2016-01-01

Complex Gleason Measures and The Nemytsky Operator

Miguel Angel Valles
University of Texas at El Paso, miglvalles@gmail.com

COMPLEX GLEASON MEASURES AND THE NEMYTSKY OPERATOR

MIGUEL A VALLES

Master's Program in Mathematical Sciences

APPROVED:

Maria Cristina Mariani, Ph.D., Chair

Joe Guthrie, Ph.D.

Hector Gonzalez-Huizar, Ph.D.

Charles Ambler, Ph.D.

Dean of the Graduate School

to my

MOTHER and FATHER

with love

COMPLEX GLEASON MEASURES AND THE NEMYTSKY OPERATOR

by

MIGUEL A VALLES, B.S.

THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at El Paso

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

Master's Program in Mathematical Sciences

THE UNIVERSITY OF TEXAS AT EL PASO

August 2016

Acknowledgements

It is in this precise moment when a Master's in English or Poetry seems like a very good idea, and it is for not other reason but that I lack the words and creativity to express my feelings of gratitude towards my advisor Dr. Maria Cristina Mariani, Chair of the Mathematics Department at the University of Texas at El Paso. For her countless hours of dedication, constant guidance and specially for introducing myself to the work that serves as a basis for this thesis, I thank her eternally.

I also wish to thank the other members of my committee, Dr. Joe Guthrie of the Mathematics Department and Dr. Hector Gonzales-Huizar of the Geology Department, both at The University of Texas at El Paso. Their availability, and comments were valuable to the completion of this work. I would also like to thank the professors and staff of the Mathematics Department who provided me with the necesasry tools to achieve my educational goals. I extend a special thank to Maria Salayandia, her dedication and eagerness to help, definitely made my journey enjoyable and pleasant.

Finally, my parents, I have no more to say but I love you, hopefully someday I will be able to pay them for all the sacrifices they have done.

Abstract

This thesis is devoted to generalize previous results on Gleason measures to complex Gleason measures, and to develop a functional calculus for complex measures in relation to the Nemytsky operator. Furthermore we present the interpretation of our results in the field of quantum mechanics, some concrete examples and further extensions of several theorems.

Contents

Chapter 1

Introduction

1.1 Mathematical Foundations of Quantum Mechanics

By the end of the 19th century, Physics had advanced in such a way that scientists were able to confront complex problems throughout vast different fields, achieve establishment of the thermodynamics theory among others, and the community agreed on the conservation laws of momentum and energy. All of the different concepts and theories were so deep that it was widely accepted that all of the physical world could be generally explained using such ideas and the only task that was left for physicists was to improve the precision of methods and measurements. A few years after, in the early 1900's, serious doubts questioning the validity of classical theories started to arise, undermining the view that physics was complete. Deficiencies in Maxwell's theories were noticed among different experiments and failure to explain under specific conditions some physical phenomena and theories that when taken to the limit generated paradoxes. These imperfections were to remain and new ideas had to be developed.

It was until the decade of the 1920 when the work of Schrödinger, Heisenberg, Dirac and others developed a new set of ideas; Quantum Mechanics, that dealt with the inadequacies of the previous theories remarkably well, to the point that Quantum Mechanics is still taught the same way this prodigious men formulated it. Nevertheless a complicated task remained; to bring the bizarre context in which the postulates of Quantum Mechanics flourished into rigorous mathematical formulations. The first physicist to address this concern, was John Von Neumann he was drawn to the task by a seminar on foundations of Quantum Mechanics conducted by Hilbert in 1926. The work published by Von Neumann

and his broad view in his "Mathematical Foundations of Quantum Mechanics" ([8]) book stand out as the first to provide a completely rigorous analysis of Quantum Mechanics. Roughly speaking, his use of abstract mathematical concepts like Hilbert spaces and operators to describe physical observables as eigenvalues of such operators greatly influenced both Quantum Mechanics and Functional Analysis.

1.2 Hidden Variables

One of the prominent mathematicians that inquired in the relatively new field of the Mathematical Foundations of Quantum Mechanics was an American mathematician from Harvard University Mathematics Department, George Mackey. Conducting investigations on application problems in Quantum Mechanics, Mackey proposed the following problem: Describe the set of all states on the quantum logic $L(H)$ of linear operators on a separable real or complex Hilbert space H. It is well known that Quantum Mechanics only permits to calculate probabilities, and at the time is was a question of debate whether the reason behind such bizarre behavior was because of the state of knowledge being incomplete due to "hidden variables" of which nothing was known or because the nature obeys Quantum Mechanics. Von Neumann showed that it is not the incompleteness of knowledge but rather something intrinsic to nature itself. A proof of Mackey's conjecture would complete his assertion and argue against the existence of hidden variables. Although Richard Kadison managed to proof the conjecture falsely for two-dimensional Hilbert spaces, one of Mackey's doctoral students, Andrew M. Gleason proved it for higher dimensions, his statement is now known as Gleason's Theorem.

1.3 Gleason's Theorem

Definition 1.1. Recall that a state, also called measure, on $L(H)$ is a mapping $m : L(H) \to [0, 1]$ such that:

$$m(H) = 1,$$

$$m\left(\bigcup_{i=1}^{\infty} M_i\right) = \sum_{i=1}^{\infty} m\left(M_i\right),$$

where $\bigcup_{i=1}^{\infty} M_i$ is the union of mutually orthogonal subspaces $\{M_t : t \in T\}$ of $L(H)$. Furthermore if x is a unit vector in H, then we define $m_x : L(H) \to [0,1]$ via:

$$m_x(M) = \|P_M x\|^2, \quad M \in L(H)$$

where P_M is the orthoprojector from H onto M, m_x is a state on $L(H)$ for any H. Moreover, given a positive Hermitian operator T on H, with $tr(T) = 1$, then:

$$m_T(M) = tr\left(TP_M\right), \quad M \in L(H) \tag{1.3.1}$$

defines a state on $L(H)$.

It is not difficult to see that class of the states on the quantum logic $L(H)$ defined by (1.3.1) is a broad class, however, in the case of H beign a two-dimensional space, as stated before, it can be shown that (1.3.1) does not determine all states.

Example 1.2. This example was taken from ([5]) to illustrate the latter. Let H_2 be a two-dimensional, real or complex Hilbert space. If:

$$m(M) = \begin{cases} 1 \ if \ M = H_2 \\ 0 \ if \ M = \{0\} \\ 0, \ or \ 1 \ if \ dim(M) = 1 \end{cases} \tag{1.3.2}$$

where in the latter case we suppose $m(M) + m(M^{\perp}) = 1$, then m is a two-valued state on $L(H_2)$. Now fix a one-dimensional subspace M_o. If we define a state m via (1.3.2) such that the one-dimensional subspaces M, $M^{\perp} \neq M_o$ attains the value 0 or 1 in m, and for $m(M_o) = 1/2$, then, those states are not expressible via (1.3.1).

Gleason's theorem, named after the mathematician who published it in 1957, was the answer to Mackey's question:

Theorem 1.3. *(Gleason's Theorem) Let H a separable, real or complex Hilbert space, $dim(H) \neq 2$, then for any state m on $L(H)$ there exists a unique positive Hermitian trace operator T on H with $tr(T) = 1$, such that:*

$$m(M) = tr(TP_M), \quad M \in L(H) \tag{1.3.3}$$

Proof. The proof of Gleason's theorem will not be provided due to its complexity and lengthiness, nevertheless a complete and detailed proof may be found in ([5] p.131). □

The proof of Mackey's conjecture by Gleason ultimately strengthen the statement made by Von Neumann before and ruled out the existence of the so-called hidden variables in the realm of Quantum Mechanics. More extensively Gleason's theorem allowed mathematicians and physicists to treat events that take place in the quantum context as a lattice, namely $L(H)$, of subspaces of a real or complex Hilbert space and assign them "probabilities" i.e. states or measures. In short, obstacles of Quantum Measure theory were reinforced by Gleason's theorem. The fact that probability measures were determined by the logical structure of quantum events was thought, by part of the scientific community, to exhibit that stochastic processes are to be encountered at the core of the physical laws that determine the structure of the universe.

1.4 Gleason Measures

Clearly, as presented in the previous section, Gleason's theorem has been profoundly studied for its close relationship with the Foundations of Quantum Mechanics, and it is the main subject of interest of this thesis to investigate the so-called Gleason measures, in particular complex Gleason measures, that came as a direct consequence of the theorem previously stated. Gleason measures are defined as follows:

Definition 1.4. Any finite (signed) measure m on $L(H)$ which can be expressed via (1.3.3) is said to be a Gleason measure.

One of the main ideas of this thesis is the vectorial character of the complex Gleason measures. Such property gives a natural mathematical interpretation of the concept. Many quantities considered in Physics have a vectorial character as well; the momentum and the angular momentum of a particle exemplify this, and in Quantum Mechanics, this kind of quantities are represented by what we call a vector operator. All this relations have triggered the desire of generalizing previous results on Gleason Measures to complex Gleason measures.

