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Investigating Math Students' Ability To Conceptualize Fractions As Two Numerals Having A Single Value

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INVESTIGATING MATH STUDENTS' ABILITY TO CONCEPTUALIZE
FRACTIONS AS TWO NUMERALS HAVING A SINGLE VALUE

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DEDICATION

This document is dedicated to Jack Scott Fanning: my grandfather. He was a mathematics teacher who taught me the density of a fraction when I was merely in middle school and sparked my interest in mathematics. He was my inspiration, my hero, and my “Pop”.

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FRACTIONS AS TWO NUMERALS HAVING A SINGLE VALUE

by

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ABSTRACT

It is no secret that math students of all ages have misconceptions about fractions. Multiple studies have shown that math students invariably see fractions as two separate and unrelated numbers, the numerator and the denominator, rather than a single number with specific value. This study is designed to investigate students' interpretations of a fraction in terms of having a single value in relation to its two numbers, the numerator and the denominator.

For the purpose of this study, the investigator developed and pilot-tested seven items that have the potential to uncover whether students attend to the value of a fraction when asked to reason about fractions in various situations.

Twenty students were interviewed using these seven items. Twelve developmental algebra students at a college and eight 7th grade students from a private school were interviewed, each for approximately thirty minutes and the conversations were audio-recorded.

Over 200 student responses were analyzed and categorized. Nearly 50 subcategories were identified. These subcategories suggested five progressive levels of understanding of fractions:

Level 1 – Fraction as merely two independent numerals almost unrelated

Level 2 – Fraction as two independent numerals related as a part-whole concept

Level 3 – Procedural conversion of fraction without attending to value

Level 4 – Fraction as having a general sense of value of fraction with some connection to the two numbers

Level 5 – Fraction as a single identifiable value with connection to the two numbers

There are some differences between the college student responses and the 7th grade student responses. The 7th graders have an overall higher level of understanding and they fared better in four of the seven items. The college students outperformed the 7th graders on only one item.

Perhaps this study will inspire further research on conceptual challenges students face in reasoning with a fraction as a conceptual entity that simultaneously has two numbers but a single value, which quantifies the multiplicative relation between the numerator and denominator.

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CHAPTER 1: INTRODUCTION

Of all the math concepts that mystify students from elementary school through adulthood, fractions take center stage. Most students do not even view fractions as numbers at all, but as “meaningless symbols that need to be manipulated in arbitrary ways to produce answers that satisfy a teacher” (Siegler et al., 2010, p. 6). There is a multitude of literature that supports the sentiment that students find fractions as one of the most difficult mathematical concepts that they are asked to learn during their academic years.

Students often confuse properties of fraction numbers with those of whole numbers. For example, some high school learners believe there is no number between $\frac{5}{7}$ and $\frac{6}{7}$, just as there is no whole number between 5 and 6 (Vamvakoussi & Vosniadou, 2004).

Some educators argue that deep understanding of fractions builds on many other foundational understanding. For example, rational numbers in algebra and derivation in calculus both require a good understanding of fractions and the relationship that the numerator has to the denominator. Thus, it is important that students learn to conceptualize fractions more deeply before they enter high school and certainly before college.

Learning about fractions usually begins in Kindergarten or 1st grade with unit fractions. Learning unit fractions is a way to help students build on what they already know about whole numbers. For example, the number 3 is built from three ones just as $\frac{3}{4}$ is built from three one-fourths. So, from the beginning of learning fractions, students are introduced to partitioning of a whole.

Common Core Standards (2011) suggests that we teach conceptual understanding, then procedural fluency, and finally application. The meaning of fractions must be mastered before

moving to reasoning, which includes equivalent fractions, and only then can arithmetic operations and making connections with fractions and decimals occur.

The essence of learning about rational numbers is about understanding that a fraction is just like any other number: it has a single value with a specific placement on a number line. I am in agreement with McLellan and Dewey's (1895) belief that "fractions are not to be regarded as something different from number – or at least a different kind of process (p. 127).

This study begins with the premise that student success with fractions is directly supported by how well they conceptualize fraction as having two numbers, a numerator and a denominator, and yet it has only one value. This study seeks to investigate how students' understanding of fractions, specifically their attendance to the value of a fraction and its connection to the multiplicative relationship between the numerator and the denominator.

CHAPTER 2: LITERATURE REVIEW

2.1 Student Poor Understanding of Fractions

It has been well reported that student understanding of fractions is weak and this lack of understanding continues even as students enter college. Much of the existing research regarding rational numbers shows us that students have a weak conceptual understanding of fractions (Kerslake, 1986; Lamon, 2001; Mack, 1993; Moss & Case, 1999; Olive, 2006; Pustejovsky 1999).

Lamon (2001) did a ten-year study of eighteen- and nineteen-year-old students at Marquette University in which she asked them seven questions to determine whether or not they understood the broad base meaning that is associated with fraction symbols. She noted that 90 percent of the students were unable to answer 50 percent of the questions. Her concern for these incoming college students is that they have yet to have had enough experience to understand rational numbers and noted that it would be difficult for them to gain that understanding in classes that assume a knowledge of rational numbers.

Research in university calculus classrooms like that done by Pustejovsky (1999) suggests that students who begin university mathematics with only part/whole interpretation of a/b may have missed their window of opportunity. Instead of blaming poor algebra skills, Lamon (2001) found that little or no understanding of rational numbers accounts for most students' conceptual difficulties when trying to understand the derivative.

2.2 Student Understanding of Fractions as Part-Whole

A student who has developed a part-whole concept of fractions immediately understands “ $3/4$ ” as three parts separated from a whole that has been partitioned into four equal parts.

However, a part-whole conception of fractions is a bit of a misnomer because, according to Steffe and Olive (2009), who observed the way students comprehend fractions in terms of levels or schemes, says the “partitive fraction scheme” is the most primitive way to comprehend fractions.

Hunting (1983) did a study with an individual student in which he found that the student was able to understand one-fourth and one-eighth as one of four and eight equal parts, respectively. However, that same student did not understand one-eighth was less than one-fourth because he did not understand the inverse nature of the numbers. Such understanding requires the iterating operation and partitive conceptions of fractions. Partitioning is in effect splitting or dividing up a whole into fractional parts, and iterating is the opposite. For example, partitioning can be explained by stating that $\frac{1}{5}$ is “the amount we get by taking a whole, dividing it into 5 equal parts, and taking one of those parts” and iterating by stating that $\frac{1}{5}$ is “the amount such that 5 copies of that amount, put together, make a whole” (Siebert & Gaskin, 2006, p. 395).

Hunting (1983) and Mack (2001) recognize the importance of part-whole reasoning in developing fraction conceptions, but highlight that to construct “genuine” fractions, students need to transcend part-whole conceptions. “Students’ informal knowledge of partitioning did not fully reflect that complexities underlying multiplication of fractions” (Mack, 2001, p.291).

Teaching efforts have focused almost entirely on the part-whole construct of a fraction (Streefland, 1991). That singular focus of curricula has “contributed to students’ difficulties in working with fractions operations and even algebraic reasoning” (Steffe & Olive, 2010, p. 343). Additionally, the concepts of mixed numbers and improper fractions confuse students in the part-whole construct since students do not understand that a numerator can be larger than the denominator, or, for that matter, a whole number can be written in front of a fraction (Neumer,

2007).

To make matters worse, teachers attempt to teach traditional algorithms to convert mixed numbers into improper fractions and vice versa while students are still struggling to grasp the concept of what mixed numbers and improper fractions really are. Students need to have a conceptual understanding of improper fractions before being taught an algorithm (Neumer, 2007) which is too abstract for them to grasp. Furthermore, Olive and Vomvori (2006) found that instruction that was restricted to part-whole concepts hindered a student's ability to meaningfully engage in classroom activities that implicitly required more advanced conceptions.

Researchers found that most students see fractions as merely a part *over* a whole rather, a two-entity conception. Consequently, they may not pay attention to the value or magnitude represented by a fraction. Simon (2002) came to the conclusion that students misconceive fractions with equivalent magnitude after asking them to compare pieces of three congruent squares partitioned into two equal parts in three different ways. The students said that certain halves were bigger than others (see Figure 1).



Figure 1: Simon (2002) Squares

The over-emphasis of the part-whole conception probably obscured students from focusing on the numerical value represented by a fraction. In an extensive study involving 200 students ranging in age from 10 to 16 years (Stafylidou & Vosniadou, 2004), students were asked questions about the numerical value of fractions. Many of the students surveyed saw a fraction as merely two independent numbers and several others were able to recognize parts of a whole. Only a few,

however, were able to understand the multiplicative relationship between the numerator and denominator.

The problems associated with the part-whole method of learning fractions is not the sole reason for student misunderstanding of fractions. Fractions are inherently difficult for learners because they can be interpreted in different ways for different purposes.

2.3 Multiple Meanings of Fractions

Fractions have multiple meanings and interpretations. Educators generally agree that there are five main interpretations: fractions as parts of wholes; fractions as the result of dividing two numbers (sometimes referred to as the quotient meaning); fractions as the ratio of two quantities; fractions as operators; and fractions as measures (Behr, Harel, Post & Lesh, 1992; Kieren, 1988; Lamon, 2006).

Fractions as operators is understood to be a number that acts on another in the sense of stretching or shrinking the magnitude of the number. A model car, for example, might be $\frac{1}{30}$ the size of the actual car.

A fraction can even be interpreted as having a distinct measure with a specific placement on a number line. The discussion of measure is usually accompanied by partitioning (Mack 2001; Siebert & Gaskin, 2006; Steffe and Olive, 2009). A unit of measure can always be partitioned into smaller and smaller subunits.

Wu (2002) argues that a major reason for students' failure to learn fractions is the “mystical and mathematically incoherent” manner in which the subject has been presented to them – i.e. with multiple meanings (part-whole, quotient, ratio). Hence, very few students can give a clear definition of a fraction that is at all related to the manipulations (reducing fractions to simplest

terms, finding equivalent fractions, determining common denominators, dividing the denominator into the numerator) they are made to learn. Wu (2002), therefore, proposes to base instruction on one clear cut definition of fraction as a point on the number line.

2.4 The Whole Number Bias

The difficulty students have with fractions should not be surprising considering that before they are exposed to rational number, students have formed an initial concept of number, which is based on the act of counting whole numbers (Vosniadou & Vershaffel, 2004). Students must adopt new rules for fractions that often conflict with well-established ideas about whole numbers. For example, when ordering fractions with like numerators, students learn that $\frac{1}{3}$ is less than $\frac{1}{2}$. With whole numbers, however, 3 is greater than 2 (Bezuk & Cramer, 1989).

