6-2014

Observable Causality Implies Lorentz Group: Alexandrov-Zeeman-Type Theorem for Space-Time Regions

Olga Kosheleva
The University of Texas at El Paso, olgak@utep.edu

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

Comments:

Technical Report: UTEP-CS-14-43

Published in *Mathematical Structures and Modeling*, 2014, Vol. 30, pp. 4-14.

Математические структуры и моделирование
2014, вып. 30, с. 1–??

УДК 000.000

OBSERVABLE CAUSALITY IMPLIES LORENTZ GROUP: ALEXANDROV-ZEEMAN-TYPE THEOREM FOR SPACE-TIME REGIONS

O. Kosheleva, V. Kreinovich

The famous Alexandrov-Zeeman theorem proves that causality implies Lorentz group. The physical meaning of this result is that once we observe which event can causally affect which other events, then, using only this information, we can reconstruct the linear structure of the Minkowski space-time. The original Alexandrov-Zeeman theorem is based on the causality relation between events represented by points in space-time. Knowing such a point means that we know the exact moment of time and the exact location of the corresponding event - and that this event actually occurred at a single moment of time and at a single spatial location. In practice, events take some time and occupy some spatial area. Besides, even if we have a point-wise event, we would not be able to know the exact moment of time and exact spatial location - since the only way to determine the moment of time and the spatial location is by measurement, and measurements are never absolutely accurate. To come up with a more realistic description of observable causality relation between events, we need to consider events which are not pointwise, but rather represented by bounded regions A in the Minkowski space-time. When we have two events represented by regions A and B, the fact that we have observed that the first event can causally influence the second one means that $a \leq b$ for some points $a \in A$ and $b \in B$. In this paper, we show that even if we only know the causal relation between such regions, we can still reconstruct the linear structure on the Minkowski space-time. Thus, already observable causality implies Lorentz group.

1. Formulation of the Problem

Causality implies Lorentz group: known result. In the Minkowski space-time of special relativity, a space-time event is described by a pair (t, x), where $t \in \mathbb{R}$ is a moment of time and $x \in \mathbb{R}^d$ is a spatial location (and d is the dimension of proper space). In this space, the causality relation $\preccurlyeq$ is described as follows: an event (t, x) can influence an event (t', x') if and only if a signal starting at location x at moment

University of Texas at El Paso, El Paso, TX 79968, USA
E-mail: `olgak@utep.edu`, `vladik@utep.edu`

t and traveling at a speed not exceeding the speed of light c, can reach the location x' at a moment t'. The speed which is needed to cover the distance $d(x, x')$ between the two spatial locations in time $t' - t$ is equal to $\dfrac{d(x, x')}{t' - t}$; therefore, the causality condition takes the form $\dfrac{d(x, x')}{t' - t} \le c$, or, equivalently,

$$(t, x) \preccurlyeq (t', x') \Leftrightarrow c \cdot (t' - t) \ge d(x, x').$$

It is easy to check that this causality relation is preserved under several coordinate transformations:

- if we shift time $t \to t + t_0$; in physical terms, this transformation corresponds to changing the starting point for measuring time;
- if we shift coordinates $x \to x + x_0$; in physical terms, this transformation corresponds to changing the starting location for measuring coordinates;
- if we perform a rotation in the proper space (e.g., in the physical 3-D space) $x \to Tx$; in physical terms, this transformation corresponds to changing the orientation of the coordinate axes;
- if we perform a Lorentz transformation; in physical terms, this transformation corresponds to switching to an observer who moves in relation to the original observer;
- if we perform a scaling $t \to \lambda \cdot t$ and $x \to \lambda \cdot x$; in physical terms, this transformation corresponds to a simultaneous changing the units for measuring time and distance – in such a way that the numerical values of velocities do not change.

Interestingly, the inverse is also true: it has been proven that every 1-1 transformation of the Minkowski space-time onto itself which preserves the causality relation is a composition of the above transformations. Such compositions form a group known as the *Lorentz group*.

This result was first proven by A. D. Alexandrov in 1949 [1,2] and became widely known after a somewhat stronger result was proven by E. C. Zeeman in 1964 [3]; the whole result is therefore often called an *Alexandrov-Zeeman theorem*. These results hold not only for the physical case of the 3-D proper space $d = 3$, they holds for all spatial dimensions $d \ge 2$.