Example 1.5. With the purpose of introducing this concept, consider the following example included in ([10]) that introduces the momentum of a particle. In order to perform a measurement of a particle's momentum in two dimensions, we need to measure the two components of the momentum; p_x , p_y relative to an (arbitrarily chosen) orthogonal coordinate system. Accordingly, in Quantum Mechanics, the momentum is represented by two observables (self-adjoint operators), P_x and P_y, and we may think of these two operators as the "components" of one vector operator:

$$P = (P_x, P_y)$$

relative to the coordinate system chosen previously. This means that if a linear change of coordinates is made, the components of P change according to the same rule as the components of a vector; but the momentum itself should be independent of the coordinate system chosen (otherwise it would not have a physical meaning).

To stress this independence of the coordinate system, we shall follow a different point of view. Notice that in order to measure a vector quantity like p, it is necessary to be able to measure the scalar product $\langle p, v\rangle$ with any given direction v.

According to the Quantum Mechanics formalism we would have an observable (self-adjoint operator) P_v for each v. Furthermore the correspondence $v \to P_v$ should be linear.

As a consequence, we make the following formal definition: Let H be a Hilbert space (to be thought as the state space of the quantum system) and V a real Hilbert space (the space of values of the vectorial quantity that we want to measure); a vectorial operator is an element of $\mathcal{L}(V, \mathcal{L}(H))$.

1.5 The Nemytsky Operator

To further complement this thesis the Nemystky operator, which is a class of nonlinear operators on L^P spaces with good continuity and boundedness properties (see [6]), will be required. They take their name from the mathematician Viktor Vladimirovich Nemytsky and are defined as follows.

Definition 1.6. Let Ω be a domain (an open and connected set) in n-dimensional Euclidean space. A function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is said to satisfy the Caratheodory's conditions if:

1. $f(x, u)$ is a continuous function of u for almost all $x \in \Omega$

2. $f(x, u)$ is a measurable function of x for all $u \in \mathbb{R}^n$

Given a function f satisfying the Caratheodory's condition and a function $u : \Omega \to \mathbb{R}^n$ define a new function $F(u) : \Omega \to \mathbb{R}$ by:

$$F(u)(x) = f(x, u(x))$$

the function F is called the Nemytsky operator.

The Nemytsky operator, in short, is a variable-coefficient composition operator of the form $\varphi(x) \to g(x, \varphi(x))$ that has been studied and used in the context of many nonlinear problems involving integrals, as well as partial and ordinary differential equations.

1.6 Structure of the Thesis

The following work concentrates in analyzing several properties of complex Gleason measures from previous works to new conclusions and possible further investigations. It focuses on developing a functional calculus that permits working with complex Gleason measures through the Nemytsky Operator. It also presents some examples and applications of the latter in Quantum Mechanics; trying to give physical interpretation of the mathematical concepts that arises. Furthermore a modest discussion on the Foundations of Quantum Mechanics is provided so that the reader understands the mathematical and physical context of the information presented. The structure of the work is as follows. Chapter 2 contains definitions and properties related to measurability of vector-valued functions. In Chapter 3 we introduce vector-valued measures and present some of its properties. Selected topics from Operator Theory are given in Chapter 4 in order to complement this thesis. A brief introduction to Hilbert Space Theory is presented in Chapter 5. Chaper 6 gives a first glance of complex Gleason measures and related topics. The construction of an integral with respect to a complex Gleason measure is provided in Chapter 7. Some results of Measure theory are rewritten for Gleason measures in Chapter 8. Chapter 9 talks about the Nemytsky operator. Next, the results of this work, that is, Theorem 9.6, in addition to applications to Quantum Mechanics are presented in Chapter 10. Finally, extensions of previous theorems are presented in Chapter 11.

Chapter 2

Measurability

In this chapter, with the intention of preparing the grounds for a better understanding of the work covered and developed in this thesis, the definitions and properties related to vector-valued measures and functions including measurability and convergence of the latter are given (for details see [11]). The definitions are arranged, so that, every new concept builds up from the previous definition provided .

2.1 Measurable Functions

Definition 2.1. A σ-algebra of subsets of S is a set $\Sigma \in \mathcal{P}(S)$ which is closed under complements and countable unions, i.e., $\Sigma \in \mathcal{P}(S)$ is a σ-algebra if and only if:

a) The empty set is in Σ, i.e., $\emptyset \in \Sigma$

a) For every element $A \in \Sigma$, the complement $X \setminus A = A^C \in \Sigma$

b) If $(A_j)_{j\in\mathbb{N}} \subseteq \Sigma$, such that $A_i \cap A_j = \emptyset$ for $j \neq i$, then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$

Some examples of σ-algebras include the power set of any set S, i.e. the set of all subsets of S, including the empty set $\emptyset$ and S itself, also the least possible σ-algebra of a set S is given by $\Sigma = \{\emptyset, S\}$ (for more details on σ-algebras consult [21]). In the remaining of the chapter we assume S is a set and Σ a σ-algebra of subsets of S.

Definition 2.2. A measure on S is a function $\mu : S \to \mathbb{R}$ satisfying the following properties:

a)$\mu(\emptyset) = 0$

b) If $(A_j)_{j\in\mathbb{N}} \subseteq \Sigma$ is a disjoint countable family in Σ then:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu\left(A_i\right) \qquad (\sigma - additivity)$$

In this case S is said to be measurable and the triplet (S, Σ, μ) is called a measure space. Also if $A \in \Sigma$ then we call A measurable. A measure μ on a set S is finite if $\mu(S) < \infty$. A measure μ on a set S is σ-finite if and only if S is the countable union of measurable sets with finite measure, i.e., $S = \bigcup_{i=1}^{\infty}(S_i)$ so that $\mu(S_i) < \infty$.

Definition 2.3. A vector space X over a field F is said to be an *inner product space* (or Euclidian, or pre-Hilbert, or unitary space) if there is a D-valued function $\langle \cdot, \cdot \rangle$ on $X \times X$ satisfying each of the following conditions for all $u, v, w \in X$ and $\alpha \in F$:

a) $\langle u, u \rangle \geq 0$, and $\langle u, u \rangle = 0$ if and only if $x = \mathbf{0}$;

b) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$;

c) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$;

d) $\langle u, v \rangle = \langle v, u \rangle^*$,

where the asterisk, $(^*)$ denotes the conjugation for $F = \mathbb{C}$ and the identity for $F = \mathbb{R}$. Such function $\langle \cdot, \cdot \rangle$ is called an *inner product.* Furthermore, common convention is that if $F = \mathbb{C}$ or $F = \mathbb{R}$ then the inner product space X is called a complex or real space, respectively. One of the simplest examples of inner product spaces is the real numbers with the inner product defined as usual multiplication, that is, $\langle u, v \rangle := uv$ for $u, v \in R$. Such concept may extended to $\mathbb{R}^n$ which will be an inner product space with the dot product as the inner product, that is:

$$\left\langle \begin{matrix} u_1 \\ \vdots \\ u_n \end{matrix}, \begin{matrix} v_1 \\ \vdots \\ v_n \end{matrix} \right\rangle := u^T v = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + \cdots + u_n v_n$$

where u^T is the transpose of u.

Even though the definition of a norm is provided next, it may also be defined, using the inner product, as $\|u\| = \langle u, u \rangle^{\frac{1}{2}}$

Definition 2.4. Given a vector space X over a subfield F of the complex numbers, a *norm* on X is a function $\|\cdot\| : X \to \mathbb{R}$ with the following properties:

a) $\|x\| \geq 0$, and $\|x\| = 0$ iff $x = \mathbf{0}$, for all $x \in X$;

b) $\|\alpha x\| = |\alpha| \, \|x\|$, $\alpha \in F$, $x \in X$;

c) $\|x + y\| \leq \|x\| + \|y\|$, $x, y \in X$ (triangle inequality).

With this in mind, a distance from a vector x to a vector y is defined to be $\|x - y\|$. Then, with respect to this distance, X is a *metric space*. Moreover a sequence $\{x_n\}$ of vectors in X converges in norm to a vector $x \in X$ if and only if $\lim_{n\to\infty} \|x_n - x\| = 0$ (in this case x is unique). A sequence $\{x_n\}$ in X is *Cauchy* if and only if for every $\epsilon > 0$, there is an integer n_0 such that for all $n, m > n_0$, $\|x_n - x_m\| < \epsilon$. Furthermore, a normed vector space is *complete* if and only if every Cauchy sequence in X converges to some vector in X. A complete normed vector space is called a *Banach space* and also a complete inner product space is called a *Hilbert space.*

In the remaining of the thesis unless otherwise stated, definitions assume that (S, Σ, μ) is a complete σ-finite measure space and X is a real Banach space with norm $\|\cdot\|$. Continuing the discussion, the chapter now focuses on the measurability of vector valued functions. Such concept is parallel to measurability of real-valued functions with obvious modifications.