When students learn about rational numbers in any form (fractions or decimals) they often use the properties of whole numbers to interpret rational numbers (Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2007). This is referred to as the “whole number bias” which, according to Ni & Zhou (2005), impedes student learning of fraction and rational numbers. Students’ initial conceptions of number constrains their interpretation of new information regarding rational number causing persistent misconceptions (Vosniadou, Vamvakoussi & Skopeliti, 2008).

2.5 Methods for Introducing Fractions

There are certainly differing thoughts concerning how to introduce fractions to students and which interpretation of fraction is most effective for deeper comprehension. Lamon (2001) tackles the following question: which “subconstruct” – part-whole, quotient, ratio, measure, or operator – is the most effective primary interpretation for students to use in building a

comprehensive understanding of rational number topics? In order to develop an answer to the question, she conducted a four-year study of 3rd through 6th graders in which five different classes studied fraction-related concepts through the “lens” of a different subconstruct. She determined that the two most effective subconstructs proved to be a) part-whole with unitizing, and b) measure – as gauged by student achievement in the area of proportional reasoning, computation, understanding of multiple rational number interpretations. These two approaches focus student attention to the inverse relation between the size of the measurement unit and the number of units that measure a given quantity; and on the successive partitioning of a unit into finer and finer subunits until one can name the amount in a given quantity.

Wu (2002) agrees. As mentioned above, he proposes to base instruction of fraction as a point on the number line, and to use reasoning to deduce, as logical consequences, all other meanings of this concept.

The Center for Improving Learning of Fractions focuses on building a solid foundation of conceptual understanding of fractional magnitudes by working on concepts and procedures involved in common fractions and decimal equivalents. Others such as Moss & Case (1999) and Donovan & Bransford (2005) suggest reversing the order in which students are introduced to fractions. They both recommend introducing students first to percent in a linear measuring context, followed by decimals, and then by fractions, as an alternative form for representing decimals. Children know a good deal about percents from their everyday experiences. By beginning with percents rather than fractions or decimals, we are able to capitalize on children’s preexisting knowledge of the meaning of these numbers and the context in which they are important (Donovan & Bransford, 2005). By introducing percents first, we allow students to make their preliminary conversions among the different rational-number representations in a direct and

intuitive fashion while developing a general understanding of how percents, decimals, and fractions are related. Moss & Case (1999) found students who received this reverse order curriculum over a 5-month period had a deeper understanding of rational numbers than those in the control group as they relied less on whole number strategies when solving non-standard problems.

Although Moss & Case (1999) and Donovan & Bransford (2005) advocate teaching percentages as the initial representation of rational numbers, there is no clear consensus yet on when and how to introduce fractions. Some suggest to introduce fraction as late as 6th grade, as opposed to introducing fractions to kindergarteners or 1st graders as is done in many American schools. Bezuk & Cramer (1989) suggests that by postponing most operations with fractions at the symbolic level until grade 6 and using instructional time in grades 4 and 5 to develop fraction concepts and the ideas of order and equivalence, teachers will find that their students will be more successful with all aspects of operations with fractions and will have a stronger quantitative understanding of them.

There is a lot of literature identifying the problem and suggestions on ways to improve our instruction. However, students in general are still struggling to intuitively see that a fraction has a single value with a specific placement on a number line (Siegler et al., 2010). Many students do not even appear to have developed an understanding that fractions are numbers (Kerslake, 1986; Domoney, 2002; Hannula, 2003). This study investigates math students' ability to conceptualize fractions as two numerals having a single value, and the ways students realize that a fraction indeed has a single value connected to the numerator and denominator.

CHAPTER 3: RESEARCH METHODS

This research study was designed to investigate students' interpretations of a fraction, specifically whether or not they recognize on their own that a fraction has a single value. This research study sought to identify levels of progression in student understanding of a fraction as having a single value that is related to the numerator and denominator. To obtain diverse responses, I chose to interview college students in developmental algebra courses and 7th graders because they had learned fractions and some of them might still be struggling with fractions. In addition, I could compare these two groups in terms of their understanding fractions.

The investigator's goal was to answer the following questions:

1. How do students respond to the items that were designed to uncover whether students interpret a fraction as having a specific value or as having two separate numbers?
2. What are the progressive levels of understanding of a fraction as having a specific value that is related to the numerator and the denominator?
3. How do the college students compare to the 7th graders in terms of the progressive levels of understanding?

3.1 Participants

The study group was comprised of two groups of participants: (a) 12 undergraduates (six males and six females) enrolled in developmental algebra classes at a university in the Southwest region, and (b) eight 7th graders (four boys and four girls) in a private school in the same region.

70% of the participants were of Hispanic origin, 20% were Caucasian and 10% were African American.

The 12 college students volunteered to take part in the study after the investigator visited their classes to recruit. They signed the consent form and agreed to participate in an interview after class. The students agreed to be interviewed for thirty minutes, agreed to be recorded, and knew there were no direct benefits to taking part in the study. Since there were only 12 volunteers, all of them were selected as participants.

Similarly, the eight 7th graders signed an assent form. The parents or guardians of the students signed the informed consent form. This group agreed to the same thirty-minute recorded interview after school and knew there were no incentives for participation. All eight volunteers were accepted as participants.

The 7th graders were partially chosen to participate as a comparative group. This grade level would have learned about fractions more recently than the college students, but they still may not understand fractions well. This group of students were partly chosen out of convenience since they were enrolled at a school where the investigator worked as a teacher.

During an eight-month period, these twenty students were individually interviewed at separate times using an in-house developed instrument that is discussed in the next section.

3.2 Instrument Items

The math items in the instrument were designed and sequenced to help reveal whether or not students intuitively recognized the value of a fraction. The instrument was designed during the course of three months utilizing a test group. This beta group had four developmental math students (two males and two females) from a university in the Southwest region of the United States. The responses from these four students helped the investigators to refine the instrument by

revising certain items to direct students to think about the single value of a fraction and removing certain items that direct students to think about fractions as merely a part-whole conception. The final instrument had seven items designed to determine whether the student was able to recognize the single value of a fraction as well as their interpretations of fractions such as merely consisting of a numerator and denominator or as “something to do” like a division problem.

The seven items on the instrument (see Appendix A for the full instrument) began with a relatively open ended prompt: **List as many things as you can about the fraction $\frac{3}{5}$.** This question helped the investigator from the beginning of the interview to uncover the student’s spontaneous interpretations of fractions without being influenced by representations in subsequent tasks.

The next item asked students to do the following: **Mark $\frac{4}{5}$ on the number line below.** During the beta phase the number line went from 0 to 6 because the first three students marked the four or between the four and five. Thus, the number line was changed from one that was from 0 to 6 to one that went from -2 to 6 to avoid the ambiguity of interpreting that 5 is the whole. As a follow up task, some of the students were asked to locate $\frac{5}{4}$ on the number line. During the interviews there was an effort to draw students (especially hesitant students) attention to two integers, -1 and 6, on the number line because students might be thinking that $\frac{4}{5}$ must be on a tick mark. This move is to differentiate between misinterpretation of the problem (must be a tick mark) and misconception ($\frac{4}{5}$ means 4 whole on a number line). This move is an indirect attempt to find out whether the student would think of $\frac{4}{5}$ as a number between 0 and 1.

Item 3 asked the students to think a bit more deeply: **Suppose x is any natural number, tell me all that you know about $\frac{x}{x+2}$.** The purpose of this item is to find out if students compare the numerator and the denominator and if they do whether they are comparing additively (i.e., a

difference of 2 units) or multiplicatively (e.g., the numerator is only a portion of the denominator). Weaker students might have a tougher time getting started on this item because of the Algebra involved and the interviewer was prepared to give some prompting about the meaning of natural numbers. If students handled this item well, then they were asked to come up with a new expression that would be more than one.

Item 4 began with the prompt: **Circle the larger fraction: $\frac{6}{8}$ or $\frac{8}{10}$.** Notice that the two fractions are specific instances of $\frac{x}{x+2}$ (the numerators are 2 less than the denominators). This item is essentially asking students to compare the values of the two fractions. This item was intended to find out whether students would really focus on the value of each fraction or on the numerators and denominators. It would be followed up with two similar questions: **Circle the larger fraction $\frac{1}{4}$ or $\frac{21}{101}$** and **Circle the larger fraction $\frac{973}{983}$ or $\frac{19}{9}$.** The former pair seeks to find out the strategies students would use to compare the two fractions (e.g. division, common denominators, estimation, etc.) noticing that the numerator 21 is about a fifth of the denominator 101 or cross-multiplying). The latter pair involves a proper fraction whose value is almost 1 and an improper fraction whose value is almost 2. These three items are sequenced such that the likelihood for a student to focus on values is greatest in the third item followed by the second item.

On Item 5 the student was asked to **Find a number that is between $\frac{2}{9}$ and $\frac{3}{9}$.** This item is included to find out whether or not the student can find a fraction or number between two consecutive fractions. This item seeks to elicit the strategy a student would use (e.g., converting each fraction to a decimal or equivalent fractions with larger denominator) and to see if the student pays attention to the values of the fractions while using the strategy.

Item 6 asked students to analyze the graph of a line, in particular its slope: This task is an attempt to have students reconcile the one value and the two numbers. Hence, this item was asked towards the end of the interview.

The task has a three parts and begins with the following: **Look at the graph. John says that the slope is $\frac{3}{12}$. Can you help me see the 3 and the 12 in this graph?**

Students were asked the next follow up question: **Mary says the slope is 0.25. Can you help me to see the 0.25 in the graph?** In order to see if students are making connections to the numbers, students were asked **Are $\frac{3}{12}$ and 0.25 the same thing?** The purpose of this questions is to find out whether students reconcile the value of 0.25 with the value of the fraction $\frac{3}{12}$.

The last item in the instrument asked students to **Approximate the following: $\frac{220}{443} + \frac{31}{59}$.** Asking for an approximate value of the sum requires students to focus on the value of each fraction. This problem is designed to find out whether students understand addition as adding values ($\frac{1}{2} + \frac{1}{2} = 1$) or merely adding numerator with numerator and denominator with denominator ($\frac{251}{502} = \frac{1}{2}$). As time permitted, students were also asked to **Approximate $\frac{71}{101} + \frac{41}{81}$.** This follow-up item is assessing whether students recognize $\frac{71}{101}$ as approximately equal to $\frac{7}{10}$ and $\frac{41}{81}$ as approximately equal to $\frac{1}{2}$.

