x_i - x_i^{(0)}$ are relatively small. In such situations, we can expand the dependence of y on $x_i = x_i^{(0)} + \Delta x_i$ in Taylor series and keep only linear terms in the resulting expansion:

$$y = f(x_1,\ldots,x_n) = f(x_1^{(0)} + \Delta x_1,\ldots,x_n^{(0)} + \Delta x_n) \approx a_0 + \sum_{i=1}^n a_i \cdot \Delta x_i,$$

where $a_0 \stackrel{\text{def}}{=} f\left(x_1^{(0)},\ldots,x_n^{(0)}\right)$ and $a_i \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial x_i}$. Substituting $\Delta x_i = x_i - x_i^{(0)}$ into this formula, we get

$$y \approx c + \sum_{i=1}^n a_i \cdot x_i,$$

where $c \stackrel{\text{def}}{=} a_0 - \sum\limits_{i=1}^n a_i \cdot x_i^{(0)}$.

Linear dependencies are approximate. Usually, in addition to the quantities $x_1,\ldots,x_n$ that provide the most influence on y, there are also many other quantities that (slightly) influence y, so many that it is not possible to take all of them into account. Since we do not take these auxiliary quantities into account, the above linear dependence is only approximate.

The corresponding approximation errors $\varepsilon \stackrel{\text{def}}{=} y - \left(c + \sum\limits_{i=1}^n a_i \cdot x_i\right)$ depend on unobserved quantities and thus, cannot be predicted based only on the values of the observed quantities $x_1,\ldots,x_n$. It is therefore reasonable to view these errors as random variables.

Skedactic functions. A natural way to describe a random variable is by its moments, starting with the mean – the first moment – and the variance – which enables us to compute the second moment. If the first moment is not 0, i.e., if the linear approximation is biased, we can always correct this bias by appropriately updating the constant c.

Next, we need to know the second moment which, since the mean is 0, coincides with the variance v. In general, for different values of x_i, we may have different values of the variance. For example, in econometrics, if we are trying to predict how investment x_1 in an industry affects its output y, clearly larger investments result result not only in larger output, but also in larger output variations.

The function $v(x_1,\ldots,x_n)$ that describes how the variance depends on the values of the quantities $x_1,\ldots,x_n$ is known as the *skedactic function.*

Which skedactic functions are empirically successful. In econometric applications, two major classes of skedactic functions have been empirically successful:

multiplicative functions (see, e.g., [2]; [3], Section 9.3; and [4])

$$v(x_1,\ldots,x_n) = c \cdot \prod_{i=1}^{n} |x_i|^{\gamma_i}$$

and exponential functions ([5], Chapter 8)

$$v(x_1,\ldots,x_n) = \exp\left(\alpha + \sum_{i=1}^{n} \gamma_i \cdot x_i\right).$$

According to the latest review [4]:

- neither of this functions has a theoretical justification, and
- in most situations, the multiplication function results in more accurate estimates.

What we do in this paper. In this paper, we use reasonable invariance ideas to provide a possible theoretical explanation for the empirical success of multiplicative and exponential skedactic functions.

We also use invariance to come up with a more general class of skedactic functions to use when neither multiplicative nor exponential functions provide a sufficiently accurate description of the desired dependence.

2 Natural Invariances

Scaling. Many economics quantities correspond to prices, wages, etc. and are therefore expressed in terms of money. The numerical value of such a quantity depends on the choice of a monetary unit. For example, when a European country switches to Euro from its original currency, the actual incomes do not change (at least not immediately), but all the prices and wages get multiplied by the corresponding exchange rate k: $x_i \to x'_i = k \cdot x_i$.

Similarly, quantities that describe the goods, such as amount of oil or amount of sugar, also change their numerical values when we use different units: for example, for the oil production, we get different numerical values when we use barrels and when we use metric tons.

When the numerical value of a quantity gets thus re-scaled (multiplied by a constant), the value of its variance gets multiplied by the square of this constant.

Scale-invariance. Since changing the measuring units for measuring $x_1,\ldots,x_n$ does not change the corresponding economic situations, it makes sense to require that the skedactic function also does not change under such re-scaling: namely, for each combination of re-scalings on inputs, there should be an appropriate re-scaling of the output after which the dependence remains the same.

