6-2014

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Comments:
Technical Report: UTEP-CS-14-32a
Published in *Proceedings of IEEE International Conference on Systems, Man, and Cybernetics SMC'2014*, San Diego, California, October 5-8, 2014, pp. 1229-1234.

r-Bounded Fuzzy Measures are Equivalent to ε-Possibility Measures

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***Abstract*—Traditional probabilistic description of uncertainty is based on *additive* probability measures. To describe non-probabilistic uncertainty, it is therefore reasonable to consider *non-additive* measures. An important class of non-additive measures are *possibility* measures, for which $\mu(A \cup B) = \max(\mu(A), \mu(B))$. In this paper, we show that possibility measures are, in some sense, universal approximators: for every $\varepsilon > 0$, every non-additive measure which satisfies a certain reasonable boundedness property is equivalent to a measure which is ε-close to a possibility measure.**

I. Additive Measures and Fuzzy (Non-Additive) Measures: Brief Reminder

Formulation of the problem. One of the main motivations behind fuzzy and other non-probabilistic uncertainty is that the traditional probability theory is sometimes not very adequate for describing uncertainty.

From the mathematical viewpoint, probability theory is based on probability (additive) measures. To describe non-probabilistic uncertainty, researchers therefore came up with a general notion of non-additive (fuzzy) measures, in particular, possibility measures.

In this paper, we show that every fuzzy measure satisfying several reasonable properties is isomorphic to an "almost"-possibility measure.

In order to formulate our result, let us first briefly recall definitions and main properties of probability measures and of fuzzy measures.

Definition 1. *Let X be a set called a* universal set. *By an* algebra of sets *(or* algebra*, for short) $\mathcal{A}$, we mean a non-empty class of subsets $A \subseteq X$ which is closed under complement and union and intersection, i.e.:*

- *if $A \in \mathcal{A}$, then its complement $-A$ also belongs to $\mathcal{A}$;*
- *if $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$;*
- *if $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.*

Definition 2. *By an* additive measure*, we mean a function μ which maps each set A from some algebra of sets to a non-negative number $\mu(A) \geq 0$ and for which*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for every two sets A and B for which $A \cap B = \emptyset$.

Comment. Example of additive measures include length of sets on a line, area of planar sets, volume of sets in 3-D space, and probability of different events.

Let us recall the main properties of additive measures. The first property is that $\mu(\emptyset) = 0$. Indeed, since the class $\mathcal{A}$ is non-empty, it contains some set A. Since $\mathcal{A}$ is an algebra, with set A, it also contains sets $-A$ and $A \cap -A = \emptyset$. Since $A \cap \emptyset = \emptyset$, additivity implies $\mu(A) = \mu(A) + \mu(\emptyset)$ and thus, $\mu(\emptyset) = 0$.

Definition 3. *A function $\mu(A)$ defined on sets is called* monotonic *if $A \subseteq B$ implies $\mu(A) \leq \mu(B)$.*

Proposition 1. *Every additive measure is monotonic.*

Proof. Indeed, if $A, B \in \mathcal{A}$, then $B - A = B \cap (-A) \in \mathcal{A}$. Due to additivity, we have $\mu(B) = \mu(A) + \mu(B - A)$. Since $\mu(B - A) \geq 0$, this implies $\mu(A) \leq \mu(B)$. The statement is proven.

Definition 4. *A function $\mu(A)$ defined on an algebra of sets is called* subadditive *if for every two sets A and B, we have*

$$\mu(A \cup B) \leq \mu(A) + \mu(B).$$

Proposition 2. *Every additive measure is subadditive.*

Proof. Indeed, we have $B - A \in \mathcal{A}$ and, due to additivity, $\mu(A \cup B) = \mu(A) + \mu(B - A)$. Since $B - A \subseteq B$, monotonicity implies that $\mu(B - A) \leq \mu(B)$ and thus,

$$\mu(A \cup B) \leq \mu(A) + \mu(B).$$

The statement is proven.