Definition 2.5. The *characteristic function* $\chi_M(x)$ of any subset M of a set S is defined by:

$$\chi_M(x) = \begin{cases} 1 & if\ x \in M \\ 0 & if\ x \notin M. \end{cases}$$

Broadly speaking the latter concept indicates membership of an element in a subset M of S. This concept helps define step functions and take a step forward in the construction of measurable vector valued functions.

Definition 2.6. A function $\varphi : S \to X$ is a *step function* if there exists a finite family $\{M_n\} \subseteq \Sigma$ of pairwise disjoint sets (i.e. every two different sets in the family are disjoint)

of finite measure and a finite family $\{e_n\} \subseteq X$ so that:

$$\varphi = \sum_n e_n \chi_{M_n},$$

where χ_{M_n} indicates the characteristic function of the set M_n.

Note that the family of all the step functions $\varphi : S \to X$ is a real vector space. Furthermore, a powerful tool that will be used in this chapter and throughout the thesis is convergence of functions, even though the work behind this concept is vast, pointwise convergence is the approach that will be used. The reader may assume that this thesis refers to pointwise convergence of functions, when convergence is indicated, unless otherwise stated.

Definition 2.7. Let $\{f_n\}$ be a sequence of functions, $f_n : S \to X$ for every n. The sequence $\{f_n\}$ *converges pointwise* to f, written as:

$$\lim_{n\to\infty} f_n = f \;\; pointwise$$

if and only if:

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for every x in the domain, i.e., $\lim_{n\to\infty} \|f_n(x) - f(x)\| = 0$.

Moreover when studying sequences of measurable functions defined on a measure space, just as the ones discussed in this thesis, it is natural to bring up almost everywhere convergence, more specifically pointwise convergence almost everywhere. To define *almost everywhere convergence,* define an equivalence class on the set of functions $f : S \to X$, by $f \sim g$ iff $f = g$ a.e. that is, the functions f and g are equal everywhere except maybe in a set with μ-measure equal to zero, i.e. $\mu(\{x : f(x) \neq g(x)\}) = 0$. By the same token, given a sequence of functions $\{f_n\}$, $f_n \to f$ μ-a.e iff $\lim_{n\to\infty} f_n(x) = f(x)$ except maybe for a set

with μ-measure equal to zero, i.e. $\mu(x : \lim_{n\to\infty} f_n(x) \neq f(x)\}) = 0$. Next, even though measurability of functions can be defined in several ways, for example, using pre-images of open sets, this research maintains the sequential definition related to step functions as standard throughout the thesis.

Definition 2.8. A function $f : S \to X$ is X-*measurable* if there is a sequence $\{\varphi_k\}$ of step functions so that $\varphi_k \to f$ in X μ-a.e. as $k \to \infty$.

Recalling the definition of a step function, it is obvious that every step function is X-measurable. The following Lemma 2.9 and Proposition 2.10 are well known results from measure theory and their proof will be omitted.

Lemma 2.9. *Let $\{S_n\}_{n\geq 1} \subseteq \Sigma$ be a countable partition of S and let $f : S \to X$ be a function. Then, the function f is X-measurable if and only if for each $n \geq 1$, the restriction $f : S_n \to X$ is X-measurable.*

Proposition 2.10. *Let $\{f_n\}$ be a sequence of X-measurable functions $f_n : S \to X$ converging μ-a.e. to a function $f : S \to X$. Then the function f is X-measurable.*

This concludes the chapter.

Chapter 3

Vector-Valued Measures

The notion of measurable functions and convergence are left behind to focus on the formal definition of vector-valued measures and some related properties. We begin this chapter with the definition of a set function. A *set function* is simply defined as a function whose input is a set and output is a value.

Definition 3.1. A set function $m : \Sigma \to X$ is called a *vector-valued measure* if:

1. $m(\varnothing) = 0$.
2. For each countable, pairwise disjoint family $\{A_j\} \subseteq \Sigma$:

   $$m\left(\bigcup_j A_j\right) = \sum_j m(A_j),$$

   where the series is commutatively convergent (i.e. any rearrangement of the series converges to the same value).

Definition 3.2. The *variation* $|m|$ of the vector-valued measure m is the set function $|m| : \Sigma \to [0, \infty]$ defined for each $A \in \Sigma$ as:

$$|m|(A) = \sup\left\{\sum_j \|m(A_j)\|\right\},$$

where the supremum is taken over all the finite partitions $\{A_j\} \subseteq \Sigma$ of A.

Given the concept of variation of a measure, it is natural to inquire on some properties of this resulting function, next some of the most important characteristics of the variation of a measure are presented.

Lemma 3.3. *The variation $|m|$ of a vector-valued measure m is a measure.*

Proof. Let $\{A_i\}$ be a partition of $A \in \Sigma$. Let t_i be real numbers such that $t_i < |m|(A_i)$. Then each A_i has a partition $\{A_{ij}\}$ such that:

$$\sum_j \|m(A_{ij})\| > t_i \quad (i = 1, 2, 3, ...)$$

Since $\{A_{ij}\}(i, j = 1, 2, 3, ...)$ is a partition of A, it follows that:

$$\sum_i t_i \le \sum_{i,j} \|m(A_{ij})\| \le |m|(A)$$

Taking the supremum of the left side of the latter inequality, over all admissible choices of $\{t_i\}$, we see that:

$$\sum_i |m|(A_i) \le |m|(A) \tag{3.0.1}$$

To prove the opposite inequality, let $\{E_i\}$ be any partition of A. Then for any fixed i, $\{E_i \cap A_j\}$ is a partition of E_i, and for any fixed j, $\{E_i \cap A_j\}$ is a partition of A_j. Hence:

$$\sum_i \|m(E_i)\| = \sum_i \left\| \sum_j m(E_i \cap A_j) \right\|$$

$$\le \sum_i \sum_j \|m(E_i \cap A_j)\|$$

$$= \sum_j \sum_i \|m(E_i \cap A_j)\| \le \sum_i |m|(A_i)$$

since the latter holds for every partition $\{E_i\}$ of A, we have:

$$|m|(A) \le \sum_i |m|(A_i) \tag{3.0.2}$$

In all, by (3.0.1) and (3.0.2) $|m|$ is countably additive. □

Note that $|m|(A) \geq |m(A)|$, but that in general $|m|(A)$ is not equal to $|m(A)|$.

Lemma 3.4. *Given two vector-valued measures $m_1, m_2 : \Sigma \to X$:*

$$|m_1 + m_2| \leq |m_1| + |m_2| .$$

Proof. The proof of this assertion follows from the previous proof and the fact that $|m|(A) \geq |m(A)|$. $\square$

Definition 3.5. Two vector-valued measures $m_1, m_2 : \Sigma \to X$ are *mutually singular*, denoted $m_1 \perp m_2$, if the measures $|m_1|$ and $|m_2|$ are mutually singular. That is to say, if there is a partition $S = A \cup B$, $A, B \in \Sigma$, such that $|m_1|(A) = 0$ and $|m_2|(B) = 0$. In particular, a vector measure $m : \Sigma \to X$ and the measure μ are mutually singular if the measures $|m|$ and μ are mutually singular.

Remark 3.6. The following statements are equivalent:

1. $|m|(A) = 0$ for some $A \in \Sigma$.

2. $m(A') = 0$ for all $A' \subseteq A$, $A' \in \Sigma$.

The following result is obtained as a consequence of the latter remark.

Lemma 3.7. *Two vector-valued measures m_1 and m_2 are mutually singular if and only if there exists a partition $S = A \cup B$, $A, B \in \Sigma$, such that:*

$$\begin{aligned} m_1(A') &= 0 \textit{ for all } A' \subseteq A,\ A' \in \Sigma, \\ m_2(B') &= 0 \textit{ for all } B' \subseteq B,\ B' \in \Sigma. \end{aligned}$$