In precise terms, this means that for every combination of numbers $k_1,\ldots,k_n$, there should exist a value $k = k(k_1,\ldots,k_n)$ with the following property:

$v = v(x_1, \ldots, x_n)$ if and only if $v' = v(x'_1, \ldots, x'_n)$, where $v' = k \cdot v$ and $x'_i = k_i \cdot x_i$.

Shift and shift-invariance. While most economic quantities are scale-invariant, some are not: e.g., the unemployment rate is measured in percents, there is a fixed unit. Many such quantities, however, can have different numerical values depending on how we define a starting point.

For example, we can measure unemployment in absolute units, or we can measure it by considering the difference $x_i - k_i$ between the actual unemployment and the ideal level $k_i > 0$ which, in the opinion of the economists, corresponds to full employment.

In general, for such quantities, we have a *shift* transformation $x_i \to x'_i = x_i + k_i$. To consider dependence on such quantities, it is therefore reasonable to consider skedactic functions which are shift-invariant, i.e., for which for every combinations of numbers $(k_1, \ldots, k_n)$, there exists a number k for which

$v = v(x_1, \ldots, x_n)$ if and only if $v' = f(x'_1, \ldots, x'_n)$, where $v' = k \cdot v$ and $x'_i = x_i + k_i$.

3 Case of Scale Invariance: Definitions and the Main Result

Definition 1. *We say that a non-negative measurable function $v(x_1, \ldots, x_n)$ is* scale-invariant *if for every n-tuple of real numbers $(k_1, \ldots, x_n)$, there exists a real number $k = k(k_1, \ldots, k_n)$ for which, for every $x_1, \ldots, x_n$ and v, the following two conditions are equivalent to each other:*

- $v = v(x_1, \ldots, x_n)$*;*
- $v' = v(x'_1, \ldots, x'_n)$*, where $v' = k \cdot v$ and $x'_i = k_i \cdot x_i$.*

Proposition 1. *A skedactic function is scale-invariant if and only it has the form $v(x_1, \ldots, x_n) = c \cdot \prod\limits_{i=1}^{n} |x_i|^{\gamma_i}$ for some values c and γ_i.*

Comment. For reader's convenience, all the proofs are placed in the last Proofs section.

Discussion. Thus, scale-invariance explains the use of multiplicative skedactic functions.

4 Case of Shift-Invariance: Definitions and the Main Result

Definition 2. *We say that a non-zero non-negative measurable function $v(x_1, \ldots, x_n)$ is* shift-invariant *if for every n-tuple of real numbers $(k_1, \ldots, x_n)$, there exists a real number $k = k(k_1, \ldots, k_n)$ for which, for every $x_1, \ldots, x_n$ and v, the following two conditions are equivalent to each other:*

- $v = v(x_1, \ldots, x_n)$;
- $v' = v(x'_1, \ldots, x'_n)$, *where* $v' = k \cdot v$ *and* $x'_i = x_i + k_i$.

Proposition 2. *A skedactic function is scale-invariant if and only it has the form* $v(x_1, \ldots, x_n) = \exp\left(\alpha + \prod_{i=1}^{n} \gamma_i \cdot x_i\right)$ *for some values* α *and* γ_i.

Discussion. Thus, shift-invariance explains the use of exponential skedactic functions. The fact that most economic quantities are scale-invariant explains why, in general, multiplicative skedactic functions are more empirically successful.

5 General Case

General case: discussion. A general case is when some of the inputs are scale-invariant and some are shift-invariant. Without losing generality, let us assume that the first m variables $x_1, \ldots, x_m$ are scale-invariant, while the remaining variables $x_{m+1}, \ldots, x_n$ are shift-invariant.

Definition 3. *Let* $m \le n$ *be an integer. We say that a non-zero non-negative measurable function* $v(x_1, \ldots, x_n)$ *is* m-invariant *if for every* n*-tuple of real numbers* $(k_1, \ldots, x_n)$*, there exists a real number* $k = k(k_1, \ldots, k_n)$ *for which, for every* $x_1, \ldots, x_n$ *and* v*, the following two conditions are equivalent to each other:*

- $v = v(x_1, \ldots, x_n)$;
- $v' = v(x'_1, \ldots, x'_n)$, *where* $v' = k \cdot v$, $x'_i = k_i \cdot x_i$ *for* $i \le m$, *and* $x'_i = x_i + k_i$ *for* $i > m$.