The deep meaning of this theorem is that all we need is the causality notion, even the linear structure – i.e., the linearity of the corresponding transformations – can be determined based only on causality.

Problem. From the mathematical viewpoint, the formulation of the Alexandrov-Zeeman theorem deals with point events, i.e., with pairs (t, x) for which we know exactly the moment of time t and the spatial location x. In practice, we can only measure the moment of time and the spatial location with some uncertainty. As a result, we only know that the actual space-time event is located in some known neighborhood region of the point $\left(\widetilde{t}, \widetilde{x}\right)$ corresponding to the measurement results $\widetilde{t}$ and $\widetilde{x}$. In other words, from the observation viewpoint, we only observe causality relation between such regions.

What would happen is instead of a transformation which preserves causality relation between *points* – as in the original Alexandrov-Zeeman theorem – we consider transformations which preserve causality relation between *regions*?

What we do in this paper. In this paper, we prove that even if we replace theoretical points with observable regions, the result will remain the same: causality still implies Lorentz group.

2. Definitions and the Main Result

Definition 1. *By a* region*, we mean a closure of a bounded open set in the Minkowski space.*

Comment.

- Boundedness comes from the fact that there should be an *a priori* upper bound on the measurement error: if there is no such upper bound, this is not a measurement, this is a wild guess.
- Openness comes from the fact that due to uncertainty, all the events in a certain neighborhood of the measured space-time location $\widetilde{x}$ are also possible.
- Finally, closeness comes from the fact that if $x_n \to x$ and all locations x_n are consistent with the given measurement, then, no matter how accurately we describe the location x, it is, within this accuracy, indistinguishable from a possible location x_n. Thus, from all practical purposes, the limit location x is also possible.

Definition 2. *We say that a set A* causally precedes *a set B – and denote it $A \preccurlyeq B$ – if there exist events $a \in A$ and $b \in B$ for which $a \preccurlyeq b$.*

Comment. All we observe is that one event – about which we know that this event is somewhere in the region A – causally affects another event b – about which all we know is that this event is somewhere in the region B. Thus, the above definition is an adequate description of *observable* causality.

It should be noticed that, in contrast to the causality relation between points – which is a (partial) order – causality relation between regions is *not* an order: for example, if two regions $A \neq B$ intersect, we have $A \preccurlyeq B$ and $B \preccurlyeq A$ but $A \neq B$.

Definition 3. *We say that a continuous 1-1 mapping f of the Minkowski space onto itself* preserves observable causality *if for every two regions A and B, $A \preccurlyeq B$ if and only if $f(A) \preccurlyeq f(B)$.*

Theorem. *Every mapping which preserves observable causality belongs to the Lorentz group.*

Comment. In other words, observable causality implies Lorentz group.

Proof. To prove our result, we will prove that every mapping that preserves observable causality also preserves the causality relation between the point events (t, x); then, our result will follow from the original Alexandrov-Zeeman theorem.

For this proof, we will need several auxiliary definitions and lemmas.

Definition 4. *A set S is called* $\preccurlyeq$-convex *if it contains, with every two points $a \preccurlyeq b$, all the points c for which $a \preccurlyeq c \preccurlyeq b$.*

One can easily check that the intersection of any family of $\preccurlyeq$-convex sets is also $\preccurlyeq$-convex. Thus, we can formulate the following definition:

Definition 5. *By a* $\preccurlyeq$-convex hull *of the closed set S, we mean the smallest $\preccurlyeq$-convex set which contains S – i.e., the intersection of all $\preccurlyeq$-convex sets containing S. This hull will be denoted by $H(S)$.*

Lemma 1. *For each bounded closed set S, its $\preccurlyeq$-convex hull has the form*

$$H(S) = \{c : a \preccurlyeq c \preccurlyeq b \textit{ for some } a, b \in S\}.$$

Proof. We need to prove that $H(S) = H_0(S)(S)$, where we denoted

$$H_0(S)(S) \stackrel{\text{def}}{=} \{c : a \preccurlyeq c \preccurlyeq b \text{ for some } a, b \in S\}.$$

1°. Let us first prove that $H_0(S) \subseteq H(S)$, i.e., that every element $c \in H_0(S)$ also belongs to $H(S)$.

