1-2014

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Comments:
Technical Report: UTEP-CS-14-08
Published in *Proceedings of the Seventh International Workshop on Constraints Programming and Decision Making CoProd'2014*, Wuerzburg, Germany, September 21, 2014.

From Global to Local Constraints: A Constructive Version of Bloch's Principle

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Abstract. Generalizing several results from complex analysis, A. Bloch formulated an informal principle – that for every global implication there is a stronger local implication. This principle has been formalized for complex analysis, but is has been successfully used in other areas as well. In this paper, we propose a new formalization of Bloch's Principle, and we show that in general, the corresponding localized version can be obtained algorithmically.

1 Bloch's Principle: Formulation of the Problem

Bloch's Principle: a brief history (see [4] for details)**.** Liouville's Theorem states that every analytical function $f(z)$ which is bounded on a whole complex plane and for which $f(0) = 0$ is equal to 0; see, e.g., [3]. This theorem requires that the constraint $|f(z)| \le C$ be satisfied *globally*, i.e., for all z. What if this constraint is only satisfied *locally*, i.e., for all z from a bounded set? Such a "localization" of Liouville's theorem was indeed proven by H. A. Schwarz: if a function $f(z)$ for which $f(0) = 0$ is analytical for all z from a disk

$$B_R(0) \stackrel{\rm def}{=} \{z : |z| < R\}$$

and $|f(z)| \le C$ for all $z \in B_R(0)$, then for all such values z, we get $|f(z)| \le \dfrac{C}{R} \cdot |z|$. When the size R increases, the bound tends to 0; so for $R \to \infty$, we get Liouville's Theorem.

Several similar localizations of global results are known. In 1926, A. Bloch, formulated a general (informal) *Bloch's Principle*: that for every global result, there is a local version from which this global result follows [2]. In complex analysis, this principle was formalized; however, there are many interesting results of the use of Bloch's Principle in other areas of mathematics.

Problem. Can we formalize Bloch's Principle in a context which is more general than complex analysis? If yes, and if the appropriate the localization always exists, can we find it algorithmically?

What we do in this paper. In this paper, we provide positive answers to both questions.

Comment. Of course, due to the informal character of Bloch's principle, no answer is final – it is always possible that our result (or a similar result) holds in a more general context.

2 Bloch's Principle: General Formalization

Analysis of the problem. In general terms, the Liouville's theorem has the form

$$\forall f \in \mathcal{F}\,(\forall x\,(f(x) \in A(x)) \Rightarrow \forall x\,(f(x) \in B(x))), \tag{1}$$

where $\mathcal{F}$ is the class of all analytical functions for which $f(0) = 0$, x goes over all complex numbers, $A(x) = \{x : |x| \le C\}$ is the set of all the valued bounded by the given constant C, and $B(x) = \{0\}$.

The implication (1) says that if the constraint $f(x) \in A(x)$ is *exactly* satisfied for all possible values x, then the conclusion holds. What we want to prove is that when the constraint is "approximately" satisfied – i.e., if it satisfied with some accuracy $\delta > 0$ for all the values x which are at a distance r form 0 – then the conclusion is also approximately satisfied, with some accuracy $\varepsilon > 0$ and for all values at a distance R from 0. We also want to make sure that when $\delta \to 0$ and $r \to \infty$, then $\varepsilon \to 0$ and $R \to \infty$. In other words, we want to prove that for every $\varepsilon > 0$ and $R > 0$, there exist $\delta > 0$ and $r > 0$ for which, if the condition is satisfied with accuracy δ for all x which are r-close to 0, then the conclusion is satisfied with accuracy ε for all x which are R-close to 0.