Proposition 3. *A skedactic function is m-invariant if and only it has the form*

$$v(x_1, \ldots, x_n) = \exp\left(\alpha + \sum_{i=1}^{m} \gamma_i \cdot \ln(|x_i|) + \sum_{i=m+1}^{n} \gamma_i \cdot x_i\right) \tag{3}$$

for some values μ *and* γ_i.

Discussion. For $m = n$, this formula leads to a multiplicative skedactic function, with $c = \exp(\alpha)$. For $m = 0$, this formula leads to the exponential skedactie function. For intermediate values $m = 1, 2, \ldots, n-1$, we get new expressions that may be useful when neither multiplicative not exponential skedactic functions work well.

6 Proofs

Proof of Proposition 1. It is easy to check that the multiplicative skedactic function is indeed scale-invariant: we can take $k = \prod_{i=1}^{n} |k_i|^{\gamma_i}$.

Let us prove that, vice versa, if a skedactic function is scale-invariant, then it is multiplicative. Indeed, the above equivalence condition means that for every $k_1,\ldots,k_n$, $v = v(x_1,\ldots,x_n)$ implies that $v' = v(x'_1,\ldots,x'_n)$, where $v' = k\cdot v$ and $x'_i = k_i\cdot x_i$. Substituting the expressions for v' and k'_i into the equality $v' = v(x'_1,\ldots,x'_n)$, we conclude that $k\cdot v = v(k_1\cdot x_1,\ldots,k_n\cdot x_n)$.

We know that $k = k(k_1,\ldots,k_n)$ and $v = v(x_1,\ldots,x_n)$. Thus, we conclude that

$$k(k_1,\ldots,k_n)\cdot v(x_1,\ldots,x_n) = v(k_1\cdot x_1,\ldots,k_n\cdot x_n). \tag{4}$$

From this equation, we infer that

$$k(k_1,\ldots,k_n) = \frac{v(k_1\cdot x_1,\ldots,k_n\cdot x_n)}{v(x_1,\ldots,x_n)}. \tag{5}$$

The right-hand side of this formula is a non-negative measurable function, so we can conclude that the ratio $k(k_1,\ldots,k_n)$ is also non-negative and measurable.

Let us now consider two different tuples $(k_1,\ldots,k_n)$ and $(k'_1,\ldots,k'_n)$. If we first use the first re-scaling, i.e., go from x_i to $x'_i = k_i\cdot x_i$, we get

$$v(x'_1,\ldots,x'_n) = v(k_1\cdot x_1,\ldots,k_n\cdot x_n) = k(k_1,\ldots,k_n)\cdot v(x_1,\ldots,x_n). \tag{6}$$

If we then apply, to the new values x'_i, an additional re-scaling $x'_i \to x''_i = k'_i\cdot x'_i$, we similarly conclude that

$$v(x''_1,\ldots,x''_n) = v(k'_1\cdot x'_1,\ldots,k'_n\cdot x'_n) = k(k'_1,\ldots,k'_n)\cdot v(x'_1,\ldots,x'_n). \tag{7}$$

Substituting the expression (6) for $v(x'_1,\ldots,x'_n)$ into this formula, we conclude that

$$v(x''_1,\ldots,x''_n) = k(k'_1,\ldots,k'_n)\cdot k(k_1,\ldots,k_n)\cdot v(x_1,\ldots,x_n). \tag{8}$$

On the other hand, we could get the values x''_i if we directly multiply each value x_i by the product $k''_i \stackrel{\text{def}}{=} k'_i\cdot k_i$:

$$x''_i = k'_i\cdot x'_i = k'_i\cdot(k_i\cdot x_i) = (k'_i\cdot k_i)\cdot x_i = k''_i\cdot x_i.$$