Definition 5. *Let X be a set called a* universal set. *By an* σ-algebra *$\mathcal{A}$, we mean a non-empty class of subsets $A \subseteq X$ which is closed under complement and countable union and intersection, i.e.:*

- *if $A \in \mathcal{A}$, then its complement $-A$ also belongs to $\mathcal{A}$;*
- *if $A_1 \in \mathcal{A}$, ..., $A_n \in \mathcal{A}$, ..., then $\bigcup\limits_i A_i \in \mathcal{A}$;*
- *if $A_1 \in \mathcal{A}$, ..., $A_n \in \mathcal{A}$, ..., then $\bigcap\limits_i A_i \in \mathcal{A}$.*

Comment. One can easily check that every σ-algebra is also an algebra: indeed, we can take $A_1 = A$ and

$$A_2 = \ldots = A_n = \ldots = B.$$

Definition 6. (see, e.g., [7], [8], [10]) *By a* σ-additive measure*, we mean a function μ which maps each set A from some σ-algebra of sets to a non-negative number $\mu(A) \geq 0$ and for which*

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$$

for every sequence of sets A_i for which $A_i \cap A_j = \emptyset$ for $i \neq j$.

Comment. One can easily check that each σ-additive measure is additive.

Definition 7. *By a* probability measure *on the universal set X, we mean a σ-additive measure μ for which $\mu(X) = 1$.*

Comment. Since every σ-additive measure is additive, each probability measure μ is monotonic and satisfies the property $\mu(\emptyset) = 1$.

These properties motivate a natural generalization of probability measures known as *fuzzy measures*; see, e.g., [1], [9].

Definition 8. *A function $\mu(A)$ defined on an algebra of subsets of a universal set X is called a* fuzzy measure *if it is monotonic and satisfies the properties $\mu(\emptyset) = 0$ and $\mu(X) = 1$.*

Comment. There is a special class of fuzzy measures known as *possibility measures*; see, e.g., [3], [4], [11].

Definition 9. (see, e.g., [6]) *A function $\mu(A)$ defined on an algebra of sets is called* maxitive *if for every two sets A and B, we have*

$$\mu(A \cup B) = \max(\mu(A), \mu(B)).$$

Proposition 3. *Every maxitive measure is monotonic.*

Proof. If $A \subseteq B$, then $A \cup B = B$, so the maxitivity takes the form $\mu(B) = \max(\mu(A), \mu(B))$, which implies $\mu(A) \leq \mu(B)$. The statement is proven.

Definition 10. *By a* possibility measure*, we mean a maxitive fuzzy measure.*

Comment. The general definition of a fuzzy measure included monotonicity. Due to Proposition 3, for maxitive measures, monotonicity automatically follows.

II. Main Result: Under Reasonable Conditions, Non-Additive Measures Are Equivalent to "Almost" Possibility Measures

What we plan to do. In this paper, we prove that fuzzy measures satisfying certain reasonable properties – and even more general measures satisfying this property – are equivalent to "almost" possibility measures. To formulate this result, we need to describe:

- what is a general measure;
- which are the reasonable properties;
- what we mean by equivalence, and
- what we mean by an "almost" possibility measure.

Let us define these notions one by one.

Definition 11. *Let X be a set called a* universal set*. By a* non-additive measure *on the set X, we mean a function μ which assigns, to some subsets $A \subseteq X$ from a certain algebra $\mathcal{A}$, a real number $\mu(A) \geq 0$.*

Comment. From this viewpoint, additive, σ-additive, probabilistic, fuzzy, maxitive, and possibility measures are particular cases of non-additive measures.

Motivations for the definition of reasonable (r-) boundedness. In general, a measure $\mu(A)$ describes how important is the set A: the larger the measure, the more important is the set A.

From this viewpoint, if we take the union $A \cup B$ of two sets of bounded size, then the size of the union cannot be arbitrarily large, it should be limited by some bound depending on the bound on $\mu(A)$ and $\mu(B)$.

Similarly, if the sizes of A and B are sufficiently small, then the size of the union is also small – i.e., for sufficiently small bounds on $\mu(A)$ and $\mu(B)$ it should be smaller than any given size.

In precise terms, we arrive at the following definition.