A natural way to describe the fact that $f(x)$ is "approximately" in the set $A(x)$ (or in the set $B(x)$) is to say that $f(x)$ is close to the set $A(x)$ in the sense of the usual distance $d(z, S) \stackrel{\text{def}}{=} \inf\{d(z, s) : s \in S\}$. In the above case, the sets $A(x)$ and $B(x)$ are compact, so $d(z, A(x)) = 0$ if and only if $z \in A(x)$. Thus, the global result (1) can be reformulated in the equivalent form

$$\forall f\,(\forall x\, d(f(x), A(x)) = 0 \Rightarrow \forall x\, d(f(x), B(x)) = 0)). \tag{2}$$

and the desired localized result has the form

$$\forall \varepsilon > 0\, \forall R > 0\, \exists \delta > 0\, \exists r > 0$$

$$\forall f\,((\forall x\,(d(x, x_0) \le r \Rightarrow d(f(x), A(x)) \le \delta)) \Rightarrow$$
$$(\forall x\,(d(x, x_0) \le R \Rightarrow d(f(x), B(x)) \le \varepsilon))). \tag{3}$$

It is worth mentioning that in the case of Liouville's Theorem (and in several similar results mentioned in [4]), not only all the sets $A(x)$ and $B(x)$ compact, but they also continuously depend on x – in the sense of the Hausdorff metric $d_H(A, B) \stackrel{\text{def}}{=} \max\left(\max\limits_{a \in A} d(a, B), \max\limits_{b \in B} d(b, A)\right)$.

The class $\mathcal{F}$ is also compact in some reasonable sense: indeed, for ever bounded set D, the set of all these functions limited to D is compact in the usual metric $d_D(f, g) = \max\limits_{x \in D} d(f(x), g(x))$. Indeed, for an analytical function $f(z)$, its value $f(z)$ can be described by a Cauchy integral over a surrounding curve γ: $f(z) = \displaystyle\int_\gamma \frac{f(t)}{z - t}\, dt$. Differentiation of this formula enables us to get a similar formula for the derivative $f'(z)$. Thus, when the analytical function is bounded, its derivative is also bounded. Due to Ascoli-Arzela theorem, this implies that the corresponding class of functions is compact – when limited to each bounded domain.

It is also important to notice that the notion of an analytical function is *locally defined*, in the sense that if a function $f(x)$ coincides with some analytical function in every neighborhood, then it is analytical itself.

Thus, we arrive at the following natural formalization of Bloch's Principle.

Definition 1. *Let us call a metric space* bounded-compact *if every closed bounded set in this space is compact.*

Comment. In particular, this implies that every closed ball

$$B_r(x_0) \stackrel{\text{def}}{=} \{x : d(x, x_0) \le r\}$$

is compact. Vice versa, if for some point x_0, every closed ball with a center at x_0 is compact then every closed bounded set is compact too: indeed, very bounded set is contained in some ball $B_r(x_0)$, and a closed subset of a compact set is also compact.

Definition 2. *Let $\mathcal{F}$ be a class of functions from a bounded-compact metric space X to a bounded-compact metric space Y. We say that the class $\mathcal{F}$ is* bounded-compact *if for every compact set $K \subset X$, this class is compact in the metric $d_K(f, g) \stackrel{\text{def}}{=} \sup\limits_{x \in K} d(f(x), g(x))$.*

Definition 3. *Let $\mathcal{F}$ be a class of a functions from X to Y, and let x_0 be a point in x_0. We say that a function $f : X \to Y$* locally belongs *to the class $\mathcal{F}$ if for every n, there exists a function $f_n \in \mathcal{F}$ which coincides with f on $B_n(x_0)$.*

Comment. This definition uses the point x_0, but one can easily check that the resulting notion does not depend on x_0.