For the new values k''_i, the formula (6) takes the form

$$v(x''_1,\ldots,x''_n) = k(k''_1,\ldots,k''_n)\cdot v(x_1,\ldots,x_n) = k(k'_1\cdot k_1,\ldots,k'_n\cdot k_n)\cdot v(x_1,\ldots,x_n). \tag{9}$$

The left-hand sides of the formulas (8) and (9) and the same, hence the right-hand sides are also equal, i.e.,

$$k(k'_1\cdot k_1,\ldots,k'_n\cdot k_n)\cdot v(x_1,\ldots,x_n) = k(k'_1,\ldots,k'_n)\cdot k(k_1,\ldots,k_n)\cdot v(x_1,\ldots,x_n). \tag{10}$$

If the skedactic function is always equal to 0, then it is multiplicative, with $c = 0$. If it is not everywhere 0, this means that its value is different from 0 for some combination of values $x_1,\ldots,x_n$. Substituting these values into the formula (10) and

dividing both sides by $v(x_1,\ldots,x_n) \neq 0$, we conclude that

$$k(k'_1 \cdot k_1,\ldots,k'_n \cdot k_n) = k(k'_1,\ldots,k'_n) \cdot k(k_1,\ldots,k_n). \tag{11}$$

When $k_i = k'_i = -1$ for some i and $k'_i = k_i = 1$ for all other i, we get

$$1 = k(1,\ldots,1) = k(k_1,\ldots,k_n) \cdot k(k_1,\ldots,k_n) = k^2(k_1,\ldots,k_n). \tag{12}$$

Since the function k_i is non-negative, this means that $k(k_1,\ldots,k_n) = 1$. Thus, from the formula (11), we can conclude that the value $k(k_1,\ldots,k_n)$ does not change if we change the signs of k_i, i.e., that

$$k(k_1,\ldots,k_n) = k(|k_1|,\ldots,|k_n|).$$

Taking logarithms of both sides of the formula (11), and taking into account that $\ln(a \cdot a') = \ln(a) + \ln(a')$, we conclude that

$$\ln(k(k'_1 \cdot k_1,\ldots,k'_n \cdot k_n)) = \ln(k(k'_1,\ldots,k'_n)) + \ln(k(k_1,\ldots,k_n)). \tag{13}$$

Let us now define an auxiliary function

$$K(K_1,\ldots,K_n) \stackrel{\text{def}}{=} \ln(k(\exp(K_1),\ldots,\exp(K_n)).$$

Since the function $k(k_1,\ldots,k_n)$ is measurable, the function $K(K_1,\ldots,K_n)$ is also measurable.

Since $\exp(a + a') = \exp(a) \cdot \exp(a')$, we conclude that when $k_i = \exp(K_i)$ and $k'_i = \exp(K'_i)$, then $k_i \cdot k'_i = \exp(K_i) \cdot \exp(K'_i) = \exp(K_i + K'_i)$. Thus, from (13), we conclude that for the new function $K(K_1,\ldots,K_n)$, we get

$$K(K'_1 + K_1,\ldots,K'_n + K_n) = K(K'_1,\ldots,K'_n) + K(K_1,\ldots,K_n). \tag{14}$$

Functions that satisfy the property (14) are known as *additive*. It is known (see, e.g., [1]) that every measurable additive function is linear, i.e., has the form

$$K(K_1,\ldots,K_n) = \sum_{i=1} \gamma_i \cdot X_i \tag{15}$$

for some values γ_i.

From $K(K_1,\ldots,K_n) = \ln(k(\exp(K_1),\ldots,\exp(K_n)))$, it follows that

$$k(\exp(K_1),\ldots,\exp(K_n)) = \exp(K(K_1,\ldots,K_n)) = \exp\left(\sum_{i=1}^{n} \gamma_i \cdot K_i\right).$$

For each $k_1,\ldots,k_n$, we have

$$k(k_1,\ldots,k_n) = k(|k_1|,\ldots,|k_n|).$$

For $K_i = \ln(|k_i|)$, we have $\exp(K_i) = |k_i|$, hence

$$k(k_1,\ldots,k_n) = \exp\left(\sum_{i=1}^n \gamma_i \cdot \ln(|k_i|)\right) = \prod_{i=1}^n |k_i|^{\gamma_i}. \tag{16}$$

From (4), we can now conclude that

$$v(x_1,\ldots,x_n) = k(x_1,\ldots,x_n) \cdot v(1,\ldots,1).$$

Substituting expression (16) for $k(x_1,\ldots,x_n)$ into this formula and denoting $c \stackrel{\text{def}}{=} v(1,\ldots,1)$, we get the desired formula for the multiplicative skedastic function $v(x_1,\ldots,x_n) = c \cdot \prod\limits_{i=1}^n |x_i|^{\gamma_i}$. The proposition is proven.