Lemma 3.8. *If two vector-valued measures $m_1, m_2 : \Sigma \to X$ are mutually singular:*

$$|m_1 + m_2| = |m_1| + |m_2| .$$

Proof. This proof follows closely the proof of Lemma 17 in ([9]). It is known from Lemma 3.4 that given two vector-valued measures $m_1, m_2 : \Sigma \to X$:

$$|m_1 + m_2| \leq |m_1| + |m_2|$$

so, it is just left to show that if $m_1 \perp m_2$:

$$|m_1 + m_2| \geq |m_1| + |m_2|.$$

Now by definition of mutual singularity, a partition $S = A \cup B$, with $A, B \in \varSigma$, such that $|m_1|(B) = 0$ and $|m_2|(A) = 0$, it follows that given $E \in \varSigma$:

$$|m_1|(E) = |m_1|(A \cap E)$$

and:

$$|m_2|(E) = |m_2|(B \cap E).$$

Now fix $\epsilon > 0$ and consider a partition $\{C_i\}_{i \in J}$ of $(A \cap E)$ such that:

$$\sum_{i \in J} \|m_1(C_i)\| \geq |m_1|(A \cap E) - \frac{\epsilon}{2}$$

and a partition $\{D_j\}_{j \in L}$ of $(B \cap E)$ such that:

$$\sum_{j \in L} \|m_2(D_j)\| \geq |m_2|(B \cap E) - \frac{\epsilon}{2}.$$

since:

$$E = \left(\bigcup_{i \in J} C_i \right) \cup \left(\bigcup_{j \in L} D_j \right)$$

is a partition of E, and:

$$|m_1 + m_2|(E) \geq \sum_{i \in J} \|m_1(C_i)\| + \sum_{j \in L} \|m_2(D_j)\|$$

$$\geq |m_1|(A \cap E) + |m_2|(B \cap E) - \epsilon.$$

In conclusion:

$$|m_1 + m_2|(E) \geq |m_1|(E) + |m_2|(E),$$

this inequality completes the proof of the Lemma. $\square$

Definition 3.9. Given two vector-valued measures $m_1, m_2 : \Sigma \to X$, we say that m_1 is *absolutely continuous* with respect to m_2, denoted $m_1 \ll m_2$, if $|m_1| \ll |m_2|$, which means:

If $A \in \Sigma$ and $|m_2|(A) = 0$, then $|m_1|(A) = 0$ as well. In particular, a vector measure $m : \Sigma \to X$ is absolutely continuous with respect to the measure μ if $|m| \ll \mu$.

Remark 3.10. According to Remark 3.6, $m_1 \ll m_2$ if and only if $A \in \Sigma$ and $m_2(A') = 0$ for all $A' \subseteq A$, $A' \in \Sigma$, implies that $m_1(A') = 0$ for all $A' \subseteq A$, $A' \in \Sigma$.

Theorem 3.11. *(Lebesgue decomposition, [11], p. 189) Let $m : \Sigma \to X$ be a vector-valued measure of σ-finite variation. Then, there exist unique vector-valued measures $m_1, m_2 : \Sigma \to X$ of σ-finite variation so that:*

$$\begin{aligned} m &= m_1 + m_2, \\ m_1 &\ll \mu,\ m_2 \perp \mu. \end{aligned}$$

The σ-finiteness of $|m|$ is necessary for the validity of Theorem 3.11 even in the case of signed measures, while the measure space (S, Σ, μ) does not need to be σ-finite.

Proof. For a complete detailed proof see [11], page 189. $\square$

Before stating the next result, the Bochner integral mentioned in the following theorem (Theorem 3.12) is defined in much the same sense and is completely analogous to the Lebesgue Integral. First a step function, is given in terms of the sum of the characteristic functions of disjoint members of a sigma-algebra and elements of a Banach space, then integrability is defined and lastly this notion is extended to a broader family of functions. Note also that given a Bochner integrable function $f : S \to X$, the "indefinite integral" $f d\mu$ indicates a vector valued measure defined on each $A \in \Sigma$ as the Bochner integral $\int_A f d\mu$. If the reader would like to further inquire on the Bochner integral refer to [22].

Theorem 3.12. *(Radon-Nikodym, [15]) For a vector valued measure $m : \Sigma \to X$, the following statements are equivalent:*

1. *There exists a unique function $f : S \to X$, Bochner integrable, such that $m = f d\mu$.*

2. *The vector valued measure m satisfies the following conditions:*

   (a) *$m \ll \mu$.*

   (b) *$|m|$ is a finite measure, that is to say $|m| : \Sigma \to [0, \infty)$.*

   (c) *For each $A \in \Sigma$ with $0 < \mu(A) < \infty$, there exists $E \subseteq A$, $E \in \Sigma$ and a compact set $K \subseteq X$ not containing zero, so that $\mu(E) > 0$ and for all $E' \subseteq E$, $E' \in \Sigma$, the set $m(E')$ is contained in the cone generated by K.*

Remark 3.13. When the space X is finite dimensional, condition 2.c) in Theorem 3.12 is satisfied by any vector valued measure $m : \Sigma \to X$. Indeed, we can select:

$$K = \{x \in X : \|x\| = 1\}.$$

Then, the cone generated by K, defined as:

$$\{\lambda x : x \in K, \lambda \geq 0\},$$

becomes X, so condition 2.c) holds. Thus, Theorem 3.12 reduces to the familiar Radon-Nikodym theorem in this case. For a detailed analysis of the conditions involved in Theorem 3.12, see Chapter 5 in [16].

Chapter 4

Some Topics in Operator Theory

The definition of Gleason Measures is once again presented in this chapter, in particular the definition of a complex Gleason Measure. Also, important theorems that aid this thesis are presented and proven, the reader may find abstract concepts within this chapter, and, even though it is not the intention of this work to deepen in such material, a precise and short definition is presented. It is left to the reader to inquire about the details. The first part of this chapter deals with some simple definitions from Operator theory that will help strengthen concepts that will be introduced in subsequent chapters.

4.1 Operator Theory

This section provides a brief introduction to operators so that the reader familiarizes with the concepts that will be used in later chapters, the main objective is to provide a good and simple explanation on specific types of operators as well as some properties of them. To begin, let U and V be two vector spaces, any mapping from U into V is called an *operator* or a *transformation*. Furthermore, one of the types of operators that is of great significance to this thesis is the class of linear operators, due to its major role in the Mathematical Foundations of Quantum Mechanics.

Definition 4.1. Let X and Y be vector spaces over the same field F. A mapping $L : X \to Y$ is called a *linear operator* (linear transformation) if for all $a, b \in F$ and $u, v \in X$ we have:

$$L(au + bv) = aL(u) + bL(v)$$

It is customary to denote the image $L(u)$ of u under a linear transformation simply by Lu. Another class of operators that is relevant to this work, is the class of continuous operators..

Definition 4.2. An operator (transformation) $T : X \to Y$ where X and Y are normed spaces is said to be *continuous* at $u_0 \in X$ if for any $\epsilon > 0$ there is $\delta > 0$ (depending on ϵ) such that:

$$\|T(u) - T(u_0)\|_Y < \epsilon \quad for \ \|u - u_0\|_X < \delta$$

The operator (transformation) L is said to be continuous if it is continuous at all points $u_0 \in X$.

Definition 4.3. A *bounded operator*, is a linear transformation $L : X \to Y$, where X and Y are vector spaces endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively, with the property that there exists some $M > 0$ such that for all $v \in X$ we have:

$$\|Lv\|_Y \leq M \|v\|_X$$

The smallest M such that the inequality holds is called the operator norm $\|L\|$ of L.

As an example, any linear transformation of a finite-dimensional Euclidean space into another Euclidean space is bounded.

Recall that a Hilbert space H is a real or complex inner product space that is complete with respect to the distance function induced by the inner product, that is, an inner product space that is also a complete metric space. A clear and important example of a Hilbert space is an Euclidean space which is complete in the norm. In particular the Euclidean space, consisting of three-dimensional vectors, denoted by $\mathbb{R}^3$, equipped with the dot product is a Hilbert space. Then a linear operator L defined in a Hilbert space is a linear transformation of a *linear subspace* (a linear subspace is simply a vector space M that is a subset of a vector

space H) $D(A)$ of H into H; $D(A)$ is called the *domain* of A and its image under A:

$$R(A) = \{Au : u \in D_A\}$$

is the *range* of A. If B is another linear operator defined on a subspace $D(B) \supset D(A)$ and it coincides with A on $D(A)$, that is, $Bu = Au$ for all $u \in D(A)$, then B is called an *extension* of A (or A a *restriction* of B), and write $A \subset B$.

Definition 4.4. Following F. Riesz and Sz. Nagy ([14], section 116), we shall say that an unbounded operator T and a bounded operator B are *permutable* (or commute) if

$$BT \subset TB.$$

in other words, TB is an extension of BT i.e. $D(TB) \subset D(BT)$ and $TBu = BTu$ for all $u \in D(BT)$ where $D(TB)$ and $D(BT)$ are the domain of the operators TB and BT respectively.

The following theorem is included so that the adjoint of an operator can be defined.

Theorem 4.5. *(Theorem 2.5, Chapter II, [13]) Let A be an operator on H, with domain $D(A)$. Denote by $D(A^*)$ the set of all vectors $v \in H$ which are such that for each v there is one and only one vector v^*which satisfies the equation:*

$$\langle Au, v\rangle = \langle u, v^*\rangle$$

for all $u \in D(A)$. The mapping:

$$A^*(v) = v^*, \quad v \in D(A^*),$$

is a linear operator, called the adjoint (also called Hermitian) of A, and it exists, i.e. $D(A^*) \neq \emptyset$ if and only if $D(A)$ is dense in H.