Indeed, if $c \in H_0(S)$, this means that $a \preccurlyeq c \preccurlyeq b$ for some $a, b \in S$. In this case, both a and b also belong to any $\preccurlyeq$-convex set S' containing S. So, c belongs to every $\preccurlyeq$-convex set containing S, and thus, belongs to their intersection $H(S)$. The statement is proven.

2°. To complete the proof, it is sufficient to show that the set $H_0(S)$ is itself $\preccurlyeq$-convex.

Since $H_0(S) \subseteq H(S)$ and $H(S)$ is the smallest $\preccurlyeq$-convex containing S, this will imply that $H_0(S)$ cannot be a proper subset of $H(S)$ and thus, that $H_0(S) = H(S)$.

We want to prove that the set $H_0(S)$ is $\preccurlyeq$-convex, i.e., that if $c, c' \in H_0(S)$ and $c \preccurlyeq d \preccurlyeq c'$, then $d \in H_0(S)$.

Indeed, the fact that c and c' are elements of $H_0(S)$ mean that $a \preccurlyeq c \preccurlyeq b$ and $a' \preccurlyeq c' \preccurlyeq b'$ for some $a, b, a', b' \in S$. Due to transitivity of the relation $\preccurlyeq$, the inequalities

$$a \preccurlyeq c \preccurlyeq d \preccurlyeq c' \preccurlyeq b'$$

imply that $a \preccurlyeq d \preccurlyeq b'$, for $a, b' \in S$. By definition of the set $H_0(S)$, this means that $d \in H_0(S)$.

The statement is proven, and so is the lemma.

Lemma 2. *For each bounded closed set S, its $\preccurlyeq$-convex hull $H(S)$ is also closed.*

Proof. Due to Lemma 1, it is sufficient to prove that the set $H_0(S)$ is closed, i.e., that if $c_n \in H_0(S)$ and $c_n \to c$, then $c \in H_0(S)$.

Indeed, by definition of the set $H_0(S)$, the fact that $c_n \in H_(S)$ means that $a_n \preccurlyeq c_n \preccurlyeq b_n$ for some $a_n \in A$ and $b_n \in B$. Since the sequence a_n is contained in a bounded closed subset of a finite-dimensional Euclidean space – i.e., in a compact set – it has a subsequence which converges to some element $a \in A$. From this subsequence, we can extra a sub-subsequence for which b_n also converges to some element $b \in A$. For this sub-subsequence, we have $a_n \preccurlyeq c_n \preccurlyeq b_n$, $a_n \to a$, $b_n \to b$, and $c_n \to c$. The relation $\preccurlyeq$ is closed. Thus, in the limit, we get $a \preccurlyeq c \preccurlyeq b$ for $a, b \in A$. By definition of the set $H_0(S)$, this means that $c \in H_0(S)$.

The lemma is proven.

The usefulness of the hulls comes from the following lemma:

Lemmas 3. For every two sets A and B, $A \preccurlyeq B \Leftrightarrow H(A) \preccurlyeq H(B)$.

Proof.

1°. Let us first prove that $A \preccurlyeq B$ implies $H(A) \preccurlyeq H(B)$.

Indeed, by definition, $A \preccurlyeq B$ means the existence of points $a \in A$ and $b \in B$ for which $a \preccurlyeq B$. Since $A \subseteq H(A)$ and $B \subseteq H(B)$, we also have $a \in H(A)$ and $b \in H(B)$; thus, we have $H(A) \preccurlyeq H(B)$.

2°. Vice versa, let us prove that $H(A) \preccurlyeq H(B)$ implies $A \preccurlyeq B$.

Indeed, by definition, $H(A) \preccurlyeq H(B)$ means that there exist values $c \in H(A)$ and $d \in H(B)$ for which $c \preccurlyeq d$. By Lemma 1, from $c \in H(A)$ and $d \in H(B)$, we conclude that $a \preccurlyeq c \preccurlyeq a'$ for some $a, a' \in A$ and $b \preccurlyeq d \preccurlyeq b'$ for some $b, b' \in B$. By transitivity, $a \preccurlyeq c \preccurlyeq d \preccurlyeq b'$ implies that $a \preccurlyeq b'$, where $a \in A$ and $b' \in B$. Thus, by definition of causal relation between sets, we indeed have $A \preccurlyeq B$.