Definition 12. *We say that a non-additive measure μ is* r-bounded *if it satisfies the following two properties:*

- *for every $\Gamma > 0$, there exists a $\Delta > 0$ such that if $\mu(A) \leq \Gamma$ and $\mu(B) \leq \Gamma$, then $\mu(A \cup B) \leq \Delta$;*
- *for every $\eta > 0$, there exists a $\nu > 0$ such that if $\mu(A) \leq \nu$ and $\mu(B) \leq \nu$ then $\mu(A \cup B) \leq \eta$.*

Proposition 4. *Every additive measure is r-bounded.*

Proof. Indeed, as we have shown, all additive measures are sub-additive: $\mu(A \cup B) \leq \mu(A) + \mu(B)$. Thus, we can take $\Delta = 2 \cdot \Gamma$ and $\nu = \dfrac{\eta}{2}$. The statement is proven.

Motivation for the definition of equivalence. The numerical values of probabilities have observable sense – the probability of an event E can be defined as the limit of the frequency with which the event E occurs. This is how, e.g., we can define the probability of a rain at a certain location: by dividing the number of days when it rained by the total number of days for which we had observations.

In contrast, e.g., possibility values do not have direct meaning, the only important thing is which values are larger and which are smaller – this describes which events are more possible and which are less possible. From this viewpoint, if two measures can be obtained from each other by a transformation that preserves the order, such measures can be considered to be equivalent.

Definition 13. *Two non-additive measures $\mu(A)$ and $\mu'(A)$ are called* equivalent *if there exists a 1-1 monotonic function $f(x)$ such that for every set A, we have $\mu'(A) = f(\mu(A))$.*

Definition 14. *Let $\varepsilon > 0$ be a real number. We will call a non-additive measure $\mu(A)$ an ε*-possibility measure *if for every two sets A and B, we have*

$$\mu(A \cup B) \leq (1+\varepsilon) \cdot \max(\mu(A), \mu(B)).$$

Comment. For monotonic measures – in particular, for fuzzy measures – due to $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we have

$$\mu(A) \leq \mu(A \cup B)$$

and

$$\mu(B) \leq \mu(A \cup B),$$

thus,

$$\max(\mu(A), \mu(B)) \leq \mu(A \cup B).$$

So, if μ is a monotonic ε-probabilistic measure, we have

$$\max(\mu(A), \mu(B)) \leq \mu(A \cup B) \leq (1+\varepsilon) \cdot \max(\mu(A), \mu(B)).$$

Thus, the value $\mu(A \cup B)$ is almost equal – with relative accuracy $\varepsilon > 0$ – to the value $\max(\mu(A), \mu(B))$ corresponding to the possibility measure.

Now, we are ready to formulate our main result.

Theorem 1. *For every $\varepsilon > 0$, every r-bounded non-additive measure is equivalent to an ε-possibility measure.*

Proof.

1°. Let us first define a doubly infinite sequence $\ldots < c_{-(k+1)} < c_k < \ldots < c_{-1} < c_0 < c_1 < \ldots < c_k < c_{k+1} < \ldots$ as follows.

We take $c_0 = 1$.

Once we have defined the value c_k for some $k \geq 0$, we define c_{k+1} as follows. By definition of an r-bounded measure, there exists a value $\Delta_k > 0$ such that if $\mu(A) \leq c_k$ and $\mu(B) \leq c_k$, then $\mu(A \cup B) \leq \Delta_k$. We then take

$$c_{k+1} \stackrel{\text{def}}{=} (1+\varepsilon) \cdot \max(c_k, \Delta_k).$$

Here, $c_0 = 1$ and $c_{k+1} \geq (1+\varepsilon) \cdot c_k$. By induction over k, we can prove that $c_k \geq (1+\varepsilon)^k$ and thus, $c_k \to \infty$ when k increases.

Similarly, once we have defined the value c_{-k} for some $k \geq 0$, we define $c_{-(k+1)}$ as follows. By definition of an r-bounded measure, there exists a value $\nu_k > 0$ such that if $\mu(A) \leq \nu_k$ and $\mu(B) \leq \nu_k$, then $\mu(A \cup B) \leq c_{-k}$. We then take

$$c_{-(k+1)} \stackrel{\text{def}}{=} (1-\varepsilon) \cdot \min(c_{-k}, \nu_k).$$

Here, $c_0 = 1$ and $0 < c_{-(k+1)} \leq (1-\varepsilon) \cdot c_{-k}$. By induction over k, we can prove that $0 < c_{-k} \leq (1-\varepsilon)^k$ and thus, $c_{-k} \to 0$ when $k \to \infty$.

