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Al-Sijistani's and Maimonides's Double Negation Theology Explained by Constructive Logic

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Abstract

Famous medieval philosophers Al-Sijistani and Maimonides argued that the use of double negation helps us to better understand issues related to theology. To a modern reader, however, their arguments are somewhat obscure and unclear. We show that these arguments can be drastically clarified if we take into account the 20 century use of double negation in constructive logic.

1 Double Negation Theology: A Brief Reminder

What is double negation theology. Abu Yakub Al-Sijistani (d. 971) and Moses ibn Maimon (1135–1204), also known as Maimonides, claimed that while while God is essentially incomprehensible, it is possible to gain some knowledge of God by using double negation; see, e.g., [15, 19]. For example, one cannot say that God is good, but it make sense to say that God is not not-good.

Why double negation? The reasoning behind the use of double negation is, to a modern reader, rather obscure and unclear.

In this short paper, we will show, however, that the use of double negation can be made much clearer to the modern reader if we take into account the 20 century developments in constructive logic.

2 What Is Constructive Logic: A Reminder

Constructivity in mathematics before the 20 century. Strictly speaking, mathematics is about proving results. However, from the ancient times, mathematicians were also interested in *constructing* objects.

The need for constructions is motivated largely by applications. For example, to predict where a satellite will be at some future moment of time, we need to

solve the system of differential equations describing the satellite's motion. From the practical viewpoint, it is not enough to prove the existence of the solution, we actually need an algorithm for producing such a solution.

Similarly, when we select a control strategy that optimizes our objective function, it is not enough to prove the existence of an optimal strategy, we need to actually construct such a strategy.

Need for constructivity has been well understood since the ancient times. For example, Euclid's *Elements* include not only geometric proofs, but also constructions of different geometric objects [8].

Until the 20 century, once the existence of an object satisfying certain properties was proven, eventually a way was found to construct this object. The resulting construction was sometimes approximate – e.g., it turned out that we cannot exactly trisect an angle by using only a compass and a rules, only approximately – but it was possible. From this viewpoint, existence and constructibility were viewed as synonyms.

This belief was best captured by David Hilbert who, in 1900, was asked, by the world's mathematical community, to select the list of most important mathematical challenges for the 20 century mathematics: "In mathematics there is no *ignorabimus*" [12].

20th century discovery: constructibility is different from existence. The famous Gödel's incompleteness theorem [10] showed that, contrary to Hilbert's expectations, there are mathematical statements about which we cannot tell whether they are true or false.

This result lead to the conclusion that in some cases, we can prove the existence of an object, but no algorithm is possible for computing this object; see, e.g., the famous Turing's paper [18]. In other words, not everything that can be proven to exist is constructible.

Two approaches to dealing with the difference between provable existence and constructibility. Until Gödel's work, the same existential quantifier $\exists x\, P(x)$ was assumed to mean two (supposedly) equivalent things:

- provable existence of an object x that satisfies the property $P(x)$, and
- a possibility to actually construct an object x with this property.

Once it was realized that these two notions are different, two approaches emerged:

- In the traditional mathematical approach, $\exists x\, P(x)$ is reserved to mean provable existence. In this approach, we need special algorithmic analysis to find out what is computable and what is not.
- An alternative approach of *constructive logic* is to use $\exists x\, P(x)$ to mean that we have a *construction* for generating an object x that satisfies the property $P(x)$.

Constructive logic approach: details. Historically, constructive logic was originated by A. N. Kolmogorov [13]; he called it *logic of problems*. In this logic, $A \vee B$ means that we know exactly which of the two statements A and B is true.

Most statements of constructive logic (except for the simplest ones that do not contain $\exists$ or $\vee$) include the existence of some objects. From this viewpoint, the truth of an implication $A \to B$ not only means that A implies B, it also means that there exists an algorithm that, given an object whose existence is implied by the truth of the statement A, constructs an object whose existence is needed to prove the truth of the statement B – and such an algorithm must be explicitly provided.

Negation $\neg A$ (as it is often done in logic) is viewed as a particular case of the implication: namely, as an implication $A \to F$, where F is a known false statement (e.g., $0 = 1$).

Constructive logic is different from the classical logic. For example, in classical logic, we have the law of the excluded middle: $A \vee \neg A$ is always true. Because of this, a double negation of each statement $\neg\neg A$ is, in classical logic, simply equivalent to the original statement A.

In the constructive logic, this would mean that we know which of the statements A and $\neg A$ is true. Since Gödel proved that it is, in general, not possible to know – even for statements of the type $\forall n\, P(n)$, where n goes over natural numbers and $P(n)$ is an easy-to-test property – we thus, in general, *do not* have the law of excluded middle. As a result, in constructive logic, double negation $\neg\neg A$ is, in general, different from the original statement A.