Proof of Proposition 2. It is easy to check that the exponential skedactic function is indeed shift-invariant: we can take $k = \exp\left(\sum\limits_{i=1}^n \gamma_i \cdot k_i\right)$.

Let us prove that, vice versa, if a skedactic function is shift-invariant, then it is exponential. Indeed, the above equivalence condition means that for every $k_1,\ldots,k_n$, $v = v(x_1,\ldots,x_n)$ implies that $v' = v(x'_1,\ldots,x'_n)$, where $v' = k \cdot v$ and $x'_i = x_i + k_i$. Substituting the expressions for v' and k'_i into the equality $v' = v(x'_1,\ldots,x'_n)$, we conclude that $k \cdot v = v(x_1 + k_1,\ldots,x_n + k_n)$.

We know that $k = k(k_1,\ldots,k_n)$ and $v = v(x_1,\ldots,x_n)$. Thus, we conclude that

$$k(k_1,\ldots,k_n) \cdot v(x_1,\ldots,x_n) = v(x_1 + k_1,\ldots,x_n + k_n). \tag{17}$$

From this equation, we infer that

$$k(k_1,\ldots,k_n) = \frac{v(x_1 + k_1,\ldots,x_n + k_n)}{v(x_1,\ldots,x_n)}. \tag{18}$$

The right-hand side of this formula is a non-negative measurable function, so we can conclude that the ratio $k(k_1,\ldots,k_n)$ is also non-negative and measurable.

Let us now consider two different tuples $(k_1,\ldots,k_n)$ and $(k'_1,\ldots,k'_n)$. If we first use the first shift, i.e., go from x_i to $x'_i = x_i + k_i$, we get

$$v(x'_1,\ldots,x'_n) = v(x_1 + k_1,\ldots,x_n + k_n) = k(k_1,\ldots,k_n) \cdot v(x_1,\ldots,x_n). \tag{19}$$

If we then apply, to the new values x'_i, an additional shift $x'_i \to x''_i = x'_i + k'_i$, we similarly conclude that

$$v(x''_1,\ldots,x''_n) = v(x'_1 + k'_1,\ldots,x'_n + k'_n) = k(k'_1,\ldots,k'_n) \cdot v(x'_1,\ldots,x'_n). \tag{20}$$

Substituting the expression (19) for $v(x'_1,\ldots,x'_n)$ into this formula, we conclude that

$$v(x''_1,\ldots,x''_n) = k(k'_1,\ldots,k'_n) \cdot k(k_1,\ldots,k_n) \cdot v(x_1,\ldots,x_n). \tag{21}$$

On the other hand, we could get the values x''_i if we directly shift each value x_i by the sum $k''_i \stackrel{\text{def}}{=} k'_i + k_i$:

$$x_i'' = x_i' + k_i' = (x_i + k_i) + k_i' = x_i + (k_i + k_i') = x_i + k_i''$$

For the new values k_i'', the formula (19) takes the form

$$v(x_1'', \dots, x_n'') = k(k_1'', \dots, k_n'') \cdot v(x_1, \dots, x_n) =$$
$$k(k_1 + k_1', \dots, k_n + k_n') \cdot v(x_1, \dots, x_n). \quad (22)$$

The left-hand sides of the formulas (21) and (22) and the same, hence the right-hand sides are also equal, i.e.,

$$k(k_1 + k_1', \dots, k_n + k_n') \cdot v(x_1, \dots, x_n) =$$
$$k(k_1', \dots, k_n') \cdot k(k_1, \dots, k_n) \cdot v(x_1, \dots, x_n). \quad (23)$$