2°. Let us now define the desired function $f(x)$. Since the sequence c_k is strictly increasing, $c_k \to \infty$ when $k \to +\infty$, and $c_k \to 0$ when $k \to -\infty$, for every positive number $x > 0$, there exists an integer k for which $c_{k-1} < x \leq c_k$. We can then define $f(x)$ as follows:

- for each integer k, we take $f(c_k) = (1+\varepsilon)^{k/2}$ and
- for each value x between c_{k-1} and c_k, we define $f(x)$ by linear interpolation: if $c_{k-1} < x \leq c_k$, then

$$f(x) = f(c_{k-1}) + \frac{x - c_{k-1}}{c_k - c_{k-1}} \cdot (f(c_k) - f(c_{k-1})).$$

Since the sequence c_k is strictly increasing, the resulting function $f(x)$ is also strictly increasing.

3°. Let us now prove that for for the new measure

$$\mu'(A) \stackrel{\text{def}}{=} f(\mu(A))$$

(which is equivalent to $\mu(A)$), for every two sets A and B, we have

$$\mu'(A \cup B) \leq (1+\varepsilon) \cdot \max(\mu'(A), \mu'(B)).$$

Without losing generality, let us assume that $\mu(A) \geq \mu(B)$. As we have mentioned in Part 2 of this proof, there exist integers k and ℓ for which $c_{k-1} < \mu(A) \leq c_{k+1}$ and $c_{\ell-1} < \mu(B) \leq c_\ell$. Since $\mu(A) \geq \mu(B)$ and c_k is an increasing sequence, we cannot have $k < \ell$, so $k \geq \ell$ and thus, $c_\ell \leq c_k$.

Hence, we have $\mu(A) \leq c_k$ and $\mu(B) \leq c_k$. By definition of Δ_k, we therefore have $\mu(A \cup B) \leq \Delta_k$. By definition of c_{k+1}, this value is always great than Δ_k, thence we have

$$\mu(A \cup B) \leq c_{k+1}.$$

Since the function $f(x)$ is increasing, we get

$$\mu'(A \cup B) = f(\mu(A \cup B)) \leq f(c_{k+1}) = (1+\varepsilon)^{(k+1)/2}.$$

On the other hand, here, $\max(\mu(A), \mu(B)) = \mu(A) > c_{k-1}$. Due to monotonicity, we have

$$\max(\mu'(A), \mu'(B)) = \mu'(A) = f(\mu(A)) >$$
$$f(c_{k-1}) = (1+\varepsilon)^{(k-1)/2}.$$

In other words, we have

$$(1+\varepsilon)^{(k-1)/2} < \max(\mu'(A), \mu'(B)).$$

Multiplying both sides of this inequality by $1+\varepsilon$, we get

$$(1+\varepsilon)^{(k+1)/2} < (1+\varepsilon) \cdot \max(\mu'(A), \mu'(B)).$$

We already know that $\mu'(A \cup B) \le (1+\varepsilon)^{(k+1)/2}$. Thus, we conclude that

$$\mu'(A \cup B) \le (1+\varepsilon) \cdot \max(\mu'(A), \mu'(B)).$$

The theorem is proven.

Comment. A natural question is: can we strengthen this result by proving that each r-bounded measure is equivalent not just to an ε-possibility measure but actually to a possibility measure? A simple answer is "No": one can easily prove that any measure which is equivalent to a maxitive one is also maxitive, so a general non-maxitive measure cannot be equivalent to a (maxitive) possibility measure.

III. First Auxiliary Result: Possibility of Uniform Equivalence

Formulation of the problem. In the previous section, we proved that each r-bounded non-additive measure can be reduced to an "almost" possibility measure.

Sometimes, we have several measures. A natural question is: can we re-scale all of them by using the same re-scaling function $f(x)$ so that all of them become ε-possibility measures?

Theorem 2. *For every $\varepsilon > 0$, and for every finite set of r-bounded non-additive measures $\mu_1(A), \ldots, \mu_n(A)$, there exists a 1-1 function $f(x)$ for which all n measures*

$$\mu_i'(A) \stackrel{\text{def}}{=} f(\mu_i(A))$$

are ε-possibility measure.