3.3 Data Collection

Twelve developmental math students (six male and six female) at a college in the Southwest region of the United States and eight 7th graders (four male and four female) from a private school in the same region were interviewed from October 2014 to May 2015. Each subject was questioned for approximately thirty minutes. The interviews all took place after regular classroom times and were done on school premises. The students were asked a series of questions

to determine their level of understanding of the single value of fractions. These sessions were recorded and transcribed. The data was collected and analyzed over several months in order to gain insights and patterns from the responses.

3.4 Data Analysis

The data analysis consists of three phases. In the initial coding phase, I tried to understand how students reasoned through each task and whether or not the students were able to conceptualize the fractions correctly or incorrectly. For example, some students may simply be looking at the magnitude of the numerator or denominator to make judgments about the size of the fraction. Others may look at the differences between the numerator and denominator to make judgements about the fraction. Still others may do something to the fraction such as division or creating an equivalent fraction with a different denominator to help them determine some traits of the fraction.

Additionally, during the first phase it was determined which responses could contribute to the study. There were a few instances of student responses that were so off track or even made up that they were excluded from the analysis because they did not contribute to this study. For Item 1 two college student responses were dropped from the analysis: one was mystified by the fraction $\frac{3}{5}$ since it was not a “normal fraction like $\frac{1}{4}$ or $\frac{1}{2}$ ” and the other student said that the only thing that came to mind when he was presented $\frac{3}{5}$ was to multiply the 3 and the 5 to get 15.

This research is about fractions rather than algebraic expressions; therefore, three responses were discarded from the results of Item 3. Two 7th grade responses were eliminated from the analysis for this task: one said that $x + 2 = 2x$ and the other interpreted $x + 2$ as being equivalent to $3x$. One response from a college student was dropped because his interpretation of the algebra

was that the x on the top and the x on the bottom could be different values. Another college student was not asked this question because of an oversight.

In order to keep the interviews from taking longer than thirty minutes, the latter tasks were not completed by some students: three college students and five 7th graders were not asked to do Item 6. One of the 7th graders who did respond to Item 6 was dropped because he decided to find his own slope ignoring the $\frac{3}{12}$ and the .25 mentioned in the problem. Four college students and two 7th graders were not asked to complete Item 7. Table 1 shows how many of the students' responses are included in the analysis:

Table 1: Number of Student Responses Included

Task	College Students	7th Grade Students	Total Students
Item 1	10	8	18
Item 2	12	8	20
Item 3	10	6	16
Item 4	12	8	20
Item 5	12	8	20
Item 6	9	2	11
Item 7	8	6	14

In the second phase of the coding, general categories were placed on each of the responses in the study. I used a spreadsheet to capture all the responses. This spreadsheet helped me to describe and identify the trends in the responses and aided in uncovering potential categories and subcategories of responses for each item. For each item, I then reexamined each student response

using these categories and subcategories and assigned a “1” if a category is evident in the response. This was a way that I could more easily consolidate the responses into groups.

In the third phase, I examined all the categories and subcategories across all the items with the goal of identifying distinctive progressive levels of understanding fractions in terms of a single value and/or two numbers. I initially identified four levels of progression but ended up splitting one of the levels into two levels.

Each student response was re-examined and assigned a level. Some students may have more than one subcategory for a response to the same task. For example, a student may initially respond within a subcategory that has level 1, but then shortly afterwards may adjust their thinking and respond at higher level. In that case, the response was re-examined to determine the appropriate level.

After assigning a level to each response, I examined how each student responded to all the items in the interview as a whole and assigned an “overall” level for the student. This assignment allowed me to compare the college students to the 7th graders in terms of their progressive levels of understanding.

CHAPTER 4: RESULTS AND DISCUSSION

In Section 4.1, student responses are discussed and organized by item. First a table of coding is presented for the item being discussed. Next a description of how the categories and subcategories emerge from the student responses is presented. This is followed by a summary of the key results for that particular item. The results in this section correspond to the first research question.

In Section 4.2, progressive levels of understanding fractions are presented and discussed. Next, two tables are presented. The first table assigns a progressive level to each of the student responses on each of the interview items and then a dominant level was assigned to each student. The second table shows the number of students operating at each level and is used to compare the two groups of students.

Section 4.3, is a discussion of the results in Chapter 4. First, a summary of the student responses to the tasks is presented. Next, is a discussion of the progressive levels of understanding fractions. Finally, a comparison of the two groups of students is given in terms of the progressive levels.

4.1 Students' Responses to Interview Tasks

In this section I will discuss each item from the interview process. I will explain the categories and subcategories that emerged from the responses to each item. Next, I will present a table that summarizes the categories of responses and then I will explain how this information was derived while giving some details from the student interviews.

4.1.1 Item 1: *List as many things you can about the fraction $\frac{3}{5}$.*

The purpose of this item is to elicit students' initial conceptions of fraction. Three categories and seven subcategories emerged from analyzing the responses from the 18 students observed on Item 1. Table 2 shows the number of student responses for each subcategory. There were a total of 27 counts because a student response may be coded in two or more subcategories. For example, one student drew a picture of $\frac{3}{5}$ then later talked about $\frac{3}{5}$ being a division problem.

Table 2: Coding for Item 1

		College	7th	All
Drawing figures	Realize the parts must be equal sized	1	1	2
	Not realizing parts must be equal size	5	4	9
Using division	Convert to decimal or % (divide 3 by 5)	2	4	6
	Incorrectly converted to decimal (divide 5 by 3)	1	3	4
Having some sense of size, magnitude, or value	$\frac{3}{5} = 0.6$ is a value (e.g. between 0.5 and 1.0)	2	1	3
	$\frac{3}{5} = 60\%$ is a portion of a whole (100%)	1	1	2
	$\frac{3}{5}$ relates 3 to 5 multiplicatively (e.g. rise of 3 for every run of 5)	1	0	1

Eleven (six college and five 7th grade) responses involved drawing pictures, mostly circles, to represent $\frac{3}{5}$. These descriptions of fractions are typically categorized as a part-whole conception. All eleven students represented the whole using a circle. One drawing by a college student was an accurate representation of $\frac{3}{5}$. Nine students began with a representation for a common fraction like half-circle and $\frac{3}{4}$ -circle and then split a piece into smaller pieces (see Figure 2). One of them (a 7th grader) realized that her drawing was not accurate and explained that the five pieces needed to be the same size. The other students thought that their drawings with non-equal sized pieces accurately represented the fraction. One of the 7th graders also drew a rectangle in which she divided it into five segments. When asked if the pieces had to be the same size, she responded that

she did not think so “just as long as you shade three of the five rectangles, it doesn’t matter that the rectangles are different sizes.” The student clearly did not realize that the number of pieces, denoting the denominator of a fraction, must be of equal size.



Figure 2: Student Drawings of $\frac{3}{5}$

Ten students (three college and seven 7th grade) tried to use division to convert $\frac{3}{5}$ into a decimal. Four students attempted to divide 5 by 3 and three of those students got an answer of 1.6; these students seemed uncomfortable with dividing the smaller number by the larger number. Two of the students who cited division and divided 3 by 5 correctly obtained 0.6 but they were not comfortable with their answer and did not make any meaningful connections to the fraction $\frac{3}{5}$. One, referring to the answer of 0.6, even said “I must have done that wrong.” Two students mentioned that they could not divide 3 by 5; one of them said “I don’t think I can divide that. It will be a huge decimal.” It is not clear whether she meant that its value would be large or if the process would render a recurring decimal. The other student who said dividing it this way would result in a “big number” explained that “if you were to see how many times five would go into three, you can’t do that since five is bigger than three.”

Six of the student responses (four college and two 7th grade) indicated that they viewed the fraction as a single value with correct connections to the two numbers (the 3 and the 5). Three of these students described 0.6 as “the equivalent amount in decimal form” and knew that amount to be between 0.5 and 1.0. One of them stated, “It’s a little over a half.” Two students (one college and one 7th grade) saw $\frac{3}{5}$ as “equal to 60%” and seemed to easily understand that 3 is 60% of 5.

One student said he “thinks in terms of percentages. That’s just how I see $\frac{3}{5}$.” One of the college students related $\frac{3}{5}$ to a rise of 3 for every run of 5 and was able to communicate that through a linear graph.

In summary, eleven students (at least initially) viewed the fraction as two independent numbers. These students drew pictures and nine of them did not even realize that the size of the parts mattered. Ten students had a sense that fractions could be converted to a decimal: four tried to divide 5 by 3 and six divided 3 by 5. Six students divided 3 by 5 to get 0.6, but only three of them could actually say with confidence that $\frac{3}{5}$ was an equivalent value to 0.6.

4.1.2 Item 2

The purpose of Item 2 is to investigate students’ conceptions of how fractions fit on a number line and to gain insights on how students interpret fractions especially as to how they compare to other numbers. Item 2 has two parts. The first part asks students to locate a proper fraction on a number line and the follow up item asks students to locate an improper fraction on a number line.

Item 2a: *Mark the $\frac{4}{5}$ on the number line below.*

For Item 2a, two main categories were identified and they differentiate student responses that were indicative of a conception that the fraction has a single value from those that were not. Five subcategories were identified. Table 3 shows the number of responses for each subcategory. Although there are 20 student responses, there are a total of 25 counts because a student response may be coded in two or more subcategories.

Table 3: Coding for Item 2a

		College	7th	All
The Fraction 4/5	Focus is on the N (4/5 is plotted at 4)	4	4	8
Not indicative of single-value conception of fractions	N is Whole and D is the number of tenths (4/5 is plotted as 4.5)	5	0	5
	Somewhere between 0 and 1	2	1	3
	Incorrect division to get 1.25	0	2	2
The Fraction 4/5	Doing division - understanding 0.8 is nearly 1	3	0	3
Indicative of single-value conception of fractions	4th tick mark with each tick mark being one-fifth	1	3	4

Item 2b: Mark the $\frac{5}{4}$ on the number line below.

Item 2b is the follow-up task where 13 of the students were asked to mark $\frac{5}{4}$ on the number line. Item 2b has the same main categories as Item 2a. Five subcategories were identified for this follow-up item. Table 4 shows student the number of responses for each subcategory.