Since the skedactic function is assumed to be non-zero, its value is different from 0 for some combination of values $x_1, \dots, x_n$. Substituting these values into the formula (23) and dividing both sides by $v(x_1, \dots, x_n) \neq 0$, we conclude that

$$k(k_1 + k_1', \dots, k_n + k_n') = k(k_1', \dots, k_n') \cdot k(k_1, \dots, k_n). \quad (24)$$

Taking logarithms of both sides of the formula (11), and taking into account that $\ln(a \cdot a') = \ln(a) + \ln(a')$, we conclude that

$$\ln(k(k_1 + k_1', \dots, k_n + k_n')) = \ln(k(k_1', \dots, k_n')) + \ln(k(k_1, \dots, k_n)). \quad (25)$$

Thus, the function $\ln(k(k_1, \dots, k_n))$ is measurable and additive, and hence ([1]) has the form

$$\ln(k(k_1, \dots, k_n)) = \sum_{i=1}^{n} \gamma_i \cdot k_i.$$

Hence, by taking exp of both sides, we conclude that

$$k(k_1, \dots, k_n) = \exp\left(\sum_{i=1}^{n} \gamma_i \cdot k_i\right). \quad (26)$$

From (17), we can now conclude that

$$v(x_1, \dots, x_n) = k(x_1, \dots, x_n) \cdot v(0, \dots, 0).$$

Substituting expression (26) for $k(x_1, \dots, x_n)$ into this formula and denoting $\alpha \stackrel{\text{def}}{=} \ln(v(0, \dots, 0))$, so that $v(0, \dots, 0) = \exp(\alpha)$, we get the desired formula for the exponential skedastic function $v(x_1, \dots, x_n) = \exp\left(\alpha + \sum_{i=1}^{n} \gamma_i \cdot x_i\right)$. The proposition is proven.

Proof of Proposition 3. It is easy to check that the skedactic function described in the formulation of Proposition 3 is indeed m-invariant: we can take

$$k = \prod_{i=1}^{m} |k_i|^{\gamma_i} \cdot \exp\left(\sum_{i=m+1}^{n} \gamma_i \cdot k_i\right).$$

Let us prove that, vice versa, if a skedactic function is m-invariant, then it has the desired form. Indeed, the above equivalence condition means that for every $k_1,\ldots,k_n$, $v = v(x_1,\ldots,x_n)$ implies that $v' = v(x'_1,\ldots,x'_n)$, where $v' = k \cdot v$, $x'_i = k_i \cdot x_i$ for $i \le m$, and $x'_i = x_i + k_i$ for $i > m$. Substituting the expressions for v' and k'_i into the equality $v' = v(x'_1,\ldots,x'_n)$, we conclude that

$$k \cdot v = v(k_1 \cdot x_1,\ldots,k_m \cdot x_m, x_{m+1} + k_{m+1},\ldots,x_n + k_n).$$

We know that $k = k(k_1,\ldots,k_n)$ and $v = v(x_1,\ldots,x_n)$. Thus, we conclude that

$$k(k_1,\ldots,k_n) \cdot v(x_1,\ldots,x_n) = v(k_1 \cdot x_1,\ldots,k_m \cdot x_m, x_{m+1} + k_{m+1},\ldots,x_n + k_n). \quad (27)$$

From this equation, we infer that

$$k(k_1,\ldots,k_n) = \frac{v(k_1 \cdot x_1,\ldots,k_m \cdot x_m, x_{m+1} + k_{m+1},\ldots,x_n + k_n)}{v(x_1,\ldots,x_n)}. \quad (28)$$

The right-hand side of this formula is a non-negative measurable function, so we can conclude that the ratio $k(k_1,\ldots,k_n)$ is also non-negative and measurable.