Proof. Consult [13] p.187. $\square$

Also, a linear operator A defined in a Hilbert space is *symmetric* if $\langle Au, v \rangle = \langle u, Av \rangle$ for all $u, v \in D(A)$. A symmetric operator A is called *self-adjoint* if $A \equiv A^*$, i.e., $A^* \subseteq A$ and $A \subseteq A^*$. An example of a symmetric operator is the so-called Schrodinger operator, more specifically such operator is linear and Hermitian (for details see [7]). Furthermore given the concept of a symmetric operator, an equivalent condition for a linear operator A to be bounded is: there exists some $M > 0$ such that for all $u \in D(A)$ we have:

$$\langle Au, u \rangle \leq M \langle u, u \rangle$$

Definition 4.6. A linear operator $L : X \to Y$ is *normal* if $\|Lv\|_Y = \|L^*v\|_Y$ for any $v \in X$. More specifically a normal operator on a complex Hilbert space H is a continuous linear operator $N : H \to H$ that commutes with its hermitian adjoint N^* ($N^* : H \to H$ such that $(Nx, y) = (x, N^*y)$ for all x and y in H) that is:

$$NN^* = N^*N$$

Some common examples of normal operators includes, hermitian, unitary, and positive operators.

One of the most important types of operators is the orthogonal projection operator, due to its essential role in Spectral Theory, which will be discussed later, and of course in Quantum Mechanics, which is the physical context in which the results of this work may be applied. As the name implies, projection operators defined on a Hilbert space are just a generalization of the familiar concept of orthogonal projections of vectors in $\mathbb{R}^3$ onto a subspace in $\mathbb{R}^3$, in particular a line or even a plane. Next, not only the definition of an orthogonal projector is provided, but it is done by means of a very important theorem.

Theorem 4.7. *(Theorem 3.1, Chapter III, [13]) Let M be a closed linear subspace of a Hilbert space H. Denote by $M^\perp$ the linear space (actually closed linear subspace) of all*

vectors orthogonal to M:

$$M^{\perp} = \{h : \langle h, g \rangle = 0,\ g \in M\}$$

Then each vector $x \in H$ can be written uniquely as a sum:

$$x = x' + x'', \quad x' \in M, \quad x'' \in M^{\perp} \tag{4.1.1}$$

and the mapping:

$$P_M(x) : H \to M, \quad x \in H \tag{4.1.2}$$

is a linear operator defined on the entire H, called the projector (or orthogonal projection operator) onto M.

Before proceeding with the proof of the theorem, the following Lemma 4.8 is presented since it takes part in the proof of Theorem 4.7.

Lemma 4.8. *(Lemma 2.1, Chapter III, [13]) If M is a closed linear subspace of a Hilbert space H and if x is a vector in H, there is a vector $x' \in M$ such that $x'' = x - x'$ is orthogonal to M.*

Proof. Denote d the distance between x and M, i.e.:

$$d = \inf_{y \in M} \|x - y\|$$

If $d = 0$, there must be a sequence $y_1, y_2, ... \in M$ such that $\|x - y_n\| \to 0$ when $n \to \infty$; since M is closed, we have in this case $x \in M$ and the lemma is proven by taking $x' = x$ and $x'' = 0$.

Assume that $d > 0$, and let $y_1, y_2, ... \in M$ be the sequence for which:

$$d = \lim_{n \to \infty} \|x - y_n\|$$

It is easy to check by using the algebraic properties of the inner product that:

$$\frac{1}{2}\|x-y_m\|^2+\frac{1}{2}\|x-y_n\|^2=\left\|x-\frac{1}{2}(y_m+y_n)\right\|^2+\left\|\frac{1}{2}(y_m-y_n)\right\|^2$$

If we let $m, n \to \infty$ above and note that:

$$\left\|x-\frac{1}{2}(y_m+y_n)\right\|^2 \geq d^2$$

because $\frac{1}{2}(y_m+y_n) \in M$, we get $\|y_m-y_n\| \to 0$. Thus, $y_1, y_2, \ldots$ is a Cauchy sequence, which has a limit $x' \in M$, since M is closed. Moreover, we have

$$\|x-x'\| = \lim_{n\to\infty}\|x-y_n\| = d$$

In order to show that $x'' \perp M$, it is sufficient to prove that $\langle x'', z\rangle = 0$ for all normalized $z \in M$. By means of the identity

$$\langle z, x'' - \langle z, x''\rangle z\rangle = 0$$

we get

$$\|x''\|^2 = \|x'' - \langle z, x''\rangle z\|^2 + \|\langle z, x''\rangle z\|^2,$$

and since $\|x'' - \langle z, x''\rangle z\| \geq d$ because $\langle z, x''\rangle z \in M$, we have

$$d^2 = \|x''\|^2 = \|x'' - \langle z, x''\rangle z\|^2 + |\langle z, x''\rangle|^2 \geq d^2 + |\langle z, x''\rangle|^2,$$

i.e., $|\langle z, x''\rangle| = 0$. Q.E.D. □

Proof. (of Theorem 4.7) The possibility of decomposition (4.1.1) for each $x \in H$ is guaranteed by Lemma 4.8. In order to prove the uniqueness of this decomposition, assume that

$$x = x_1' + x_1'', \quad x_1' \in M \quad x_1'' \in M^{\perp}$$

Then we have

$$x' - x_1' = -(x'' - x_1'').$$

Since, on the other hand, $x' - x_1' \in M$, $x'' - x_1'' \in M^\perp$, and therefore $x' - x_1' \perp x'' - x_1''$, we get

$$\langle x' - x_1', x' - x_1' \rangle = - \langle x' - x_1', x'' - x_1'' \rangle = 0$$

which implies $x' - x_1' = \mathbf{0}$ and $x'' - x_1'' = \mathbf{0}$. Thus, the uniqueness of x' and x'' for each $x \in H$ is established, and consequently (4.1.2) defines a mapping of H into M.

To prove that this mapping is a linear operator, note that if

$$y = y' + y'', \quad y' \in M, \quad y'' \in M^\perp$$

we have

$$ax + by = (ax' + by') + (ax'' + by''),$$

where, due to the linearity of M and $M^\perp$,

$$ax' + by' \in M \quad \textit{and} \quad ax'' + by'' \in M^\perp.$$

Since the decomposition (4.1.1) of any vector in H is unique, we get

$$(ax + by)' = P_M(ax + by) = ax' + by' = aP_M(x) + bP_M(y),$$

which proves the linearity of P_M. Q.E.D. □

Lastly, if M is a closed linear subspace of a Hilbert space H and P_M is the projector on M, then $x' = P_M x$ is called the *projection* of the vector $x \in H$ onto the subspace M. This finalizes the chapter.

Chapter 5

Hilbert Space Theory

This chapter is dedicated to some results in Hilbert spaces. Such results hold great value within the present work since the mathematics done in this thesis are directly or indirectly related to Hilbert spaces. Now, one of the concepts that lies at the core of many theorems presented here, is the set theoretic concept of separability; in order to introduce such concept, some definitions are presented.

Recall that an *orthogonal system* of vectors is a set of vectors in which any two vectors are *orthogonal*, i.e., given a vector space X, $\{x_n\}_{n=1}^{\infty} \subseteq X$, with $x_i \perp x_i$ ($\langle x_i, x_j \rangle = 0$) for all $i, j \in \mathbb{N}$, then $\{x_n\}$ is called an orthogonal system, furthermore if each vector in the system is *normalized*, that is, $\|x_i\| = 1$ for all $i \in \mathbb{N}$, then $\{x_n\}$ is called an *orthonormal system*. Other definitions will be presented before characterizing separability for Hilbert spaces.

Definition 5.1. Given a vector space V over a field F, the *span* of a set S of vectors is defined to be the intersection W of all subspaces of V containing S. Alternatively the span S may be defined as the set of all finite linear combinations of elements in S:

$$W = span(S) = \left\{ \sum_{i=1}^{n} \lambda_i u_i : n \in \mathbb{N}, u_i \in S, \lambda_i \in F \right\}$$

In this case W is referred to as the subspace spanned by S, conversely S is called a spanning set of W or simply said S spans W. Also the *closed linear span* of S, denoted by $[S]$ is the intersection of all closed linear subspaces of V which contain S:

$$[S] = \{u \in V : \forall \epsilon > 0, \exists v \in span(S); \|u - v\| < \epsilon\}$$

Definition 5.2. An orthonormal system S in a inner product space X is called an *orthonormal basis* (or a complete orthonormal basis) in the inner product space X if the closed linear subspace $[S]$ spanned by S is identical to the entire inner product space X, i.e., $[S] = X$.

With this definition separability can be stated specifically for Hilbert spaces.

Theorem 5.3. *A Hilbert space H (that is a complete Euclidean space) is separable if and only if there is a countable orthonormal basis in H.*

This definition may also be stated as follows: The Euclidian space $\mathcal{E}$ is called separable if there is a countable everywhere dense subset of vectors of $\mathcal{E}$. Common examples of separable Hilbert spaces include, the real line. The following theorem encompasses the importance of the previous concepts given, the Theorem provides *"a simple recipe for computing projections in practically important cases"* ([13] p.198)

Theorem 5.4. *(Theorem 3.2, Chapter III, [13]) If M is a separable closed linear subspace of a (not necessarily separable) Hilber space H, and if $\{e_i\}_{i=1}^{n}$ is an orthonormal basis in M, then:*

$$E_M u = \sum_i \langle e_i, u \rangle e_i, \quad u \in H,$$

where E_M is the projector on M.