The lemma is proven.

Notation 1. *We denote* $C^-(b) \stackrel{\text{def}}{=} \{c : c \preccurlyeq b\}$ *and* $C^+(b) \stackrel{\text{def}}{=} \{c : b \preccurlyeq c\}$.

Lemma 4. *For every two regions A and B,*

$$H(A) \subseteq H(B) \Leftrightarrow \forall C\,((A \preccurlyeq C \Rightarrow B \preccurlyeq C)\,\&\,(C \preccurlyeq A \Rightarrow C \preccurlyeq B)),$$

where the quantifier goes over all regions C.

Proof.

1°. Let us first assume that $H(A) \subseteq H(B)$. Then, from $A \preccurlyeq C$, we conclude that $a \preccurlyeq c$ for some $a \in A$ and $c \in C$. Since $A \subseteq H(A) \subseteq H(B)$ and $C \subseteq H(C)$, we conclude that $a \preccurlyeq c$ for $a \in H(B)$ and $c \in H(C)$, hence $H(B) \preccurlyeq H(C)$. By Lemma 3, this implies that $B \preccurlyeq C$.

Similarly, $C \preccurlyeq A$ implies that $C \preccurlyeq B$.

2°. Let us now show that if $H(A)$ is not a subset of $H(B)$, then:

- either there exists a region C for which $A \preccurlyeq C$ but $B \not\preccurlyeq C$,
- or there exists a region C for which $C \preccurlyeq A$ but $C \not\preccurlyeq B$.

Indeed, $H(A) \not\subseteq H(B)$ means that $A \not\subseteq H(B)$ – since if we had $A \subseteq H(B)$, then for the hull $H(A)$, we would have $H(A) \subseteq H(H(B)) = H(B)$. Thus, there exists an element $a \in A$ for which $a \notin H(B)$.

By Lemma 1, the hull $H(B)$ consists of all the elements x for which $b \preccurlyeq x \preccurlyeq b'$ for some $b, b' \in B$. Thus, the fact that $a \notin H(B)$ means that:

- either $a \not\preccurlyeq b$ for all $b \in B$
- or $b \not\preccurlyeq a$ for all $b \in B$.

2.1°. Let us first consider the case when $a \not\preccurlyeq b$ for all $b \in B$.

Since the set B is compact, the set

$$\{c : c \preccurlyeq b \text{ for some } b\} = \bigcup_{b \in B} C^-(b)$$

is closed. Thus, the complement C to this set is open. Every point a from the Minkowski space has the form $a = (t, x)$. The fact that the point $a = (t, x)$ belongs to this complement means that for a sufficiently small $\varepsilon > 0$, the points $\underline{a} \stackrel{\text{def}}{=} (t - \varepsilon, x)$ and $\overline{a} \stackrel{\text{def}}{=} (t + \varepsilon, x)$ also belong to the complement C.

For the region $C \stackrel{\text{def}}{=} \{c : \underline{a} \preccurlyeq c \preccurlyeq \overline{a}\}$, we clearly have $C \preccurlyeq A$, since $a \preccurlyeq a$ for $a \in A$ and $a \in C$. However, $C \preccurlyeq B$ is impossible, because this would imply that $x \preccurlyeq b$ for some $b \in B$, and we selected the set C in such a way that this set C does not contain such points.

2.2°. In the second case, when $b \not\preccurlyeq a$ for all $b \in B$, we can similarly find a region C for which $A \preccurlyeq C$ but not $B \preccurlyeq C$.

The lemma is proven.

Corollary. *If f is a continuous 1-1 mapping of the Minkowski space onto itself which preserves observable causality, then*

$$H(A) \subseteq H(B) \Leftrightarrow H(f(A)) \subseteq H(f(B)).$$

Proof. Indeed, as Lemma 4 shows, the relation $H(A) \subseteq H(B)$ can be described in terms of the observable causality relation. Thus, any mapping that preserves the observable causality relation also preserves the relation $H(A) \subseteq H(B)$.