Table 4: Coding for Item 2b

		College	7th	All
The Fraction 5/4	Focus is on the N (5/4 is on the 5)	3	1	4
Not indicative of single-value conception of fractions	N is Whole and D is the number of tenths (5/4 is plotted as 5.4)	1	0	1
	Somewhere between 0 and 1	1	2	3
The Fraction 5/4	5/4 is equivalent to 1.25 and is slightly more than 1	3	0	3
Indicative of single-value conception of fractions	5/4 is equivalent to 1 1/4 because each tick mark is one-fourth	1	1	2

For the original task of marking $\frac{4}{5}$ on the number line, there are four subcategories for responses that are not indicative of $\frac{4}{5}$ having a single value. Eight students (four college and four 7th grade) plotted the $\frac{4}{5}$ at the 4; these students seemed to disregard the denominator 5 or they might

have assumed the whole was 5 in a part-whole conception of fraction. Five students (all college) marked a point half way between 4 and 5. The five college students who marked between the 4 and 5 probably interpreted $\frac{4}{5}$ as 4.5 thinking that the numerator was the starting point and the denominator denoted the number of tenths as shown in Figure 3.

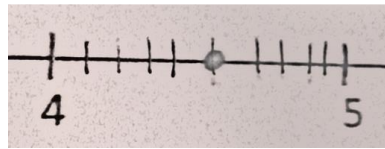


Figure 3: $\frac{4}{5}$ with 4 as the Numerator and 5 as the Tenths

Three students (two college and one 7th grade) said that $\frac{4}{5}$ had to be between 0 and 1 but they were unclear, even if they correctly marked the $\frac{4}{5}$ on the number line, as to why $\frac{4}{5}$ was between 0 and 1. As one student put it, “I don’t know how to explain it. That is just how we were taught. I know it is not bigger than one, because there is no whole number in front.” This student was probably aware that a mixed number has a fractional component and a whole number component. Nevertheless, it is not clear whether he really thought that a fraction has a specific value other than the fraction must be between 0 and 1.

In an attempt to figure out the decimal value for $\frac{4}{5}$, two 7th graders divided 5 by 4 to get 1.25. They seemed to be merely using a division procedure, thinking that they need to divide the larger number by the smaller.

Seven students seemed to know $\frac{4}{5}$ has a specific value. Three college students correctly thought of dividing 4 by 5 to get 0.8 and described 0.8 as close to 1 on the number line. Four other students (one college and three 7th grade) explained $\frac{4}{5}$ as being the fourth of four tick marks between the 0 and 1 because each tick mark represented a one-fifth. All these seven responses were also categorized as conceptualizing the fraction as having a single value.

For follow-up task Item 2b, thirteen students were asked to locate $\frac{5}{4}$ on the number line. For these students, the conceptual basis for interpreting $\frac{5}{4}$ was the same as that for interpreting $\frac{4}{5}$. Consequently, the five subcategories for this task are analogous to those for the original task Item 4a.

Four students (three college and one 7th grade) focused on the numerator and marked the 5 on the number line. One (a college student) interpreted $\frac{5}{4}$ as 5.4.

Three students (one college and two 7th grade) said that $\frac{5}{4}$ was between the 0 and 1. As mentioned earlier, some students simply believed *all* fractions were between 0 and 1 without having a rationale for why. A student said when referring to the $\frac{5}{4}$, “the only way that would be bigger than 1 is if it were a mixed fraction.”

Five students (four college and one 7th grade) were able to find the correct placement of the $\frac{5}{4}$ on the number line accompanied by correct rationale that it was slightly more than 1.

In summary, there were 18 instances where a student response is not indicative of conceptualizing a fraction as having a specific value. Seven out of 20 students seemed to conceptualize a fraction as having a single value; two of these students also have comments that were indicative of not thinking a fraction as having a single value.

4.1.3 Item 3: *Suppose x is a natural number, tell me all that you know about $\frac{x}{x+2}$.*

The purpose of this item is to find out how students compare the numerator and the denominator and to see if students are able to interpret the result(s) of an expression in a manner that shows an understanding of value of fraction. There were two main categories identified and

three subcategories identified for each main category (see Table 5). Although there were 16 student responses for this task, there were a total of 21 counts.

Table 5: Coding for Item 3

		College	7th	All
Not indicative the expression as having a value	Crossed out the two x 's	3	1	4
	Replacing x with a number but not attending to value	3	2	5
	Can be greater than 1	2	1	3
Indicative the expression as having a value	Increases as x increases	1	1	2
	Never reaches 1	2	2	4
	Found an expression that is greater than 1	2	1	3

There are three subcategories of responses that are interpretations of the expression that does not seem to indicate that the expression has a value, which depends on the value of x . Three college students and one 7th grader crossed out the x on the top and the x on the bottom of the expression with two students obtaining 2 and one student obtaining $\frac{1}{2}$. One may consider this an error in simplifying an algebraic expression in algebra but underlying this computational error is a lack of understanding of fraction as a relationship between the numerator and the denominator. It is possible that these students are relating the numerator and denominator additively rather than multiplicatively, in that the difference between the denominator $x + 2$ and the numerator x is equivalent to the difference between the new denominator 2 and the new numerator 0.

Five students (three college and two 7th grade) replaced the x with at least two different values but did not notice any patterns because they were not really attending to value nor were they paying attention to the relationship between the numerator and the denominator.

Three (two college and one 7th grade) claimed, after replacing the x with at least two natural numbers, that the expression could be a whole or greater than 1. Even when the students could not show an example that validated this sentiment, they still said that the expression could be more

than 1. As one student put it, “there would most definitely be numbers bigger than one because of the realm of possibilities.” This indicates that the student was not aware that a fraction that is greater than 1 would have its numerator being greater than its denominator, and probably did not pay attention to the multiplicative relationship between the numerator and the denominator.

Six student responses seem to suggest that they are paying attention to the value of the expression $\frac{x}{x+2}$. Two students (one college and one 7th grade) noticed that as x increases, the value of the expression also increases. Four students (two college and two 7th grade) noticed that the expression would never reach one.

To find out whether these six students understood why $\frac{x}{x+2}$ is less than 1, they were asked to create an expression that would be more than 1. Three of the students could not produce an answer to this follow up question. However, the other three students (two college and one 7th grade) said to simply turn the expression upside down like this: $\frac{x+2}{x}$. The 7th grader who came to this conclusion explained it this way: “Oh my gosh!” she exclaimed. “I can just flip it.” However, she seemed disappointed in her result. She replaced the x with a 2 and she wrote $\frac{4}{2} = 2$. She said, “It’s bigger than 1, but it’s not as big as I wanted it to be.” This may indicate that she may not have fully conceptualized the relationship of the numerator and denominator.

In summary, students were asked to consider the algebraic expression $\frac{x}{x+2}$ while supposing that x could be a natural number. Twelve of the students’ responses were instances where students did not seem to be aware that the expression has a value, whose value depends on the value of x . Three said the expression could be equal to or greater than 1 even though could not find an example. Two students could tell that the fraction kept getting larger with each natural number and four more realized that the expression would never become 1. As a follow-up question to Item

3, six students were asked if they could come up with an expression that was more than a whole. Three students were able to answer the question in a manner that was indicative of understanding, at least as some level, the multiplicative relationship between the numerator and denominator.

4.1.4 Item 4

Unlike the earlier items, Item 4 explicitly asked students to pay attention to the value of each fraction to determine which is larger. Item 4 has three parts in which students were asked to compare fractions: $\frac{6}{8}$ versus $\frac{8}{10}$, $\frac{1}{4}$ versus $\frac{21}{101}$, and $\frac{973}{983}$ versus $\frac{19}{9}$. Note that both $\frac{6}{8}$ and $\frac{8}{10}$ are specific instances of $\frac{x}{x+2}$.

Item 4a: Circle the larger fraction $\frac{6}{8}$ or $\frac{8}{10}$.

The responses corresponding to the first pair of fractions (Item 4a) are presented in Table 6. There were two main categories (these remain the same for all three parts of Item 4) identified that differentiate student responses that were indicative of a conception that the fractions have unique and comparable values from those that were not. There were five subcategories identified with 20 coded responses.

Table 6: Coding for Item 4a

		College	7th	All
6/8 & 8/10	Same difference = Same size	3	3	6
Fractions <u>do not</u> have unique or comparable values	Big N or Big D = Big Fraction	5	2	7
	Just chose the smaller D without knowing why	1	1	2
	Used procedure without understanding	1	2	3
6/8 & 8/10				
Fractions <u>do</u> have unique or comparable values	Changed both to decimals or percentages to compare	2	0	2

Item 4b: Circle the larger fraction $\frac{1}{4}$ or $\frac{21}{101}$.

The responses corresponding to the next pair of fractions (Item 4b) that students were asked to compare are presented in Table 7. There were four subcategories identified and 18 student responses were coded.

Table 7: Coding for Item 4b

1/4 & 21/101		College	7th	All
Fractions <u>do not</u> have unique or comparable values	Big N or Big D = Big Fraction	1	1	2
	Just chose the smaller D without knowing why	5	3	8
	Used procedure without paying attention to value	2	3	5
1/4 & 21/101				
Fractions <u>do</u> have unique or comparable values	Changed both to decimals or percentages to compare	3	0	3

Item 4c: Circle the larger fraction $\frac{973}{983}$ or $\frac{19}{9}$.

The responses corresponding to the last pair of fractions (Item 4c) that students were asked to compare are presented in Table 8. This pair of fractions includes an improper fraction. Six subcategories of responses emerged with 23 student responses coded.

Table 8: Coding for Item 4c

		College	7th	All
973/983 & 19/9	Same difference = Same Size	3	3	6
Fractions <u>do not</u> have unique or comparable values	Big N or Big D = Big Fraction	1	3	4
	Just chose the smaller D without knowing why	1	2	3
	Incorrect concept of an improper fraction	2	3	5
973/983 & 19/9				
Fractions <u>do</u> have unique or comparable values	Could correctly describe the relationship of N to D	2	1	3
	All improper fractions are bigger than all proper fractions	1	1	2

In regard to the first pair of fractions, six students (three college and three 7th grade) focused on the difference between the numerator to the denominator in both fractions. They said that, since 6 and 8 had a difference of two, and 8 and 10 also had a difference of two, the fractions must be equal. One 7th grade student simplified the fractions to get $\frac{3}{4}$ and $\frac{4}{5}$ and remained steadfast in her thinking that the same difference equals same size (i.e., both fractions have a difference of 1 between the denominator and the numerator). Another college student showed pictorially that $\frac{6}{8}$, $\frac{8}{10}$ and $\frac{10}{12}$ were equal because the differences from the numerator to the denominator were the same (see Figure 4).