Proof. Refer to [13] page 198. $\square$

The previous theorem concludes the section, this chapter now focuses on presenting applications to Quantum Mechanics.

5.1 Some Applications of Measure Theory and Hilbert Spaces

In the present section, the counter part of some concepts that were covered in Measure, Hilbert and Operator theory are presented in the context of Quantum Mechanis, more

specifically, Wave Mechanics. The beginning of modern Quantum Mechanics, in which the famous physicist Schrödinger was a major exponent, was marked by the paper in which he proposed the formalism of Wave Mechanics (see [13]). In his paper it can be observed that when discussing the physical interpretation of the so-called Schrödinger equation it is known that for each interval I:

$$P_t(I) = \int_I |\psi(x,t)|^2 \, dx$$

is the probability of finding a system in the state $\psi(x,t)$ within I at a given time t.

Let $(\Omega, \mathcal{B}, \mu)$ be a measure space, with the previous definition of probability it is natural to investigate *square integrable functions* i.e. an extended complex-valued function $f(x)$, $x \in \Omega$, defined almost everywhere on Ω such that $f(x)$ is measurable and $|f(x)|^2$ is integrable on Ω, that is:

$$\int_\Omega |f(x)|^2 \, d\mu(x)$$

exist and is finite ([13]). We consider the space $L_{(2)}(\Omega, \mu)$ of all complex-valued functions which are square integrable on Ω with the equivalence relation given by almost everywhere equality; moreover denote the family of all equivalence classes with the usual symbol $L^2(\Omega, \mu)$. Furthermore $L^2(\Omega, \mu)$ becomes a vector space and a Hilbert space with the inner product of $f, g \in L^2(\Omega, \mu)$ defined by:

$$\langle f \mid g \rangle = \int_\Omega f^*(x) g(x) d\mu(x)$$

where $f^*(x)$ denotes the complex conjugate.

With these concepts we shall introduce the structure used in Quantum Mechanics, more specifically in wave mechanics, to describe a system of n particles in which each particle is distinct. To start, we assume that the n particles in our system move in three dimensions, and we will denote $\mathbf{r_k}$ the position vector of the k^{th} particle. We can expand $\mathbf{r_k}$ in terms of three orthonormal vectors $\mathbf{p}_x$, $\mathbf{p}_y$, $\mathbf{p}_z$, in a reference system of coordinates in the real

Euclidean space $\mathbb{R}^3$ as follows:

$$\mathbf{r}_k = x_k\mathbf{p}_x + y_k\mathbf{p}_y + z_k\mathbf{p}_z, \qquad k = 1,\, 2,\, 3, \ldots,\, n$$

In wave mechanics it is postulated that the state of a system of n particles is given by the function $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)$ at any given time t, defined on the configuration space $\mathbb{R}^{3n}$ of coordinate vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$. Furthermore we shall assume that $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)$ is once continuously differentiable in t, and square integrable with respect to the measure space $(\mathbb{R}^{3n}, \mathcal{B}^{3n}, \mu_l^{3n})$ (where $\mathcal{B}^{3n}$ denotes the family of Borel sets in $\mathbb{R}^{3n}$) i.e., $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t) \in L_{(2)}(\mathbb{R}^{3n})$, and normalized to verify:

$$\int_{\mathbb{R}^{3n}} |\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)|^2 \, d\mathbf{r}_1 \ldots d\mathbf{r}_n = 1$$

In general, if a given function $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)$ represents a state of the system in question, then $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)$ is called a *wave function* of the corresponding system.

Note that the wave function $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)$ by itself does not have any physical meaning, rather we have that the measure given by:

$$P_t(B) = \int_B |\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)|^2 \, d\mathbf{r}_1 \ldots d\mathbf{r}_n, \qquad B \in \mathcal{B}^{3n}$$

is interpreted as a probability measure $P_t(B)$ represents for every Borel set B, the probability of having the outcome of a measurement at time t of the positions $\mathbf{r}_1, \ldots, \mathbf{r}_n$ of the n particles of a system's state is given by $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)$ within B.

In the context defined above it turns out that if a state is described by the wave function $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)$ at some given time t, then the function defined by $c\psi_1(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)$, with the assumption that $|c| = 1$ and $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t) = c\psi_1(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)$ almost everywhere with respect to the Lebesgue measure on $\mathbb{R}^{3n}$, describes the same state. Furthermore, this implies that each function in the equivalence class $\psi(t) \in L^2(\mathbb{R}^{3n})$, in which $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)$ is contained, would be described by the same wave function $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_n;\, t)$. Finally, this previous result allows us to systematically formulate one of the basic assumptions of wave

mechanics in a convenient way:

Postulate 1: The state of a system of n different particles is described at any time t by a normalized vector $\psi(t)$ from the Hilbert space $L^2(\mathbb{R}^{3n})$. The time-dependent vector function $c\psi(t)$, $|c| = 1$, represents the same state as $\psi(t)$. (see Chapter II, section 5, pg, 120 in [13])

5.2 Some Examples of Projectors in Hilbert Spaces

This thesis takes a few pages to inquire in the behaviour of projectors acting on a Hilbert space, this is done by presenting examples, that were taken from exercises in [7].

Exercise 5.5. By definition, a projector P_i is less than or equal to another projector P_j if $P_iP_j = P_i$; one then uses the notation $P_i \leq P_j$. Show that if $P_i \leq P_j$, one necessarily has $\langle u \mid P_i \mid u\rangle \leq \langle u \mid P_j \mid u\rangle$ for any $\mid u\rangle$, and conversely. Show that the inequality thus defined actually satisfies the characteristic axioms of an inequality, namely that i) $P_i \leq P_j$ and $P_j \leq P_i$ imply $P_i = P_j$; ii) $P_i \leq P_j$ and $P_j \leq P_k$ imply $P_i \leq P_k$.

Proof. Let $\mathscr{E}_j, \mathscr{E}_i$ be subspaces related to P_j and P_i respectively. Suppose first that $P_i = P_j$ then clearly $\langle u \mid P_i \mid u\rangle = \langle u \mid P_j \mid u\rangle$. Now if $P_i < P_j$ i.e. $P_iP_j = P_i$ then

$$\langle u \mid P_i \mid u\rangle = \langle u \mid P_iP_j \mid u\rangle < \langle u \mid P_j \mid u\rangle$$

Conversely if $\langle u \mid P_i \mid u\rangle = \langle u \mid P_j \mid u\rangle$ then $P_i = P_j$. If $\langle u \mid P_i \mid u\rangle < \langle u \mid P_j \mid u\rangle$; note that:

$$\mid u\rangle = P_i \mid u\rangle + (I - P_i) \mid u\rangle \quad \text{and} \quad \mid u\rangle = P_j \mid u\rangle + (I - P_j) \mid u\rangle$$

also:

$$\mid u\rangle = (P_i \setminus P_j) \mid u\rangle + (P_j \setminus P_i) \mid u\rangle + (P_iP_j) \mid u\rangle + (I \setminus P_i)(I \setminus P_j) \mid u\rangle$$

so:

$$P_i \mid u\rangle = P_i(P_i \setminus P_j) \mid u\rangle + P_i(P_j \setminus P_i) \mid u\rangle + P_i(P_iP_j) \mid u\rangle + P_i(I \setminus P_i)(I \setminus P_j) \mid u\rangle$$

but:

$P_i(P_j \setminus P_i) \mid u\rangle = 0$ since $(P_j \setminus P_i) \mid u\rangle$ lies entirely in $\mathscr{E}_j$;

$P_i(P_i \setminus P_j) \mid u\rangle = (P_i \setminus P_j) \mid u\rangle$ since $(P_i \setminus P_j) \mid u\rangle$ lies entirely in $\mathscr{E}_i$;

$P_i(P_iP_j) \mid u\rangle = (P_iP_j) \mid u\rangle$ since $(P_iP_j) \mid u\rangle$ lies entirely in $\mathscr{E}_i$;

$P_i(I \setminus P_i)(I \setminus P_j) \mid u\rangle = 0$ since $(I \setminus P_i)(I \setminus P_j) \mid u\rangle$ lies entirely in the complement of $\mathscr{E}_i$.