Let us now consider two different tuples $(k_1,\ldots,k_n)$ and $(k'_1,\ldots,k'_n)$. If we first use the transformation corresponding to the first tuple, i.e., go from x_i to $x'_i = k_i \cdot x_i$ for $i \le m$ and to $x'_i = x_i + k_i$ for $i > m$, we get

$$v(x'_1,\ldots,x'_n) = v(k_1 \cdot x_1,\ldots,k_m \cdot x_m, x_{m+1} + k_{m+1},\ldots,x_n + k_n) =$$
$$k(k_1,\ldots,k_n) \cdot v(x_1,\ldots,x_n). \quad (29)$$

If we then apply, to the new values x'_i, an additional transformation $x'_i \to x''_i = k'_i \cdot x'_i$ for $i \le m$ and $x'_i \to x''_i = x'_i + k'_i$ for $i > m$, we similarly conclude that

$$v(x''_1,\ldots,x''_n) = v(k'_1 \cdot x'_1,\ldots,k'_m \cdot x'_m, x'_{m+1} + k'_{m+1},\ldots,x'_n + k'_n) =$$
$$k(k'_1,\ldots,k'_n) \cdot v(x'_1,\ldots,x'_n). \quad (30)$$

Substituting the expression (29) for $v(x'_1,\ldots,x'_n)$ into this formula, we conclude that

$$v(x''_1,\ldots,x''_n) = k(k'_1,\ldots,k'_n) \cdot k(k_1,\ldots,k_n) \cdot v(x_1,\ldots,x_n). \quad (31)$$

On the other hand, we could get the values x''_i if we directly apply to the tuple x_i the transformation corresponding to the product $k''_i = k'_i \cdot k_i$ for $i \le m$ and to the sum $k''_i = k_i + k'_i$ for $i > m$: b

$$x''_i = k'_i \cdot x'_i = k'_i \cdot (k_i \cdot x_i) = (k'_i \cdot k_i) \cdot x_i = k''_i \cdot x_i$$

for $i \le m$ and

$$x_i'' = x_i' + k_i' = (x_i + k_i) + k_i' = x_i + (k_i + k_i') = x_i + k_i''$$

for $i > m$.

For the new values k_i'', the formula (29) takes the form

$$v(x_1'', \ldots, x_n'') = k(k_1'', \ldots, k_n'') \cdot v(x_1, \ldots, x_n) =$$

$$k(k_1' \cdot k_1, \ldots, k_m' \cdot k_m, k_{m+1} + k_{m+1}', \ldots, k_n + k_n') \cdot v(x_1, \ldots, x_n). \quad (32)$$

The left-hand sides of the formulas (31) and (32) and the same, hence the right-hand sides are also equal, i.e.,

$$k(k_1' \cdot k_1, \ldots, k_m' \cdot k_m, k_{m+1} + k_{m+1}', \ldots, k_n + k_n') \cdot v(x_1, \ldots, x_n) =$$

$$k(k_1', \ldots, k_n') \cdot k(k_1, \ldots, k_n) \cdot v(x_1, \ldots, x_n). \quad (33)$$

Since we assume that the skedactic function is non-zero, its value is different from 0 for some combination of values $x_1, \ldots, x_n$. Substituting these values into the formula (33) and dividing both sides by $v(x_1, \ldots, x_n) \neq 0$, we conclude that

$$k(k_1' \cdot k_1, \ldots, k_m' \cdot k_m, k_{m+1} + k_{m+1}', \ldots, k_n + k_n') = k(k_1', \ldots, k_n') \cdot k(k_1, \ldots, k_n). \quad (34)$$

When $k_i = k_i' = -1$ for some $i \leq m$ and $k_i' = k_i = 1$ for all other i, we get

$$1 = k(1, \ldots, 1) = k(k_1, \ldots, k_n) \cdot k(k_1, \ldots, k_n) = k^2(k_1, \ldots, k_n). \quad (35)$$

Since the function k_i is non-negative, this means that $k(k_1, \ldots, k_n) = 1$. Thus, from the formula (34), we can conclude that the value $k(k_1, \ldots, k_n)$ does not change if we change the signs of k_i for $i \leq m$, i.e., that

$$k(k_1, \ldots, k_m, k_{m+1}, \ldots, k_n) = k(|k_1|, \ldots, |k_m|, k_{m+1}, \ldots, k_n).$$