Definition 4. *We say that a bounded-compact class of functions $\mathcal{F}$ is* locally defined *if it contains all the functions that locally belong to this class.*

Definition 5. *Let $\mathcal{F}$ be a bounded-compact locally defined class of functions. By an $\mathcal{F}$-*constraint *A, we mean a (Hausdorff)-continuous function that map each point $x \in X$ into a compact set $A(x) \subseteq Y$.*

Definition 6. *Let $\mathcal{F}$ be a bounded-compact locally defined class of functions, and let A and B be $\mathcal{F}$-constraints.*

- *We say that the constraint A* globally implies *the constraint B if for every function $f \in \mathcal{F}$, the condition $\forall x\,(f(x) \in A(x))$ implies $\forall x\,(f(x) \in B(x))$.*
- *We say that the constraint A* locally implies *the constraint B for ε, R, δ, and r if for every function $f(x)$ for which $d(f(x), A(x)) \le \delta$ for all x with $d(x, x_0) \le r$, we have $d(f(x), B(x)) \le \varepsilon$ for all x with $d(x, x_0) \le R$.*
- *We say that the constraint A* locally implies *the constraint B if for every $\varepsilon > 0$ and for every $R > 0$, there exist real numbers $\delta > 0$ and $r > 0$ such that A locally implies B for ε, R, δ, and r.*

Comment. The definition of local implication uses a point x_0, but one can easily see that the corresponding property does not change if we replace this point with any other point from the metric spaces X.

Proposition 1. *Let $\mathcal{F}$ be a bounded-compact locally defined class of functions, and let A and B are $\mathcal{F}$-constraints. Then, if A globally implies B, then A locally implies B.*

Proof. We will prove the result by contradiction. Let us assume that A does not locally imply B. This means that there exist $\varepsilon > 0$ and $R > 0$ such that for every n, there is a function $f_n \in \mathcal{F}$ for which $\max\limits_{x \in B_n(x_0)} d(f_n(x), A(x)) \leq 1/n$ but $d(f_n(x_n), B(x_n)) > \varepsilon$ for some $x_n \in B_{x_0}(R)$. Since the sequence x_n is contained in a compact set $B_R(x_0)$, it has a subsequence which converges to some limit ℓ. Without losing generality, we can assume that $x_n \to \ell$.

Since $\mathcal{F}$ is compact relative to each metric $d_{B_k(x_0)}$, from the sequence f_n, we can extract a subsequence $n(1,i)$ convergent for $k = 1$; from this subsequence, we can extract a subsequence $n(2,i)$ which is convergent for $k = 2$, etc. The diagonal subsequence $f_{n(i,i)}$ then converges for all k. This convergence is for all x, no matter how far from x_0 we are, so we can defining a point-wise limit function $f(x)$. On each ball $B_k(x_0)$, this limit coincides with the corresponding limit from $\mathcal{F}$ limited to this ball. Thus, the limit function $f(x)$ locally belongs to $\mathcal{F}$; since the class $\mathcal{F}$ is locally defined, this means that $f \in \mathcal{F}$.

For the limit function f, for every x, the condition $d(f_n(x), A(x)) \leq 1/n$ in the limit tends to $d(f(x), A(x)) = 0$. Since A globally implies B, we conclude that we have $d(f(x), B(x)) = 0$ for all x, in particular, that we have $d(f(\ell), B(\ell)) = 0$. However, from $d(f_n(x_n), B(x_n)) > \varepsilon$, in the limit $x_n \to \ell$, we get $d(f(\ell), B(\ell)) \geq \varepsilon > 0$. This contradictions shows that our assumption is wrong, and A does locally imply B. The proposition is proven.

3 Bloch's Principle: A Constructive Version

Towards an algorithmic version. In this paper, we will use the usual definitions of computable numbers, functions, compact spaces, etc.; see, e.g., [1, 5].

Proposition 2. *If spaces X and Y are computable and computably bounded-compact, and if A and B are computable functions for which A globally implies B, then there exists an algorithm that, given rational numbers $\varepsilon > 0$ and $R > 0$, produces computable numbers $\delta > 0$ and $r > 0$ for which A locally implies B for ε, R, δ, and r.*

Proof. From the proof of Proposition 1, we can conclude that that for $\varepsilon_0 = \varepsilon/3$ and for $R_0 = R + 1$, there exists an integer $n = n_0$ for which $r = n$ and $\delta = 1/n$ satisfy the desired property. Let us show how to algorithmically find this n. For that, we will repeat the below computations for $n = 1, 2, \ldots$ until we find the value n for which the desired condition is satisfied.