Definition 6. *We say that a sequence of regions $\{A_n\}$ is* $\preccurlyeq$-embedded *if $H(A_i) \supseteq H(A_{i+1})$ for all i.*

Lemma 5. *Let $\{A_n\}$ be a $\preccurlyeq$-embedded sequence of regions, and let B be a region. Then:*

$$\forall n (B \preccurlyeq A_n) \Leftrightarrow H(B) \bigcap \left(\bigcup \left\{ C^-(c) : c \in \bigcap_n H(A_n) \right\} \right) \neq \emptyset.$$

Proof.

1°. Let us first assume that $B \preccurlyeq A_n$ for all n. Let us then prove that the set B have a common point with the union $\bigcup \left\{ C^-(c) : c \in \bigcap\limits_n H(A_n) \right\}$.

The fact that $B \preccurlyeq A_n$ means that $b_n \preccurlyeq a_n$ for some $b_n \in B$ and $a_n \in A_n \subseteq H(A_n)$. Since all the elements of the sequence b_n belong to the compact set B, this sequence has a convergent subsequence.

Each element of the sequence a_n belongs to the set $H(A_n)$, and thus, to the compact set $H(A_1)$. Thus, we can extract a sub-subsequence for which the sequence a_n also converges. For this sub-subsequence, we have $b_n \to b$ and $a_n \to a$. From $b_n \preccurlyeq a_n$, we conclude that $b \preccurlyeq a$.

Since the sequence of regions is $\preccurlyeq$-embedded, for each n, all the elements a_n, a_{n+1}, ... belong to the closed set $H(A_n)$. Thus, their limit a also belongs to this set. This is true for every n, so a belongs to the intersection $\bigcap\limits_n H(A_n)$ of all these sets. Since $b \preccurlyeq a$, we have $b \in C^-(a)$. Thus, the point $b \in B$ belongs to the union $\bigcup \left\{ C^-(c) : c \in \bigcap\limits_n H(A_n) \right\}$. The statement is proven.

2°. Let us now assume that B have a common point with the union $\bigcup \left\{ C^-(c) : c \in \bigcap\limits_n H(A_n) \right\}$, in other words, that there exists an element $b \in B$ for which, for some $c \in \bigcap\limits_n H(A_n)$, we have $b \in C^-(c)$ (i.e., $b \preccurlyeq c$). Let us prove that in this case, $B \preccurlyeq A_n$ for all n.

Indeed, for each n, since $c \in \bigcap\limits_n H(A_n)$, we have $c \in H(A_n)$. Since we have $b \preccurlyeq c$ for two elements $b \in B$ and $c \in H(A_n)$, we can thus conclude that $B \preccurlyeq H(A_n)$. Similarly to Lemma 3, we can then conclude that $B \preccurlyeq A_n$.

The statement is proven, and so is the lemma.

Lemma 6. *Let $\{A_n\}$ be a $\preccurlyeq$-embedded sequence of regions, and let B be a region. Then:*

$$\forall n (A_n \preccurlyeq B) \Leftrightarrow H(B) \bigcap \left(\bigcup \left\{ C^+(c) : c \in \bigcap_n H(A_n) \right\} \right) \neq \emptyset.$$

Proof is similar to the proof of Lemma 5.

Notation 2. *For each $\preccurlyeq$-embedded sequence of regions $\mathcal{A} = \{A_n\}$, by $B(\mathcal{A})$, we will denote the class of all the regions B for which, for every n, we have $B \preccurlyeq A_n$ and $A_n \preccurlyeq B$.*

Lemma 7. *For each $\preccurlyeq$-embedded sequence of regions $\mathcal{A} = \{A_n\}$, the following two conditions are equivalent to each other:*

- *the intersection $\bigcap\limits_n H(A_n)$ consists of a single point;*
- *the class $B(\mathcal{A})$ is* minimal *in the sense that no other $\preccurlyeq$-embedded sequence of regions $\mathcal{A}'$ has a set $B(\mathcal{A}')$ which is a proper subclass of $B(\mathcal{A})$.*

Proof. Let us denote $h \stackrel{\text{def}}{=} \bigcap\limits_n H(A_n)$. In these terms, by Lemmas 5 and 6, the class $B(\mathcal{A})$ consists of all the regions B for which the closure $H(B)$ has common elements with both unions $\bigcup\{C^-(c) : c \in h\}$ and $\bigcup\{C^+(c) : c \in h\}$.