Thus

$$\langle u \mid P_i \mid u\rangle = \langle u \mid P_i \setminus P_j \mid u\rangle + \langle u \mid P_iP_j \mid u\rangle < \langle u \mid P_j \mid u\rangle$$

Now if $\mathscr{E}_i = \mathscr{E}_j$ then $\langle u \mid P_i \setminus P_j \mid u\rangle = 0$ and $\langle u \mid P_i \mid u\rangle = \langle u \mid P_i \setminus P_j \mid u\rangle + \langle u \mid P_iP_j \mid u\rangle = \langle u \mid P_iP_j \mid u\rangle \leq \langle u \mid P_j \mid u\rangle$;

if $\mathscr{E}_j \subset \mathscr{E}_i$ then since $\langle u \mid P_i \mid u\rangle = \langle u \mid P_j \mid u\rangle + \langle u \mid P_i \setminus P_j \mid u\rangle$ thus $\langle u \mid P_j \mid u\rangle < \langle u \mid P_i \mid u\rangle$ which is a contradiction;

finally if $\mathscr{E}_i \subset \mathscr{E}_j$ then $\langle u \mid P_i \setminus P_j \mid u\rangle = 0$ and $\langle u \mid P_i \mid u\rangle < \langle u \mid P_j \mid u\rangle$.

Thus $\mathscr{E}_i \subseteq \mathscr{E}_j$, i.e., $P_i = P_iP_j$; and by definition $P_i \leq P_j$.

For the second part of the problem, suppose $P_i \leq P_j$ and $P_j \leq P_i$ then $P_i = P_iP_j$ and $P_j = P_jP_i$ thus since $P_jP_i = P_iP_j$ we obtain $P_i = P_j$; finally if $P_i \leq P_j$ and $P_j \leq P_k$ hence $P_i = P_iP_j = P_iP_jP_k = P_iP_k$ and then $P_i \leq P_k$. □

Exercise 5.6. $P_1, P_2, \ldots, P_k$ being projectors, show that their sum is likewise a projector if and only if:

$$\sum_{i=1}^{k} \langle u \mid P_i \mid u\rangle \leq \langle u \mid u\rangle$$

for any vector $\mid u\rangle$ of a Hilbert space.

Proof. Suppose that $\mathbf{P} = P_1 + P_2 + \cdots + P_k$ is also a projector, then $P_i, P_j$are orthogonal

projectors, that is:

$$P_iP_j = 0 \quad \text{i.e.} \quad \mathscr{E}_i \cap \mathscr{E}_j = \emptyset \qquad \text{for all } i \neq j$$

Let H be a Hilbert space and $\mathbf{P}$ be the sum of the orthogonal projectors upon $\mathscr{E}_{\mathbf{P}}$ now;

$$| u\rangle = \mathbf{P} | u\rangle + (I \setminus \mathbf{P}) | u\rangle$$

where $(I \setminus \mathbf{P})$ is the projector upon $\mathscr{E}_{(I \setminus \mathbf{P})}$ the complement of $\mathscr{E}_{\mathbf{P}}$ hence;

$$\langle u | P_1 | u\rangle + \langle u | P_2 | u\rangle + \cdots + \langle u | P_k | u\rangle = \sum_{i=1}^{k} \langle u | P_i | u\rangle$$

$$= \langle u | \mathbf{P} | u\rangle \leq \langle u | \mathbf{P} | u\rangle + \langle u | (I \setminus \mathbf{P}) | u\rangle = \langle u | u\rangle$$

Conversely suppose that:

$$\sum_{i=1}^{k} \langle u | P_i | u\rangle \leq \langle u | u\rangle .$$

Let:

$$| u\rangle = P_1 | u\rangle + P_2 | u\rangle + \cdots + P_k | u\rangle - \sum_{i,j=1}^{k} P_iP_j | u\rangle$$

that is:

$$\sum_{i=1}^{k} \langle u | P_i | u\rangle \leq \langle u | u\rangle = \sum_{i=1}^{k} \langle u | P_i | u\rangle - \sum_{i,j=1}^{k} \langle u | P_iP_j | u\rangle$$

that is:

$$\sum_{i,j=1}^{k} \langle u | P_iP_j | u\rangle \leq 0$$

since $\langle u | P_iP_j | u\rangle \geq 0$ for all $0 < i,j \leq k$ then $\langle u | P_iP_j | u\rangle = 0$ thus $P_iP_j = 0$ for all $i \neq j$, hence $\mathbf{P}$ is a projector. $\square$

Chapter 6

Gleason Measures

As previously stated in the introduction, Gleason measures were defined as a direct consequence of the so-called Gleason's theorem. In this chapter, we now move on to present one of the central topics of this thesis; complex Gleason measures.

6.1 Complex Gleason Measures

Let H be a Hilbert space, $\mathcal{A} \subset \mathcal{L}(H)$ a C^*-algebra of bounded normal operators in H and $\mathcal{P}$ the set of orthogonal projectors in H. (A C^*-algebra is a Banach $*$-algebra $\mathcal{A}$ with the property $\|AA^*\| = \|A^*A\|$ for all $A \in \mathcal{A}$).

Definition 6.1. A *complex Gleason Measure* is a function $\mu : \mathcal{P} \to \mathbb{C}$ which is σ-additive on orthogonal families of projections in $\mathcal{P}$, i.e. if $(S_n)_{n\in\mathbb{N}}$ is a countable orthogonal family of subspaces of H with a closed liner span S then:

$$\mu(S) = \sum_{n\in\mathbb{N}} \mu(S_n)$$

Note that complex Gleason measures are defined in the space of orthogonal projectors; this is equivalent to defining complex Gleason measures from the subspaces $S \subseteq H$ to the complex, since the orthogonal projections ultimately act on a Hilbert space into a specific given subspace, that is $\mu : S \to \mathbb{C}$ such that S is a subspace of the Hilbert space H.

Definition 6.2. [[23] chapter VII, definition 2.E.1] Let $(X, \mathcal{M})$ be a measurable space (i.e.: $\mathcal{M}$ is a σ-algebra of subsets of X). A *spectral measure* E is a mapping $E : \mathcal{M} \to \mathcal{P}$ such that:

1. $E(U)$ is an orthogonal projector for every $U \in \mathcal{M}$.

2. $E(X) = I$, $E(\emptyset) = 0$

3. If $U = \bigcup_{n\in\mathbb{N}} U_n$ and the sets (U_n) are disjoint, then $E(U) = \sum_{n\in\mathbb{N}} E(U_n)$ (where the series is convergent in the strong operator topology).

4. If $U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots$ and $\bigcap_{n\in\mathbb{N}} U_n = \emptyset$, then $E(U_n) \to 0$ in the strong operator topology, i.e.: $E(U_n)x \to 0$ for all $x \in H$.

5. If $U = U_1 \cap U_2$, then $E(U) = E(U_1) \cdot E(U_2)$. In particular if U_1 and U_2 are disjoint, then $E(U_1)$ and $E(U_2)$ are orthogonal.

The usual techniques applied for complex-valued measures can be used to demonstrate many basic properties of spectral measures. Note that Property 3 of Definition 6.2 is obvious given Property 4. Moreover, it is easy to show that given $U \subset W$ then $E(U) \leq E(W)$. Lastly modularity is an interesting fact that spectral measures share with their complex counterparts, we first observe that $E(U \cup W) + E(U \cap W) = E(U) + E(W)$ from this and from the observation that $E(U)E(U \cup W) = E(U)$ we conclude Property 5.

Definition 6.3. Let $\mu : \mathcal{P} \to \mathbb{R}$ be a Gleason measure. μ is said to be *concentrated* on a subspace S_0 if $S \subset S_0^\perp$ ($S_0^\perp$ is the orthogonal complement, i.e. the set of all vectors of S_0 which are orthogonal to any other vector of H) implies that $\mu(S) = 0$. In terms, of projections, we can express the same idea by saying that μ is concentrated on a projector P_0 if for any projector $P \in \mathcal{P}(H)$ $P_0P = 0$ implies that $\mu(P) = 0$. We note this by $\mu \subset S_0$ or $\mu \subset P_0$. Furthermore, if the set $\{P \in \mathcal{P} : \mu(P) = 0\}$ has a greatest element, P_0, then $I - P_0$ is called the strong support of μ. Evidently, $\mu(P) = 0$ if and only if $P(I - P_0) = 0$ (see [19]).

Even though absolute continuity and mutual singularity were already presented; definitions 3.5 and 3.9 respectively, the notions of such concepts are introduced next for Gleason measures specifically.

Definition 6.4. Let $\lambda, \alpha : \mathcal{P} \to \mathbb{R}$ be two Gleason measures. The measure λ is said to be absolutely continuous with respect to α and we write $\lambda \ll \alpha$, if $\alpha(P) = 0$ implies $\lambda(P) = 0$. Two Gleason measures λ and α are said to be mutually singular (written $\lambda \perp \alpha$), if there exists an orthogonal decomposition $I = P_0 + Q_0$ with P_0, Q_0 orthogonal projections such that, $P_0Q_0 = Q_0P_0 = 0$ and $\lambda \subset P_0$, $\mu \subset Q_0$.

This chapter will now take a short deviation from Gleason measures to introduce an important class of bounded operators in order to define the so-called trace class operators, such operators will be used throughout this thesis.