In these computations, we will use the fact that there are algorithms for computing the maximum and minimum of a computable function over a computable compact. We will also use the fact that for a computable function $F(x)$ on a computable compact set K, for every two computable numbers $z^- < z^+$ within the range of $F(x)$ on K, we can compute an intermediate value $z \in (z^-, z^+)$ for which the set $\{x : F(x) \le z\}$ is a computable compact.

Before we go through $n = 1, 2, \ldots$, we use the intermediate-value algorithm to compute a value $R' \in (R, R+1)$ for which the ball $B_{R'}(x_0)$ is computably compact.

Then, for each n, we compute a value $r_n \in (n-1, n)$ for which the closed ball $B_{r_n}(x_0)$ is a computable compact. Since this ball is a computable compact, the value $v(f) \stackrel{\text{def}}{=} \max\limits_{x \in B_{r_n}(x_0)} d(f(x), A(x))$ is also computable – and is, therefore, a computable function of $f \in \mathcal{F}' \stackrel{\text{def}}{=} \mathcal{F}_{|B_{x_0}(R)}$.

The restriction $\mathcal{F}'$ is a computable compact. Thus, by the same intermediate-value result, we can compute a value $\delta_n \in (1/n, 1/(n-1))$ for which the set $S \stackrel{\text{def}}{=} \{f : v(f) \le \delta_n\}$ is a computable compact. We can therefore compute the maximum M of a computable function $d(f(x), B(x))$ over all $x \in B_{R'}(x_0)$ and all $f \in S$ with any given accuracy. Let us compute it with accuracy $\varepsilon/3$. If the resulting estimate $\widetilde{M}$ is $\le (2/3) \cdot \varepsilon$, we stop.

Let us show that if we stop, then we get the desired n. Indeed, in this case, if for some f, we have $d(f(x), A(x)) \le 1/n < \delta_n$ for all $x \in B_n(x_0)$, then (since $r_n < n$) this inequality is also true for all $x \in B_{r_n}(x_0)$, hence $v(f) < \delta_n$. Every $x \in B_R(x_0)$ belongs to $B_{R'}(x_0)$ and thus, for this x, we have $d(f(x), B(x)) \le M$. Since $M \le \widetilde{M} + \varepsilon/3$ and $\widetilde{M} \le (2/3) \cdot \varepsilon$, we conclude that $d(f(x), B(x)) \le \varepsilon$.

Let us now show that the above algorithm will stop for $n = n_0 + 1$. By definition of n_0, if $x \in B_{n_0}(x_0)$ and $d(f(x), A(x)) \le 1/n_0$, then $d(f(x), B(x)) \le \varepsilon/3$ for all $x \in B_{R_0}(x_0)$. Here, $R' < R + 1 = R_0$, so $x \in B_{R'}(x_0)$ implies that $x \in B_{R_0}(x_0)$. Similarly, since $r_n > n - 1 = n_0$, we conclude that $\max\limits_{x \in B_{n_0}(x_0)} d(f(x), A(x)) \le v(f) = \max\limits_{x \in B_{r_n}(x_0)} d(f(x), A(x))$ and thus, $v(f) \le \delta_n < 1/n_0$ implies that $\max\limits_{x \in B_{n_0}(x_0)} d(f(x), A(x)) < \dfrac{1}{n_0}$. Thus, indeed, for all such x and f, we have $d(f(x), B(x)) \le \varepsilon/3$; hence, the largest value M is $\le \varepsilon/3$, so $\widetilde{M} \le (2/3) \cdot \varepsilon$, and the algorithm will stop. The proposition is proven.

Acknowledgments. This work was supported in part by the National Science Foundation grants HRD-0734825, HRD-124212, and DUE-0926721.

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