For a sequence $\mathcal{A}_0 = \left\{A_n^{(0)}\right\}$ for which the intersection $h_0 \stackrel{\text{def}}{=} \bigcap\limits_n H\left(A_n^{(0)}\right)$ consists of a single element s, the set $B(\mathcal{A}_0)$ thus consists of all the regions B for which the closure $H(B)$ has common elements with both sets $C^-(s)$ and $C^+(s)$.

If for a sequence $\mathcal{A}$, the intersection h has more than one element, this means that it contains some elements $s \neq s'$. We can then easily form an $\preccurlyeq$-embedded sequence of $\preccurlyeq$-convex regions $\mathcal{A}_0$ whose intersection is exactly $\{s\}$: e.g., if $s = (t, x)$, we can take, as A_n, all the elements which casually follow from $(t - 2^{-n}, x)$ and which causally precede $(t + 2^{-n}, x)$. Clearly, if $H(B)$ has common elements with both sets $C^-(s)$ and $C^+(s)$, then it also has common elements with both unions $\bigcup\{C^-(c) : c \in h\}$ and $\bigcup\{C^+(c) : c \in h\}$. Thus, $B(\mathcal{A}_0) \subseteq B(\mathcal{A})$.

To complete our proof, let us show that if the intersection $\bigcap\limits_n H(A_n)$ contains an element $s' = (t', x') \neq s = (t, x)$, then $B(\mathcal{A}_0)$ is a proper subset of $B(\mathcal{A})$. In other words, we will prove that the class $B(\mathcal{A})$ contains a region B which is not contained in $B(\mathcal{A})$. To find this region B, let us consider, for each integer n, the region

$$B_n \stackrel{\text{def}}{=} \{c : (t' - 2^{-n}, x') \preccurlyeq c \preccurlyeq (t' + 2^{-n}, x')\}.$$

For each n, the region B_n contains the point $s' = (t', x')$ and thus, has a common point s' with both sets $C^-(s')$ and $C^+(s')$ which are part of the corresponding unions. Thus, $B_n \in B(\mathcal{A})$.

On the other hand, if we had $B_n \subseteq B(\mathcal{A}_0)$ for all n, this would means that each B_n contains a common element b_n^- with the set $C^-(s)$ and a common element b_n^+ with the set $C^+(s)$. Thus, for each n, we would have $b_n^- \preccurlyeq s$ and $s \preccurlyeq b_n^+$. Since b_n^- and b_n^+ are elements of B_n, we have $b_n^- \to s'$ and $b_n^+ \to s'$. In the limit, we thus get $s' \preccurlyeq s$ and $s \preccurlyeq s'$, thus $s = s'$, which contradicts to our assumption that $s \neq s$. This contradiction shows that it is not possible to have $B_n \subseteq B(\mathcal{A}_0)$ for all n. Thus, for some n, we have $B_n \subseteq B(\mathcal{A})$ but $B_n \not\subseteq B(\mathcal{A}_0)$ – and therefore, $B(\mathcal{A}_0)$ is a proper subset of $B(\mathcal{A})$.

The lemma is proven.

Notation 3. *According to Lemma 6, for each $\preccurlyeq$-embedded sequence of regions $\mathcal{A} = \{A_n\}$ for which the class $B(\mathcal{A})$ is minimal, the intersection $\bigcap\limits_n H(A_n)$ consists of a single point. Let us denote this point by $s(\mathcal{A})$.*

Discussion. Due to Lemma 7, points in the Minkowski space can be described in terms of the observable causality relation between regions – as $\preccurlyeq$-embedded sequence of regions $\mathcal{A}$ for which the corresponding class $B(\mathcal{A})$ is minimal. Thus, the mapping that preserve observable causality also preserves the corresponding points $s(\mathcal{A})$.