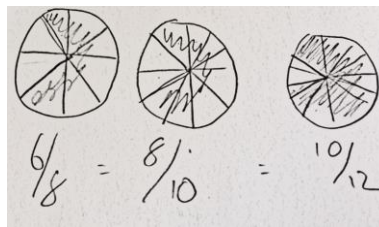


Figure 4: Student Showing Three Fractions as Equivalent

Seven students (five college and two 7th grade) said that $\frac{8}{10}$ is the larger fraction just because their focus was on the larger individual numbers. This is a misconception that large numerators and/or denominators means large numbers and indicates that students are not paying attention to the single value of the fraction but rather the individual numerals.

Two students (one college and one 7th grade) simply selected the fraction that had the smaller denominator as the one with the larger value because, as one of them put it, “that’s just the way we were taught.” This student merely memorized a technique that he believed to be true about fractions.

Three responses from students were procedurally correct without understanding what makes a fraction’s value larger than another. Two 7th graders used cross multiplication to determine that $\frac{8}{10}$ is the larger fraction. A college student simplified the fractions then he got a

common denominator of 20. He selected $8/10$ as the larger fraction without really paying attention to the value of each fraction.

Two students (both college) said that $6/8$ is equal to 0.75 or 75% and $8/10$ is equal to 0.8 or 80%. One explained $8/10$ was the larger fraction because it was closer to 1. The other student said that $6/8 = 3/4$ and was thus “75% of a whole.” Similarly, he said that $8/10 = 4/5$ and thus “80% of a whole.”

For Item 4b, students were asked to compare $1/4$ and $21/101$ (refer to Table 7). Two students (one college and one 7th grade) selected $21/101$ because the individual numerals were bigger.

Eight (five college and three 7th grade) said $1/4$ has the larger value because it has a smaller denominator. These students could not articulate any other reason why $1/4$ was larger than $21/101$. Three of them indicated that the smaller number in the denominator meant “bigger pieces” but they were not paying attention to the number in the numerator.

Five students used a procedure, with no attention to values, to help them select a fraction. Two 7th graders used cross multiplication to help them compare the fractions. Three (two college and one 7th grade) tried to get a common denominator to determine which one is bigger. These students were not able to determine a common denominator. One made a pure guess and the other two said they did not know which fraction was the larger.

Three college students determined that $1/4 = 0.25$ or 25%. One noticed that $21/100$ was very similar to $20/100$ and said that was equal to 0.20. One described $21/100$ pictorially (see Figure 5) showing $21/100$ as slightly less than $25/100 = 1/4$. Another college student said that $1/4$ is equivalent to 25% and $21/100$ is “roughly 20.9%.” All three chose $1/4$ as the larger number and seemed to make connections to the single value associated with the fractions.

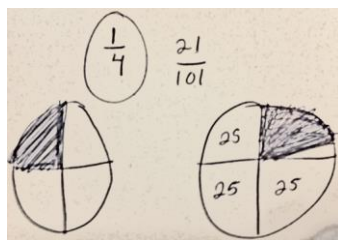


Figure 5: Student Showing Why $\frac{1}{4} > \frac{21}{100}$

For Item 4c, students were asked to compare a proper fraction, $\frac{973}{983}$, to an improper fraction, $\frac{19}{9}$.

Six students (three college and three 7th grade) indicated that the fractions were the same size because they both have a difference of 10 from the numerator to the denominator.

Four students (one college and three 7th grade) selected the $\frac{973}{983}$ as the larger fraction because the individual numerals, the 973 and the 983, in the first fraction are larger than the 19 and 9 in the second fraction.

Three students (one college and two 7th grade) circled the $\frac{19}{9}$ because, as they had done in Item 4a and Item 4b, they selected the fraction with the smaller denominator.

Five students (two college and three 7th grade) noticed that the numerator in $\frac{19}{9}$ was larger than its denominator although failed to correctly conceptualize that an improper fraction has a value greater than one. One of the 7th graders even tried to describe it with a drawing. He began by drawing a circle then he started drawing lines through the circle. When asked what he was thinking, he said, “ $\frac{19}{9}$ is a circle with 19 pieces and 9 of them shaded.” Two students said that the only way that a fraction is larger than 1 is for it to be a mixed number which indicates that these students are not paying attention to the relationship of the numerator and denominator.

Three students (two college and one 7th grade) were able to recognize the correct relationship of the numerator and denominator. One college student said $\frac{973}{983}$ was “close but not even 1” and he described $\frac{19}{9}$ as “more than 1, no its actually more than 2.” His responses

indicate that he recognizes the closer the numerator and denominator are to each other, the closer the fraction is to a value of 1, and similarly, as the numerator approaches twice as much as the denominator, the closer the fraction gets to a value of 2.

Two students (one college and one 7th grade) described $19/9$ as an improper fraction and said $973/983$ was proper. They said that all improper fractions were larger than all proper fractions which denotes some sense of value to the fractions.

In summary, the responses for Item 4 are more informative on students' conceptions of fractions than those for the other items for this study. Consolidating the counts in Table 6, Table 7 and Table 8, 26 of the 61 counts focus on the numerator and/or the denominator rather than the value of the fraction. Twelve responses were misconceptions of the relationship between the numerator and the denominator (e.g. same difference = same size). Eight student responses dealt with procedural understanding (or misunderstanding) that did not help them attend to the value to the fraction. Only ten responses relate the fractions to its value or magnitude.

4.1.5 Item 5: *Find a number between $\frac{2}{9}$ and $\frac{3}{9}$.*

The purpose of this item is to elicit the strategy a student would use (e.g., converting each fraction to a decimal or equivalent fractions with larger denominator) and to see if the student pays attention to the values of the fractions while using the strategy. Four strategies were identified: focusing on whole numbers, thinking of $2.5/9$, using division to convert fractions to decimals, and finding a common denominator. Other than the first category, some students are attending to values and some are not. For consistent presentation of results, I will use “did not attend to the value of the fractions” and “did attend to the value of the fractions” as the main categories. Table 9 shows the 21 counts from the 20 students who were asked to do this task.

Table 9: Coding for Item 5

		College	7th	All
Did <u>not</u> attend to the value of the fractions	Focusing on whole numbers	3	4	7
	Thinking of 2.5/9	4	1	5
	Using division to convert the fractions to decimals	0	2	2
	Finding a new common denominator	1	1	2
Did attend to the value of the fractions	Thinking of 2.5/9	1	0	1
	Using division to convert the fractions to decimals	0	1	1
	Finding a new common denominator	3	0	3

Seven students (three college and four 7th grade) said there were no numbers between the two fractions because they are “consecutive” fractions. These students were probably thinking that there was not a number between 2 and 3, without realizing that $\frac{2}{9}$ and $\frac{3}{9}$ are values that could have other values in between.

Five students (four college and one 7th grade) offered unconventional fractions such as $\frac{2\frac{1}{2}}{9}$ and $\frac{2.5}{9}$ but they were not really attending to its value. As one of the college students put it, “I have never seen a fraction like that before, but that’s all I know to do.” She was focusing on the two numerators instead of the fractions as having specific values.

Two 7th grades divided and got 0.22... and 0.33... for the two fractions but had no understanding of how that helped them to find a number between the fractions.

One college student and one 7th grader found equivalent fractions with new common denominators. Although they found fractions that were in between the two given fractions, they did not attend to the values of the fractions. For example, the 7th grader multiplied both numerators and denominators by 5 and obtained $\frac{10}{45}$ and $\frac{15}{45}$ and said that the answer was $\frac{11}{45}$ through $\frac{14}{45}$. However,

she was not able to explain how those values related to the original pair of fractions and it is not even clear that she understood why $\frac{11}{45}$ through $\frac{14}{45}$ were indeed between $\frac{2}{9}$ and $\frac{3}{9}$.

Among those students who attended to value, a college student explained that $\frac{2.5}{9}$ was equivalent to $\frac{5}{18}$ and that was a number “exactly in the middle.” One 7th grader used division by changing $\frac{2}{9}$ to 0.2 with a line over the 2 and $\frac{3}{9}$ to 0.3 with a line over the 3. Although she incorrectly rounded those answers to 0.23 and 0.34 she knew that 0.234 and 0.33 were between the two fractions. Among the three college students who used new common denominators to find numbers between the $\frac{2}{9}$ and $\frac{3}{9}$, two of them explained that there “many answers” between the two fractions and one said that there were an “infinite number” of numbers between the two fractions.

As an interesting side note, four of the students drew $\frac{2}{9}$ and $\frac{3}{9}$ on a number line with tick marks between the two fractions as in Figure 6. The fact that students put tick marks between the two fractions may indicate that they have some sense that each fraction has a single value. However, three of the students who made this drawing were recorded as “did not attend to the value of the fractions” because they disregarded their drawings and explained that there were no numbers between $\frac{2}{9}$ and $\frac{3}{9}$.

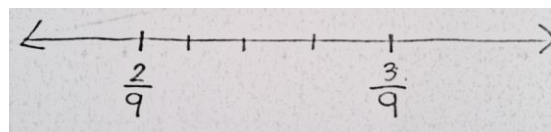


Figure 6: Student Drawing Tick Marks Between $\frac{2}{9}$ and $\frac{3}{9}$

In summary, there were 16 responses are not indicative of students paying attention to the values of the fractions although three of the students had responses that could be indicative of

some level of understanding of value. Five student responses seemed to indicate students were attending to the value of the fractions.

4.1.6 Item 6: *Look at the graph above. John says that the slope is $\frac{3}{12}$. Can you help me see the 3 and the 12 in this graph? Mary says the slope is 0.25. Can you help me see the 0.25 in the graph? Are $\frac{3}{12}$ and 0.25 the same thing?*

Item 6 asks students to analyze the graph of a line, in particular its slope. The series of questions is an attempt to have students reconcile the one value and the two numbers. Only 11 out of the 20 students worked on this item. Two categories of responses were identified and they differentiate student responses that were indicative of a conception that the numbers are related to the graph and those that were not. Four subcategories were identified. Table 10 shows the 13 coded responses.