Proof. This theorem can be proven in a way which is similar to the proof of Theorem 1, the only difference is how the sequence c_k is built.

We still take $c_0 = 1$.

Once we have defined the value c_k for some $k \ge 0$, we define c_{k+1} as follows. By definition of an r-bounded measure, for each i from 1 to n, there exists a value $\Delta_{ki} > 0$ such that if $\mu_i(A) \le c_k$ and $\mu_i(B) \le c_k$, then $\mu_i(A \cup B) \le \Delta_{ki}$. We then take

$$c_{k+1} = (1+\varepsilon) \cdot \max(c_k, \Delta_{k1}, \ldots, \Delta_{kn}).$$

Similarly, once we have defined the value c_{-k} for some $k \ge 0$, we define $c_{-(k+1)}$ as follows. By definition of an r-bounded measure, for each i from 1 to n, there exists a value $\nu_{ki} > 0$ such that if $\mu_i(A) \le \nu_{ki}$ and $\mu_i(B) \le \nu_{ki}$, then $\mu_i(A \cup B) \le c_{-k}$. We then take

$$c_{-(k+1)} = (1-\varepsilon) \cdot \min(c_{-k}, \nu_{k1}, \ldots, \nu_{kn}).$$

The rest of the proof is the same as for Theorem 1.

IV. Second Auxiliary Result: Case of Generalized Metric

Motivation. Similarly to the fact that measures describe size, metrics describe distance. Usually, we considers metrics $d(a,b)$ which satisfy the triangle inequality

$$d(a,c) \le d(a,b) + d(b,c).$$

However, it does not have to be this particular inequality.

What is important is that if $d(a,b)$ and $d(b,c)$ are bounded by some constant $\Gamma > 0$, then the distance $d(a,c)$ cannot be arbitrarily large, it should be limited by some bound depending on the bound on $d(a,b)$ and $d(b,c)$.

Similarly, if the distance $d(a,b)$ and $d(b,c)$ are sufficiently small, then the distance $d(a,c)$ is also small – i.e., for sufficiently small bounds on $d(a,b)$ and $d(b,c)$ it should be smaller than any given size.

In precise terms, we arrive at the following definition.

Definition 15. *Let X be a set. A function $d : X \times X \to R_0^+$ which assigns a non-negative number $d(a,b)$ to every pair (a,b) is called an* r-bounded metric *if it satisfies the following two properties:*

- *for every $\Gamma > 0$, there exists a $\Delta > 0$ such that if $d(a,b) \le \Gamma$ and $d(b,c) \le \Gamma$, then $d(a,c) \le \Delta$;*
- *for every $\eta > 0$, there exists a $\nu > 0$ such that if $d(a,b) \le \nu$ and $d(b,c) \le \nu$ then $d(a,c) \le \eta$.*

Comment. A similar notion has been proposed in [2].

Proposition 5. *Every metric satisfying the triangle inequality is r-bounded.*

Proof. Indeed, due to the triangle inequality

$$d(a,c) \le d(a,b) + d(b,c),$$

we can take $\Delta = 2 \cdot \Gamma$ and $\nu = \dfrac{\eta}{2}$. The statement is proven.

Definition 16. (see, e.g., [5]) *A function $d : X \times X \to R_0^+$ is called an* ultrametric *if it satisfies the inequality*

$$d(a,c) \le \max(d(a,b), d(b,c))$$

for all a, b, and c.

Definition 17. *Let $\varepsilon > 0$ be a positive real number. A function $d : X \times X \to R_0^+$ is called an* ε-ultrametric *if it satisfies the inequality*

$$d(a,c) \le (1+\varepsilon) \cdot \max(d(a,b), d(b,c))$$

for all a, b, and c.

Comment. One can easily check that for every $\varepsilon > 0$, each ultrametric is an ε-ultrametric.

Proposition 6. *For every $\varepsilon > 0$, each ε-ultrametric is an r-bounded metric.*

Proof. Indeed, due to the inequality

$$d(a,c) \le (1+\varepsilon)\cdot \max(d(a,b), d(b,c)),$$

we can take $\Delta = (1+\varepsilon)\cdot\Gamma$ and $\nu = \dfrac{\eta}{1+\varepsilon}$. The statement is proven.