To complete the proof of our main result, we need to show that the causality relation between these points can also be described in terms of observable causality. Thus, every mapping which preserves observable causality preserve the original causality – and therefore, due to the original Alexandrov-Zeeman theorem, belongs to the Lorentz group. This is proven in the following two lemmas.

Lemma 8. *For each $\preccurlyeq$-embedded sequence of regions $\mathcal{A}$ for which the class $B(\mathcal{A})$ is minimal, this class $B(\mathcal{A})$ consists of all the regions B for which the $\preccurlyeq$-convex hull $H(B)$ contains the point $s(\mathcal{A})$:*

$$B(\mathcal{A}) = \{B : s(\mathcal{A}) \in H(B)\}.$$

Proof. If $B \in B(\mathcal{A})$, then, by Lemmas 5 and 6, this means that for $c \stackrel{\text{def}}{=} s(\mathcal{A})$, the set $H(B)$ contains a point $b^- \in C^-(c)$ and a point $b^+ \in C^+(c)$. Thus, $b^- \preccurlyeq c \preccurlyeq b^+$. Since the set $H(B)$ is $\preccurlyeq$-convex, we conclude that $c \in H(B)$.

Vice versa, let B be a set for which $c \in H(B)$. Then, clearly, $H(B)$ has a common element c with both $C^-(c)$ and $C^+(c)$, and thus, $B \in B(\mathcal{A})$.

Lemma 9. *Let $\mathcal{A}$ and $\mathcal{A}'$ are $\preccurlyeq$-embedded sequence of regions for which the classes $B(\mathcal{A})$ and $B(\mathcal{A}')$ are minimal, and let $s(\mathcal{A})$ and $s(\mathcal{A}')$ be the corresponding points in the Minkowski space. Then, the following two conditions are equivalent to each other:*

- *$s(\mathcal{A}) \preccurlyeq s(\mathcal{A}')$;*
- *$B \preccurlyeq B'$ for every two regions $B \in B(\mathcal{A})$ and $B' \in B(\mathcal{A}')$.*

Proof.

1°. Let us first assume that $c = s(\mathcal{A}) \preccurlyeq c' = s(\mathcal{A}')$. Let $B \in B(\mathcal{A})$ and $B' \in B(\mathcal{A}')$. Let us then show that $B \preccurlyeq B'$.

Indeed, by Lemma 8, the set B contains the point c, and the set B' contains the point c'. Thus, $c \preccurlyeq c'$ implies $B \preccurlyeq B'$.

2°. Let us now assume that $B \preccurlyeq B'$ for every two sets $B \in B(\mathcal{A})$ and $B' \in B(\mathcal{A}')$. Let us then prove that $c = s(\mathcal{A}) \preccurlyeq c' = s(\mathcal{A}')$.

Indeed, let $c = (t, x)$ and $c' = (t', x')$. For every integer n, the us consider the regions

$$B_n \stackrel{\text{def}}{=} \{z : (t - 2^{-n}, x) \preccurlyeq z \preccurlyeq (t + 2^{-n}, x)\}$$

and

$$B'_n \stackrel{\text{def}}{=} \{z : (t' - 2^{-n}, x') \preccurlyeq z \preccurlyeq (t' + 2^{-n}, x')\}$$

By Lemma 8, we have $B_n \in B(\mathcal{A})$ and $B'_n \in B(\mathcal{A}')$. Thus, we have $B_n \preccurlyeq B'_n$. This means that there exist points $b_n \in B_n$ and $b'_n \in B_n$ for which $b_n \preccurlyeq b'_n$. In the limit when $n \to \infty$, we have $b_n \to c$ and $b'_n \to c'$, thus $c \preccurlyeq c'$.

The statement is proven, and so is the lemma and our main result.

Acknowledgments

This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721.

References

1. Alexandrov A.D. On Lorentz transformations // Uspekhi Math. Nauk 1950, V. 5, No. 1, P. 187 (in Russian).
2. Alexandrov A.D., and Ovchinnikova V.V. Remarks on the foundations of Special Relativity // Leningrad University Vestnik 1953, No. 11, P. 94-110 (in Russian).
3. Zeeman E.C. Causality implies the Lorentz group // Journal of Mathematical Physics 1964, V. 5, No. 4, P. 490-493.