Table 10: Coding for Item 6

		College	7th	All
No conception of how the numbers relate to the graph	Has no clear strategy	2	0	2
	Mentions formula or procedure with little understanding	3	1	4
Has a sense of how the numbers and/or values relate to the graph	Showed 3 and 12 on the graph but not 0.25	4	0	4
	Reconciled 0.25 with the 3 and 12	2	1	3

Two college students presented no clear strategy with their response. One said that slope is m in the equation $y = mx + b$. She replaced the m with $\frac{3}{12}$ and said that was a way to show the 3 and the 12 in the graph. Another student simply said he did not know where the 3 and the 12 were on the graph.

Four students (three college and one 7th grader) mentioned either a rise and run strategy or the slope formula with little understanding of how that may help them to find the 3 and the 12 on the graph. One college student wrote $\frac{y_2 - y_1}{x_2 - x_1}$ but was unable to use the formula to find the 3 and the 12. One college student did a rise of 3 and a run of 12, but instead of beginning somewhere on the graph, he started at the origin. Thus, his efforts did not help him to illustrate where the 3 and the 12 were on the graph.

Four college students were able to locate the point that corresponds to (14, 3). They each started with the point (2,0) in the graph and from that position moved up 3 and to the right 12. They did not, however, reconcile the 0.25 with the $\frac{3}{12}$.

Three other students (two college and one 7th grade) found the 3 and the 12 on the graph. They used the same methods of the four college students in the previous subcategory, but were also able to interpret 0.25 as being equivalent to $\frac{1}{4}$. They said that 0.25 would therefore be a rise of 1 and a run of 4. One student exhibited his answers by circling points on the graph (see Figure 7). None of the students, however, explained 0.25 as a rise of 0.25 for a run of 1.

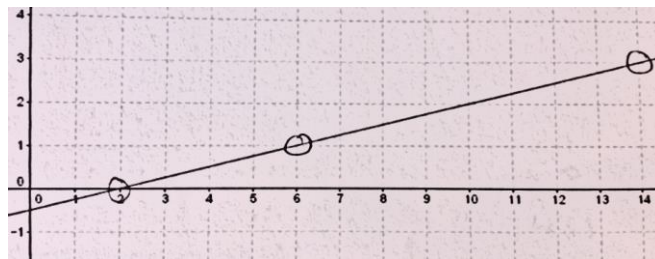


Figure 7: A Student Showing $0.25 = \frac{1}{4}$ and $\frac{3}{12}$ on the Graph

In summary, eleven students were asked to respond to Item 6. There were 13 coded responses. Six of the responses were not indicative of a conception the numbers relate to the graph. Seven responses indicate some level of a conception the numbers relate to the graph, however,

only three of those responses were indicative that students were able to reconcile the value of 0.25 with the value of the fraction $\frac{3}{12}$ by interpreting 0.25 as $\frac{1}{4}$, which is equivalent to $\frac{3}{12}$. None of the students interpreted 0.25 as $\frac{0.25}{1}$ or interpreted $\frac{3}{12}$ as being equal to $\frac{3 \div 12}{12 \div 12}$ which can be simplified to $\frac{0.25}{1}$.

4.1.7 Item 7

This item was included to elicit responses that required students to focus on the single value of the fractions and to determine whether students recognize the multiplicative relationship between the numerator and the denominator. Item 7 has two parts, 7a and 7b, in which there were two main categories identified that differentiate student responses that were indicative of attending to the multiplicative relationship between the numerator and denominator from those that were not.

Item 7a: *Approximate the following:* $\frac{220}{443} + \frac{31}{59}$.

There were 14 students who responded to this item and three subcategories were identified.

Table 11 shows the 17 coded responses.

Table 11: Coding for Item 7a

220/443 + 31/59		College	7th	All
Not attending to multiplicative relationship	Added the Ns and Ds (e.g. $220 + 31/443 + 59 = 251/502$)	3	4	7
	Tried finding a common denominator	4	3	7
220/443 + 31/59				
Attending to multiplicative relationship	Approximated as $1/2 + 1/2$	2	1	3

Seven students (three college and four 7th grade) added the numerators to the numerators

and the denominators to the denominators and five of them obtained 251/502. One 7th grader rounded 443 to 440, 31 to 30 and 59 to 60 then added the 220 to 30 and the 440 to 60 to obtain 250/500. Another 7th grader rounded the fractions similarly to obtain $200/400 + 30/60 = 230/460$.

Seven students (four college and three 7th grade) attempted to get a common denominator. Four merely mention the process of getting a common denominator, but were unable to demonstrate what to do with these fractions. Three of the students attempted to multiply numbers to get a common denominator. One of the 7th graders multiplied 59 by 7 to obtain 413. She also attempted to multiply 59 by 8 but obtained an incorrect result of 481. She stated that she was “trying to get as close as possible to 443,” but she seemed unclear what to do with these products.

Three students (two college and one 7th grader) recognized both fractions in the expression were near $\frac{1}{2}$ and said the sum was near 1. These responses were indicative of their awareness that the denominator is about twice of the numerator, which implies that they attended to the multiplicative relationship between the numerator and denominator.

Item 7b: *Approximate the following:* $\frac{71}{101} + \frac{41}{81}$.

Only six students were asked to respond to Item 7b, a follow up task. The categories and subcategories for this item are consistent with Item 7a. Table 12 shows the six coded responses.

Table 12: Coding for Item 7b

71/101 + 41/81		College	7th	All
Not attending to multiplicative relationship	Added the Ns and Ds (e.g. $71+41/101+81 = 112/182$)	0	1	1
	Tried finding a common denominator	3	1	4
71/100 + 41/81				
Attending to multiplicative relationship	Approximated as $0.7 + 0.5$	1	1	2

One 7th grader added the numerators and denominators after rounding the individual numbers like this: $70/100 + 40/80 = 110/180$. This strategy did not adhere to the values of the fractions.

Four students (three college and one 7th grader) tried to find a common denominator in order to add the two fractions. One of the three college students, approximated the two fractions in Item 7a but decided to find a common denominator for this item, probably because the relationships between the numerator and denominator is less obvious for this item. The 7th grader wrote $70/100 + 40/80 = 7/10 + 5/10 = 12/10$. Although this student found a correct approximation, she might be merely following a procedure without really attending to the meaning of the values of the fraction.

Two students (one college and one 7th grader) were able to determine the approximate sum as being about 1.2. The college student merely looked at the expression and said the sum was approximately 1.2; he must have noticed the values of the fractions as 0.7 and 0.5 respectively. The 7th grader, mentioned in the previous subcategory, took her sum of 12/10 and said that was 1.2; it is less apparent whether this student focused on the relationship between 12 and 10, or merely recognized 12/10 as being equal to 1.2.

In summary, students were asked to approximate two different sums. There were 19 instances where students did not pay attention to the multiplicative relationship between the numerator and denominator. Only five instances seemed to that students solved this problem by relating the numerator and denominator of each fraction.

4.2 Categorizing Responses in Terms of Understanding Fractions as Having Specific Values

Remember one of the goals of this study is to help identify levels of understanding of a

fraction; the highest level being understanding a fraction as having a single value that is related to the numerator and denominator. Based on the categories and subcategories presented in Section 4.1, I identified five progressive levels of understanding a fraction as having a specific value that is related to the numerator and the denominator. The five levels are:

Level 1 – Fraction as merely two independent numerals almost unrelated

Level 2 – Fraction as two independent numerals related as a part-whole concept

Level 3 – Procedural conversion of fraction without attending to value

Level 4 – Fraction as having a general sense of value of fraction with some connection to the two numbers

Level 5 – Fraction as a single identifiable value with connection to the two numbers

Students operating at Level 1 means that they are making no discernable connection to the numerator and denominator. The numerator and denominator are viewed as merely two independent numerals. The subcategories associated with Level 1 are: *Not realizing the parts must be equal size* (Item 1), *Incorrect conversion to decimal* (Item 1), *N is the number of whole and D is the number of tenths* (Item 2), *Incorrect Division* (Item 2), *Crossed out the two x's* (Item 3), *Same difference = Same size* (Item 4), *Big N or Big D = Big Fraction* (Item 4), *Focusing on the whole numbers* (Item 5), *Thinking of 2.5/9 without attending to the values of the fraction* (Item 5), *Has no clear strategy* (Item 6), *Mentions formula or procedure with little understanding* (Item 6), and *Added the Ns and Ds* (Item 7).

Students operating at Level 2 understand the connection between the numerator and denominator as a part-whole conception. They interpret the denominator as the whole and they interpret the numerator as the parts or “pieces” of the whole. The subcategories associated with

Level 2 are: *Realizing the parts must be equal* (Item 1), *Focus is on the N ($4/5$ is plotted at 4)* (Item 2), *Replacing x with a number but not attending to value* (Item 3), and *Just chose the smaller D without knowing why* (Item 4). There are no subcategories from items 5, 6 and 7 for this level.

Students operating at Level 3 can make conversions from fractions to decimals or equivalent fractions but they do not make connections to the single value associated with the fraction. The subcategories associated with Level 3 are: *Using division to convert to a decimal* (Item 1), *Used procedure without understanding* (Item 2), *Can be greater than 1* (Item 3), *Used procedure without understanding* (Item 4), *Using division to convert the fractions to decimals without attending to the value of the fractions* (Item 5), *Finding a common denominator without attending to the value of the fractions* (Item 5) and *Tried finding a common denominator* (Item 7).

Students operating at Level 4 have a general sense of value based on what they have been told about fractions; for example, they may know that all proper fractions are between zero and 1 and all improper fractions are larger than proper fractions. The subcategories associated with Level 4 are: *$3/5$ relates 3 to 5 multiplicatively* (Item 1), *Somewhere between 0 and 1* (Item 2), *4th tick mark with each tick mark being one-fifth* (Item 2), *$5/4$ is equivalent to $1\frac{1}{4}$ because each tick mark is one-fourth* (Item 2), *Increases as x increases* (Item 3), *All improper fractions are bigger than all proper fractions* (Item 4), *Finding a common denominator while attending to the value of the fractions* (Item 5), and *Showed 3 and 12 on the graph but not 0.25* (Item 6).

Students operating at Level 5 make discernable connections from the numerator and denominator to the single value of the fraction. They have a sense of the value even without having to resort to processes or procedures. The subcategories associated with Level 5 are: *$3/5 = 0.6$ is a value (e.g. between 0.5 and 1.0)* (Item 1), *Doing division - understanding 0.8 is nearly 1* (Item 2), *$5/4$ is equivalent to 1.25 and is slightly more than 1* (Item 2), *Never reaches 1* (Item 3), *Found an*

expression that is greater than 1 (Item 3), Changed both to decimals or percentages to compare (Item 4), Could correctly describe the relationship of N to D (Item 4), Thinking of 2.5/9 while attending to the value of the fractions (Item 5), Using division to convert the fractions to decimals while attending to the value of the fractions (Item 5), Reconciled 0.25 with the 3 and 12 (Item 6), Approximated as $1/2 + 1/2$ (Item 7) and Approximated as $0.7 + 0.5$ (Item 7).