Definition 18. *Two r-bounded metrics $d(a,b)$ and $d'(a,b)$ are called* equivalent *if there exists a 1-1 monotonic function $f(x)$ such that for every two points a and b, we have*

$$d'(a,b) = f(d(a,b)).$$

Theorem 3. *For every $\varepsilon > 0$, every r-bounded metric is equivalent to an ε-ultrametric.*

Proof is similar to the proof of Theorem 1.

Theorem 4. *For every $\varepsilon > 0$, and for every finite set of r-bounded metrics $d_1(a,b)$, …, $d_n(a,b)$, there exists a 1-1 function $f(x)$ for which all n functions*

$$d'_i(a,b) \stackrel{\text{def}}{=} f(d_i(a,b))$$

are ε-ultrametrics.

Proof is similar to the proof of Theorem 2.

V. General Result

Motivations. Let us now formulate a general result which includes both above results – about measures and about metrics – as particular cases.

Definition 19. *By a* domain*, we mean a set S with a binary operation $\circ : S\times S \to S$ – which is, in general, partially defined.*

Comment.

- For measures, S is an algebra of sets, and $\circ$ is the union.
- For metrics, S is the set of all pairs (a,b), and the binary operation transforms pairs (a,b) and (b,c) into the pair (a,c) – and it is undefined if the second component of the first pair is different from the first component of the second pair.

Definition 20. *By a* characteristic*, we mean a function*

$$F : S \to R_0^+.$$

Definition 21. *A characteristic $F(x)$ is called* r-bounded *if it satisfies the following two properties:*

- *for every $\Gamma > 0$, there exists a $\Delta > 0$ such that if $F(x) \le \Gamma$, $F(x') \le \Gamma$, and $x\circ x'$ is defined, then $F(x\circ x') \le \Delta$;*
- *for every $\eta > 0$, there exists a $\nu > 0$ such that if $F(x) \le \nu$ and $F(x') \le \nu$ then $F(x\circ x') \le \eta$.*

Definition 22. *Let $\varepsilon > 0$ be a positive real number. A characteristic $F(x)$ is called an ε-*maxitive *if it satisfies the inequality*

$$F(x\circ x') \le (1+\varepsilon)\cdot\max(F(x), F(x'))$$

for all x and x' for which the operation $x\circ x'$ is defined.

Definition 18. *Two characteristics $F(x)$ and $F'(x)$ are called* equivalent *if there exists a 1-1 monotonic function $f(x)$ such that for every $x \in S$, we have*

$$F'(x) = f(F(x)).$$

Theorem 5. *For every $\varepsilon > 0$, every r-bounded characteristic is equivalent to an ε-maxitive one.*

Proof is similar to the proof of Theorem 1.

Theorem 6. *For every $\varepsilon > 0$, and for every finite set of r-bounded characteristics $F_1(x)$, …, $F_n(x)$, there exists a 1-1 function $f(x)$ for which all n characteristics*

$$F'_i(x) \stackrel{\text{def}}{=} f(F_i(x))$$

are ε-maxitive.

Proof is similar to the proof of Theorem 2.

VI. Summary

The traditional probabilistic description of uncertainty uses additive probability measures. For describing non-probabilistic uncertainty, it is therefore reasonable to use non-additive measures. The most well-known example of such measures are possibility measures $\mu(A)$, for which, for every two sets A and B, we have $\mu(A\cup B) = \max(\mu(A), \mu(B))$.

In this paper, we show that the wide use of possibility measures may be explained by the fact that, under some reasonable conditions, these measures can approximate any non-additive measures: namely, we prove that for every $\varepsilon > 0$, every non-additive measure is isomorphic to an *ε-possibility measure*, i.e., to a measure $\mu'(A)$ for which $\max(\mu(A), \mu(B)) \le \mu(A\cup B) = \max(\mu(A), \mu(B)) \le (1+\varepsilon)\cdot\max(\mu(A), \mu(B))$.

It turns out that if we have several measures, then the tuple consisting of these measures is isomorphic to a tuple of ε-possibility measures. A similar result is also proven for generalized metrics.

Acknowledgments

This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721.

The authors are thankful to M. Amine Khamsi, Ilya Molchanov, and Hung T. Nguyen for valuable discussions, and to the anonymous referees for valuable suggestions.

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