In constructive logic, we do not have the law of the excluded middle, but what we do have – as was already proven in [13] – is the *double negation* of this law, i.e., a statement $\neg\neg(A \vee \neg A)$.

Moreover, it has been proven [9] that if instead of the logical operations $\vee$ and $\exists$, we take their double negations $A \widetilde{\vee} B \stackrel{\text{def}}{=} \neg\neg(A \vee B)$ and $\widetilde{\exists} x\, P(x) \stackrel{\text{def}}{=} \neg\neg(\exists x\, P(x)$, then for these new operations $\widetilde{\vee}$ and $\widetilde{\exists}$, we have classical logic.

Constructive logic approach: successes. Constructive logic has been successfully used to describe what can be computed and constructed and what cannot. This area of research is known as *constructive mathematics*; see, e.g., [1, 4, 5, 7, 14].

3 How Constructive Logic Can Explain Double Negation Theology

Finite domain vs. infinite domain. The main reason why it is not possible to have an algorithm deciding which mathematical statements are true and which are not is that mathematics deals with infinite domains.

If we limit ourselves to a finite domain D, then it is easy to check whether in this domain, a statement like $\forall x_{x \in D} P(x)$ or $\exists x_{x \in D} P(x)$: it is sufficient to simply check the corresponding statement $P(x)$ for all finitely many objects

$x \in D$. In an infinite domain – e.g., when D is the set N of all natural numbers – such direct checking is not possible.

Al-Sijistani's and Maimonides's God as a master of an infinite domain. Where does God fit into this picture? According to both philosophers, God is a master of an infinite domain, he is aware of what is going on in the infinite Universe.

Thus, whether we are talking about a set of natural numbers or more complex infinite sets, Al-Sijistani's and Maimonides's God can actually check whether a property $P(x)$ holds for all the objects x from this infinite set. This is the only feature of God that we will use in this explanation; from this viewpoint, Al-Sijistani's and Maimonides's God can be viewed simply as a creature that can check the truth of infinitely many statements in a finite time.

All other aspects of God we can, for our purposes, safely ignore.

Constructive logic from the viewpoint of Al-Sijistani's and Maimonides's God. To understand God means to view things from God's viewpoint. If we use constructive logic to describe God's viewpoint, then we do have each statement $A \vee \neg A$ true: since, with an ability to check infinitely many statements, Al-Sijistani's and Maimonides's God can check, for each statement A, whether this statement is true or its negation $\neg A$ is true. This is possible when A is a statement of the type $\forall n\, P(n)$, this is possible for more complex statements such as $\forall m\, \exists n\, P(m,n)$, etc.

Thus, constructibility for this superior creature is equivalent to classical logic, with the law of excluded middle.

Resulting explanation to double negation theology. We humans are only capable of making finitely many computational steps in a finite amount of time. Since our ability is thus limited, to describe what is constructible and what is not, we can use constructive logic, with its lack of law of excluded middle.

Al-Sijistani's and Maimonides's God is, by definition, a creature who can check infinitely many statements in finite time. As a result, if we want to describe what is constructible for the creature with such ability, then the resulting logic becomes *classical* logic, *with* the law of excluded middle.

In particular, the fact that the law of excluded middle $A \vee \neg A$ holds in God's constructive logic means that Al-Sijistani's and Maimonides's God is capable, given any arithmetic statement A with any number of quantifiers, to check whether this statement is true or not. Gödel's result shows that, in general, we cannot check this – which explains Al-Sijistani's and Maimonides's statement that God is, in general, incomprehensible to us.

While (due to our lack of God's infinite abilities) we cannot check which statements are true or not, we can, according to the double negation result, simulate God's constructive logic by interpreting each of the statement with a double negation. This is exactly what Al-Sijistani's and Maimonides's double negation theology suggest.

So, constructive logic indeed provides for an explanation of the double negation theology.

Beyond the double negation theology. A related logic-related theological idea can be found in *Zohar* [3], the main book of Kabbalah. According to this book, in Binah (Understanding), one of ten emanations (*sefirot*) through which God reveals himself, God exists as the great "Who" who stands at the beginning of each question.

This is perfectly in line with the main idea behind Hilbert's *epsilon calculus*, when we introduce a symbol $\varepsilon x\, P$ whose meaning is as follows: if $\exists x\, P(x)$, then $\varepsilon x\, P$ is an object for which the property $P(x)$ is true, i.e., for which $P(\varepsilon x\, P)$; see, e.g., [17]. Of course, this formal idea does not help us actually construct this object in situations when not such construction is possible, but it helps us apply techniques of constructive logic to such situations – i.e., in effect, similar to double negation, it helps us apply techniques of constructive logic in situations when usual finite algorithms do not lead to constructions.

Comment. While epsilon calculus is mostly familiar to logicians, many mathematicians are familiar with it via N. Bourbaki's books, where $\varepsilon x\, P$ is denoted as $\tau_x(P)$ [6].

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