Whereas Section 4.1 identifies categories and subcategories for each item based on student responses, this section assigns each student response a level. Each subcategory is associated with a particular level and student responses that are exclusively in that subcategory are simply assigned that level. For each student response that has been coded in two or more subcategories in Section 4.1, I reanalyzed the response to determine the level at which the student seemed to be operating while working on each item.

Table 13 presents the number of student responses for each item. Although a student may be operating at more than one level across the items, I assigned one dominant level that the student seemed to be operating at across all items (the mode), and this information is presented in the *overall* column. If a dominant level is not clear, then if the student had the understanding required to operate at the highest level, then that level was assigned (denoted by an asterisk), else the next highest level was considered and so forth.

Table 13: Levels Assigned to Each Student Response for Each Item

	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6	Item 7	Overall
College 1	N/A	1	1	1	1	N/A	N/A	1
College 2	1	2	N/A	1	1	1	N/A	1
College 3	N/A	1	3	1	1	1	1	1
College 4	1	2	4	3	1	4	4	4*
College 5	1	1	1	3	1	4	1	1
College 6	3	1	1	3	1	1	3	3*
College 7	4	4	1	1	1	4	N/A	4*
College 8	3	3	5	2	4	N/A	3	3
College 9	1	1	1	2	1	4	3	1
College 10	5	5	N/A	4	4	5	1	5
College 11	1	1	1	2	3	N/A	N/A	1
College 12	5	5	5	5	5	5	5	5
7 th Grader 1	2	2	4	4	3	5	3	3
7 th Grader 2	3	2	N/A	1	1	1	5	1
7 th Grader 3	1	3	N/A	3	3	N/A	1	3
7 th Grader 4	1	1	3	2	1	N/A	1	1
7 th Grader 5	3	4	5	3	5	N/A	N/A	4*
7 th Grader 6	4	4	3	2	1	N/A	1	4*
7 th Grader 7	1	1	1	1	1	N/A	1	1
7 th Grader 8	1	3	4	2	3	N/A	N/A	3*
Total Responses	18	20	16	20	20	11	14	20
Means	2.28	2.35	2.69	2.30	2.10	3.18	2.36	

For a comparison between the groups of students, I created Table 14 to show the number of students operating at each level. It is divided into three sub-tables: for all students, for college student, and for 7th graders.

Table 14: Number of Students at Each Level

	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6	Item 7	Overall
All Students	18	20	16	20	20	11	14	20
Level 1	9 (50%)	8 (40%)	7 (44%)	6 (30%)	12 (60%)	4 (36%)	7 (50%)	9 (45%)
Level 2	1 (6%)	4 (20%)	0	6 (30%)	0	0	0	0
Level 3	4 (22%)	3 (15%)	3 (19%)	5 (25%)	4 (20%)	0	4 (29%)	5 (25%)
Level 4	2 (11%)	3 (15%)	3 (19%)	2 (10%)	2 (10%)	4 (36%)	1 (7%)	4 (20%)
Level 5	2 (11%)	2 (10%)	3 (19%)	1 (5%)	2 (10%)	3 (27%)	2 (14%)	2 (10%)
College Students	10	12	10	12	12	9	8	12
Level 1	5 (50%)	6 (50%)	6 (60%)	4 (33%)	8 (67%)	3 (33%)	3 (38%)	6 (50%)
Level 2	0	2 (17%)	0	3 (25%)	0	0	0	0
Level 3	2 (20%)	1 (8%)	1 (10%)	3 (25%)	1 (8%)	0	3 (38%)	2 (17%)
Level 4	1 (10%)	1 (8%)	1 (10%)	1 (8%)	2 (17%)	4 (44%)	1 (12%)	2 (17%)
Level 5	2 (20%)	2 (17%)	2 (20%)	1 (8%)	1 (8%)	2 (22%)	1 (12%)	2 (17%)
7th Graders	8	8	6	8	8	2	6	8
Level 1	4 (50%)	2 (25%)	1 (17%)	2 (25%)	4 (50%)	1 (50%)	4 (66%)	3 (38%)
Level 2	1 (12%)	2 (25%)	0	3 (38%)	0	0	0	0
Level 3	2 (25%)	2 (25%)	2 (33%)	2 (25%)	3 (38%)	0	1 (17%)	3 (38%)
Level 4	1 (12%)	2 (25%)	2 (33%)	1 (12%)	0	0	0	2 (25%)
Level 5	0	0	1 (17%)	0	1 (12%)	1 (50%)	1 (17%)	0

4.3 Discussion

In this section, I will use the results to answer the three research questions:

1. How do students respond to the items that were designed to uncover whether students interpret a fraction as having a specific value or as having two separate numbers?
2. What are the progressive levels of understanding of a fraction as having a specific value that is related to the numerator and the denominator?
3. How do the college students compare to the 7th graders in terms of the progressive levels of understanding?

4.3.1 Student Responses to Items

In Section 4.1, I examined the student responses for each interview item and identified a total of 43 subcategories for the items based on the responses. Some of these subcategories revealed students' misconceptions of fractions and some showed how students attended to the value of fractions and related it to the numerator and denominator. In this section, I present what I found interesting or noteworthy and report the struggles students had in answering some of the items.

For instance, the first item is so open ended that it may have caused students to become unsure or even nervous. Students seemed to be more at ease with questions that appear to have correct and incorrect answers to those that asks students to explain what they are thinking. Many of the students who struggled with this first item seemed to become more comfortable with tasks that included prompts such as the two items with graphs.

For some students, their difficulty with conceptualizing fraction meaningfully is related to their difficulty with whole-number division that results in a remainder. For example, a student did not think that she could divide 3 by 5 because for her the divisor must be smaller than the dividend. Another student thought that 3 divided by 5 would be a huge decimal, probably because it is a recurring decimal. It appears that understanding the meaning of division is necessary for students to understand fractions.

Neither one of the groups in this study had a strong foundation in algebra, thus it is not surprising that three student responses from Item 3 had to be discarded. However, it is surprising that this item produced some of the richer conversations about fractions. In fact, one student even spent a third of his time in the interview talking about this item.

Similarly, some of the struggles the students had with Item 6 had to do with misconceptions

about the algebra. It is clear that many of the students attended more to the formulas associated with slope rather than really attending to the numbers presented in this problem. On the other hand, the students who did attend to the numbers on Item 6 were able to make good connections to the values as well.

It is unfortunate that more students were not asked to participate in Items 7a and 7b. The students who were asked to do these two items gave insightful responses. This task truly reveals whether or not students attend to the multiplicative relationship between the numerator and the denominator. Interestingly, of the three students who correctly interpreted the first set of fractions as $1/2 + 1/2$, only two of them correctly interpreted the second as $0.7 + 0.5$. This indicates how doubling and halving may be easier for students to determine than it is to determine the sevenths in Item 7b where the denominator is not a multiple of the numerator

4.3.2 Progressive Levels of Understanding Fractions

In Section 4.2, I introduced the five progressive levels of understanding a fraction as having a single value that is related to the numerator and the denominator. Table 13 gives some insights on how students responded to certain items. For ease of comparison, I calculated the means of the levels for each item (see Table 13).

Item 1 and Item 5 have the two lowest averages of the seven items. It is not surprising that Item 1, *tell me all that you know about the fraction $3/5$* , had some of the lower leveled responses because it was the first task and the least directed item in the instrument. The low scores on Item 5, *find a number that is between $2/9$ and $3/9$* , are also not surprising. It asks students to find a number between two consecutive fractions, a typically daunting task mentioned in other studies.

For example, a study conducted by Pantziara & Philippou (2012) determined that finding a number between two consecutive fractions is one of the most difficult activities related to fractions.

Item 3 and Item 6 have the two highest averages of the seven items. Item 3, *suppose x is a natural number, tell me all you know about $\frac{x}{x+2}$* , only had 16 student responses that were counted which may have skewed the results. Item 3 was a task that prompted some of the longer conversations because some students wanted to replace the x with several different numbers in order to obtain a pattern. As evident in the results, this desire to find a pattern might be a cause for some students to think about what was occurring and consequently led to some of the higher leveled responses for this item. Recall that Item 6 asks students to locate the 3 and the 12 on the graph and the 0.25 on the graph. Also, recall that only 11 student responses were counted for this item. Thus, the results may be skewed; however, this task has the most visual prompts which may account for some of the higher ratings.

It is also noteworthy that Level 2 has fewer associated subcategories than do the other levels. In fact, Level 2 does not have a subcategory associated with the last three items, perhaps because these types of tasks do not lend themselves to the part-whole conception and perhaps also because these tasks were at the end of the interview. Although, there *is* a category associated with Level 2 for Item 3, *replacing x with a number but not attending to value*, the students who were associated with this category were also associated with another more dominate category. Thus, there were no Level 2 responses for Item 3 as well. This may be another reason for Item 6 and Item 3 having the highest averages as mentioned earlier.

One may argue that Level 2 is more sophisticated than Level 3 because part-whole interpretation of fraction is conceptually correct whereas Level 3 is procedural. For the purpose of this study, it makes sense to have Level 3 as more advanced than Level 2 in terms of progression

to understanding a fraction as having a single value. Level 3 responses are indicative of students not making connection to the single value but they have procedurally found the representation of that single value, while Level 2 responses are still merely attending to the two numbers.

It is interesting to observe instances of additive reasoning versus multiplicative reasoning during some of the interviews. When students crossed out the x 's in Item 3 and the part-whole misconception displayed in Figure 4 are examples of additive reasoning. The "same difference = same size" subcategory in Item 4 shows additive thinking instead of multiplicative thinking. When students think multiplicatively, they see how many times larger or smaller the numerator was compared to the denominator. Thinking of $1/n$ as "one out of n parts" is to think of fractions additively. A sophisticated student can say "one out of n " and mean it additively at the moment of saying it (like, one of those six cars is grey), and yet flip it effortlessly to "the number of cars over there is six times as large as the number of grey cars" (Thompson & Saldanha, 2003, p.25). It is not that additive thinking is bad. It is just that students who are able to make the transition from additive thinking to multiplicative thinking have an easier time seeing the specific value of the fraction. While the subcategory "same difference = same size" found in 4a is an example of additive thinking, that same subcategory in Item 4c may introduce another misconception.