Taking logarithms of both sides of the formula (35), and taking into account that $\ln(a \cdot a') = \ln(a) + \ln(a')$, we conclude that

$$\ln(k(k_1' \cdot k_1, \ldots, k_m' \cdot k_m, k_{m+1} + k_{m+1}', \ldots, k_n + k_n')) =$$

$$\ln(k(k_1', \ldots, k_n')) + \ln(k(k_1, \ldots, k_n)). \quad (36)$$

Let us now define an auxiliary function

$$K(K_1, \ldots, K_n) \stackrel{\text{def}}{=} \ln(k(\exp(K_1), \ldots, \exp(K_m), K_{m+1}, \ldots, K_n)).$$

Since the function $k(k_1, \ldots, k_n)$ is measurable, the function $K(K_1, \ldots, K_n)$ is also measurable.

Since $\exp(a + a') = \exp(a) \cdot \exp(a')$, for $i \leq m$, we conclude that when $k_i = \exp(K_i)$ and $k_i' = \exp(K_i')$, then $k_i \cdot k_i' = \exp(K_i) \cdot \exp(K_i') = \exp(K_i + K_i')$. Thus, from (36), we conclude that for the new function $K(K_1, \ldots, K_n)$, we get

$$K(K'_1+K_1,\ldots,K'_n+K_n)=K(K'_1,\ldots,K'_n)+K(K_1,\ldots,K_n). \tag{37}$$

The function $K(K_1,\ldots,K_n)$ is measurable and additive and hence [1] has the form

$$K(K_1,\ldots,K_n)=\sum_{i=1}^n \gamma_i\cdot X_i \tag{38}$$

for some values γ_i.

From

$$K(K_1,\ldots,K_n)=\ln(k(\exp(K_1),\ldots,\exp(K_m),K_{m+1},\ldots,K_n)),$$

it follows that

$$k(\exp(K_1),\ldots,\exp(K_m),K_{m+1},\ldots,K_n)=\exp(K(K_1,\ldots,K_n))=\exp\left(\sum_{i=1}^n \gamma_i\cdot K_i\right).$$

For each $k_1,\ldots,k_n$, we have

$$k(k_1,\ldots,k_m,k_{m+1},\ldots,k_n)=k(|k_1|,\ldots,|k_m|,k_{m+1},\ldots,k_n).$$

Let us take $K_i=\ln(|k_i|)$ for $i\le m$ and $K_i=k_i$ for $i>m$, then we have $\exp(K_i)=|k_i|$ for $i\le m$ and $K_i=k_i$ for $i>m$. Hence,

$$k(k_1,\ldots,k_n)=\exp\left(\sum_{i=1}^m \gamma_i\cdot\ln(|k_i|)+\sum_{i=m+1}^n \gamma_i\cdot k_i\right). \tag{39}$$

From (27), we can now conclude that

$$v(x_1,\ldots,x_n)=k(x_1,\ldots,x_n)\cdot v(1,\ldots,1,0,\ldots,0).$$

Substituting expression (39) for $k(x_1,\ldots,x_n)$ into this formula, we get the desired formula for the skedastic function

$$v(x_1,\ldots,x_n)=\exp\left(\alpha+\sum_{i=1}^m \gamma_i\cdot\ln(|x_i|)+\sum_{i=m+1}^n \gamma\cdot x_i\right),$$

with $\alpha=\ln(v(1,\ldots,1,0,\ldots,0))$. The proposition is proven.

Acknowledgments

We acknowledge the partial support of the Center of Excellence in Econometrics, Faculty of Economics, Chiang Mai University, Thailand.

This work was also supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721.

References

1. J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 2008.
2. A. C. Harvey, "Estimating regression models with multiplicative heteroscedasticity", *Econometrica*, 1976, Vol. 44, pp. 461–465.
3. G. G. Judge, R. C. Hill, W. E. Griffiths, H. Lütkepohl, and T.-C.Lee, *Introduction to the Theory and Practice of Econometrics*, John Wiley & Sons, New York, 1988.
4. J. P. Romano and M. Wolf, *Resurrecting Weighted Least Squares*, University of Zurich, Department of Economics, Working Paper No. 172, August 2015, available at SSRN http://wwrn.com/abstract=2491081.
5. J. M. Wooldridge, *Introductory Econometrics*, South-Western, Mason, Ohio, 2012.