Students who said that the fractions in Item 4c were equal may have used additive reasoning but if the students truly subtracted, they should have gotten 10 and -10 as differences. These students seemed to focus only on the bigger number minus the smaller. The tendency to subtract the smaller number from the larger number seems to be related to the fact that some students conceptualized a fraction as being division of the bigger number by the smaller number which happened with Items 1 and 2. It may be noteworthy that students' misconceptions about addition, subtraction, multiplication and division may be a source of misunderstandings such as

“subtractions always makes smaller”, “division always makes smaller” and “you can’t divide a smaller number by a bigger number”. Graeber, D., Tirosh, D. & Glover, R. (1989) describe these types of misunderstandings as “implicit primitive behavior models” which they say that some teachers inadvertently contribute to perpetuating (p. 95). Clearly, the students in this study had these as well as other misconceptions of basic operations related and unrelated to fractions.

Looking at Table 13, it is easy to see that Level 1 is the most common across the board. Nevertheless, generally speaking, it shows that these students attend to the two numbers more often than the single value. This may be due to the specific group of students who volunteered for this study.

Students who rated more often on Level 5 have the ability to see a fraction as having a single value that is connected to the numerator and denominator. Only two (both college students) of the twenty students were found to be operating on Level 5 overall.

4.3.3 College Students Versus 7th Graders

Table 14 helped me to compare the college students and the 7th graders in terms of how each group interprets fractions as having a single value. The “overall” column shows that 50% of the college students are in Level 1 whereas the 7th graders have 38% in that same level. Both groups have 0% at Level 2 in the “overall” column. College students have a combined 34% in Levels 3 and 4 as compared to 63% for the 7th graders. The 7th graders do not have any “overall” ratings at Level 5, while the college students have 17% in that level.

The two groups of students seemed to have fairly similar lower-leveled results. Remember that the sample of students may not be reflective of the true population because I accepted all

volunteers as participants. Four items (Item 2, Item 3, Item 5 and Item 7) do appear to have results that favor one of the groups over the other.

The 7th graders seem to have better results than the college students on Item 2, Item 3 and Item 5. On Item 2, 50% of the 7th graders are rated above Level 2 while only 33% of college students are rated above Level 2. The 7th graders also outscored the college students on Item 3. They have 83% of their ratings above Level 2 while the college students only have 40% above Level 2. On Item 5, The 7th graders have 50% of their ratings above Level 2 while the college students only have 33% of their ratings above Level 2.

The only item that the college students significantly outscored the 7th graders is on Item 7. The college students had 62% of their ratings above Level 2 while the 7th graders only had 34% of their ratings above Level 2.

Overall it appears that the 7th graders rated slightly better than the college students with 63% of their ratings above Level 2 as compared to 50% for the college students. The 7th graders may have had more recent instruction concerning fractions than the college students. However, it may simply be that the college students, who were enrolled in developmental math courses, were weaker students in general.

It is interesting that the two students who had an overall rating for Level 5 were both college students. One would think that the college students would attend to the value of a fraction more than the 7th graders, however these two students happened to be the exception not the rule for the group of college students who participated in this study.

CHAPTER 5: CONCLUSION

The aim of this study was to investigate how students conceptualize fractions, specifically as it pertains to the specific value of the fraction. After analyzing the results of the interviews, clear levels of progression were identified which show the different levels of sophistication in terms of seeing fraction as a single value. More specifically, it shows a progression from a conception that a fraction is made up of two seemingly unrelated numerals to a single value conception of a fraction where the numerator and denominator are multiplicatively related. These five levels of progression are intended to contribute to future research:

Level 1 – Fraction as merely two independent numerals almost unrelated

Level 2 – Fraction as two independent numerals related as a part-whole concept

Level 3 – Procedural conversion of fraction without attending to value

Level 4 – Fraction as having a general sense of value of fraction with some connection to the two numbers

Level 5 – Fraction as a single identifiable value with connection to the two numbers

The results of this study helped me to uncover some important recommendations going forward. First, and perhaps most importantly, I recommend for future study group to be made up of a more diverse study group. The students in this study, especially the developmental math students, may have only contributed to rich responses for the lower levels but not the higher levels. In fact, certain students rated as Level 1 for every item or nearly every item. I recommend that further research should include students who have a good mastery of fractions and are able to articulate their understanding of a fraction as having a value that is connected to the multiplicative relationship between the numerator and denominator.

As noted earlier, Level 2 subcategories were more scarce. Although the goal of this study is mainly to obtain responses that are indicative of the single value concept, it may be valuable to capture various kinds of responses as a contrast. This may mean that additional tasks that may capture some Level 2 responses may need to be added to the instrument.

Furthermore, I recommend that Item 2 be altered to include a fraction other than $\frac{4}{5}$. It may be beneficial to use a fraction that cannot so easily be converted to an equivalent fraction with a denominator of 10 or 100. This may cause students to attend to value rather than simply having them find a common denominator.

It is important that the teaching community recognize that merely teaching processes and procedures related to fractions is not enough for students to conceptualize the specific values associated with fractions. This study highlights problems that many students face while working with fractions. It shows that typically students, even if directly asked to determine the value of a fraction, rely heavily on the processes and procedures that are usually unrelated to the specific value of the fraction rather than intuitively recognizing the single value of a fraction. If we make an effort to point students to the multiplicative nature of fractions from the very beginning of fraction instruction, then students may conceptualize fractions more clearly and may be more successful even in high school and college math courses.

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APPENDIX

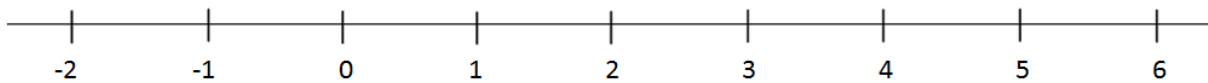
INTERVIEW ITEMS 1 THROUGH 7

ITEM 1

List as many things as you can about the fraction $\frac{3}{5}$.

ITEM 2

2a: Mark $\frac{4}{5}$ on the number line below:



2b: Mark $\frac{5}{4}$ on the number line.

ITEM 3

Suppose x is a natural number, tell me all you know about $\frac{x}{x+2}$.

ITEM 4

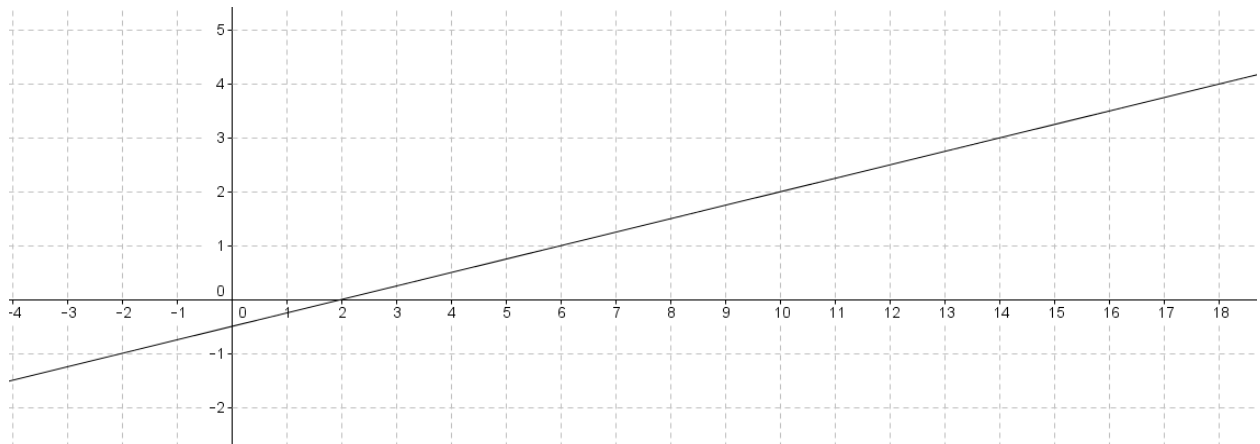
4a: Circle the larger fraction: $\frac{6}{8}$ or $\frac{8}{10}$

4b: Circle the larger fraction: $\frac{1}{4}$ or $\frac{21}{101}$

4c: Circle the larger fraction: $\frac{973}{983}$ or $\frac{19}{9}$

ITEM 5

Find a number that is between $\frac{2}{9}$ and $\frac{3}{9}$.

ITEM 6

Look at the graph above. John says that the slope is $\frac{3}{12}$. Can you help me see the 3 and the 12 in this graph? Mary says the slope is 0.25. Can you help me see the 0.25 in the graph? Are $\frac{3}{12}$ and 0.25 the same thing?

ITEM 7

7a: Approximate the following: $\frac{220}{443} + \frac{31}{59}$

7b: Approximate the following: $\frac{71}{101} + \frac{41}{81}$

CURRICULUM VITA

William “Bill” Fanning was born in Meridian, Mississippi in February 1969 to William and Annette Fanning. He graduated from Hillwood High School, Nashville, Tennessee, in May 1987 and began attending Belmont University, Nashville, Tennessee, in the Fall 1987 semester. Bill Fanning completed a Bachelor of Science degree in Mathematics with a heavy emphasis on secondary education in December 1991. In March of 1992 he began a 14-year career with Sylvan Learning Center in Nashville, Tennessee. Bill began teaching high school Algebra and Geometry at Columbia Central High, in the Fall of 2006 and Developmental College Algebra as an adjunct professor at Columbia State Community College in the Spring 2008. After teaching mathematics for three years, Bill joined the military to help him to pursue a graduate level degree. While actively serving in the United States Army at Fort Bliss, Texas, Bill was admitted to The University of Texas at El Paso’s Master of Arts in Teaching (MAT) program in the Spring semester of 2011. After nearly three years of service, he was honorably discharged from the United States Army in May 2014. In the Fall semester of 2014, he began teaching developmental Algebra as an adjunct professor at the University of Texas at El Paso. He also taught General Mathematics as an adjunct professor at Western Technical College from June 2014 to July 2015. Bill Fanning graduated in May of 2016 with a Master of Arts in Teaching (MAT) degree with 18 hours of graduate level mathematics. Shortly thereafter, he was offered a teaching position with Parry McCluer High School in Buena Vista, Virginia, where he will begin a mathematics teaching career in the Fall 2016 semester.

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