Definition 6.5. For $1 \leq p < \infty$, we denote by L_p the class of bounded operators T which satisfy the following condition: for each orthonormal system $\{\varphi_k, k \in K\}$ in H, $\sum_{k\in K} |\langle T\varphi_k, \varphi_k\rangle|^p < \infty$.

As an important and curious detail, the class L_p yields a two sided ideal in $\mathcal{L}(H)$, thus it is contained in the ideal of compact operators (for details see[17]).

Remark 6.6. In order to define the trace of an operator A, we need the series:

$$Tr(T) = \sum_{k\in K} \langle T\varphi_k, \varphi_k\rangle$$

to be absolutely convergent. So it is natural to define the trace for operators in L_1. We call the operators in L_1, operators of *trace class*. Then if A is a trace class operator and B is bounded, it is easy to show that AB is also of trace class. Moreover, we have that:

$$|Tr(AB)| \leq \|B\| Tr(|A|).$$

where $\|B\|$ denotes the norm of the bounded operator B. Also recall that a *frame function* of weight w in H is a real-valued function f defined on the unit sphere of H such that if $\{\varphi_n\}$ is an orthonormal basis of H then:

$$\sum_n f(\varphi_n) = w.$$

We denote f_μ the frame function associated to μ ([2]).

As a direct result of Gleason's theorem 1.3.3, it is known that a real Gleason's measure μ verifying $|\mu(S)| \leq K$, $S \in \mathcal{S}$, (where $\mathcal{S}$ is the collection of subspaces in H) is represented by a self-adjoint operator ([18]). An analogous result holds for a complex Gleason measure:

Proposition 6.7. *(Gleason's theorem for vector valued measures) Let H be a separable Hilbert space, and $\mu : \mathcal{S} \to X$ a vector-valued measure. Assume that:*

$$|\mu(P_x)| \leq K \ \forall x \in H$$

with P_x the orthogonal projection on the one-dimensional subspace $S = \langle x \rangle$, or equivalently, that the frame function f_μ corresponding to μ, is bounded. Then, there exists a normal operator of trace class $\rho \in L_1$ such that:

$$\mu(P) = Tr(\rho P) \ \forall P \in \mathcal{P}$$

Proof. Without loss of generality, by decomposing the vector-valued measure into its components, one may assume that each component of μ is a real measure. Observe that a bounded frame function f, on the unit sphere S is regular, i.e., $f(x) = (Tx, x)$ with T a self-adjoint operator on $\mathbb{R}^3$: if $|f(x)| \leq M$, $f + M$ is a positive frame function. From Lemma 2.8 in [2], $f + M$ is continuous and then f is regular. Considering this result, in a similar way to [2], section 4.1, we obtain that μ is represented by ρ in $\mathcal{L}(H)$. Since for every orthonormal system $\{e_n, n \in N\}$, $\sum_n \langle \rho e_n, e_n \rangle < \infty$, and for every rearrangement $\{\varphi_k, k \in N\}$ of $\{e_n, n \in N\}$:

$$\sum_k \langle \rho\varphi_k, \varphi_k \rangle = \sum_n \langle \rho e_n, e_n \rangle$$

we get that:

$$\sum_n |\langle \rho e_n, e_n \rangle| < \infty.$$

In this case $\rho \in L_1$ ([17]), and in particular it is compact. It follows that μ is represented by an operator ρ in L_1. $\square$

From now on, a complex measure μ is bounded if the corresponding frame function is bounded, or equivalently, if it is bounded on the one-dimensional subspaces of H.

Remark 6.8. Let B and C be two positive, self-adjoint operators and let $A = B + iC$, then A is a normal operator. Moreover if B, $C \in L_1$ then $A \in L_1$.

Proof. Let $A = B + iC$ and $A^* = (B + iC)^* = B^* - iC^*$. Now

$$AA^* = (B + iC)(B^* - iC^*) = BB^* + CC^*$$

and

$$A^*A = (B^* - iC^*)(B + iC) = B^*B + C^*C$$

since both B and C are self-adjoint operators we obtain

$$BB^* + CC^* = B^*B + C^*C$$

thus $AA^* = A^*A$, i.e., A is a normal operator. $\square$

Furthermore the binomial form of a complex bounded measure μ can be written as $\mu = \lambda + i\nu$ where λ, ν are two positive bounded measures, in this case λ and ν would be represented by B and C, two positive hermitian operators of trace class, respectively. If we let $A = B + iC$ be an operator, then by the previous proposition A represents the measure μ and by the previous remark , we get a normal operator $A \in L_1$ that is bounded by a positive measure. For μ a real measure this result provides a decomposition in positive and negative parts.

Theorem 6.9. *Consider $\rho \in L_1$, normal and $\rho = |\rho|u$ the polar decomposition of ρ. Then*

$|\rho|$ defines a positive measure and

$$|Tr(\rho P)| \leq Tr(|\rho|P) \;\; P \in \mathcal{P}.$$

Proof. $Tr(\rho P)$ and $Tr(|\rho|P)$ are well defined since $\rho \in L_1$ [S]. $\rho = u|\rho| = |\rho|u$ with $|\rho| > 0$ and $u^* = u^{-1}$. Let a be the positive square root of $|\rho|$. Consider $\{e_i\}$ an orthonormal basis of H and $P \in \mathcal{P}$ then

$$|Tr(\rho P)| = |Tr(ua^2P^2)| = |Tr(PauaP)| \leq$$

$$\sum_i |\langle uaPe_i, aPe_i\rangle| \leq \sum_i \|aPe_i\|^2 = Tr(|\rho|P).$$

□

Remark 6.10. From theorem 6.9, a bounded complex Gleason measure μ is bounded by a positive measure. In this case, if μ is real the following decomposition is obtained

$$\mu = \mu^+ - \mu^-$$

with $\mu^+ = \frac{\mu_{\|\rho\|}+\mu}{2}$ and $\mu^- = \frac{\mu_{\|\rho\|}-\mu}{2}$.

Now let us consider a complex Gleason measure $\mu = \mu_1 + i\mu_2$. Without loss of generality we may assume both μ_1 and μ_2 to be positive measures. Now by Gleason's theorem we have

$$\mu_1(P) = Tr(AP),$$

and

$$\mu_2(P) = Tr(BP),$$

for density operators A and B. Therefore, using the linearity of trace operators we obtain

$$\mu(P) = Tr((A + iB)P).$$

Clearly the operator $A + iB$ is normal if $AB = BA$. To see this observe

$$(A + iB)(A^* - iB^*) = (A^* - iB^*)(A + iB).$$

Since the density operator is self-adjoint we have

$$(A + iB)(A - iB) = (A - iB)(A + iB).$$

This gives

$$i(BA - AB) = i(AB - BA).$$

Hence

$$AB = BA.$$

6.2 Some Examples of the Trace of an Operator

This section concentrates on analyzing the behaviour of the trace of operators acting on a Hilbert Space, this is done by presenting examples, taken from proposed exercises in [7].

Exercise 6.11. Let $\mid u\rangle$ and $\mid v\rangle$ be two vector of finite norm. Show that:

$$Tr\left(\mid u\rangle \langle u \mid\right) = \langle u \mid u\rangle$$

$$Tr\left(\mid u\rangle \langle v \mid\right) = \langle v \mid u\rangle$$

Proof. Let $\{|i\rangle\}$ be an orthonormal basis for the Hilbert space of the system. As a first

step it will be shown first that the trace of an operator A is given by:

$$Tr(A) = \sum_i \langle i \mid A \mid i \rangle$$

For simplicity assume the vector space is finite-dimensional. Now if A is a linear operator on a vector space V, and if $\{|i\rangle\}$ is a basis for V, then the matrix elements A_{ij} of the matrix given by A with respect to this basis are defined by it's action on this basis as follows:

$$A\,|i\rangle = \sum_j A_{ji}\,|j\rangle \tag{6.2.1}$$

The trace of the linear operator with respect to this basis is then defined as the sum of its diagonal entries:

$$Tr(A) = \sum_i A_{ii} \tag{6.2.2}$$

Now, it is known that the trace is a basis-independent value, so we can simply refer to the trace of the linear operator; it's just the trace with respect to any chosen basis. Suppose that V is equipped with an inner product, just as in the case of a Hilbert space and let $\{|i\rangle\}$ be an orthonormal basis for V, then it is possible to take the inner product of both sides of (6.2.1) with respect to an element $|k\rangle$ of the basis to obtain:

$$\langle k \mid A \mid i\rangle = \sum_j \langle k \mid A_{ji} \mid j\rangle = A_{ji}\,\langle k \mid j\rangle = \sum_j A_{ji}\delta_{kj} = A_{ki}$$

In other words $\langle k \mid A \mid j\rangle$ gives precisely the matrix elements A_{kj} of the matrix related to A in the given basis. In particular the diagonal entries are given by $\langle i \mid A \mid i\rangle$. Thus:

$$Tr(A) = \sum_i \langle i \mid A \mid i\rangle\,.$$

Now, let $\{|i\rangle\}$ be an orthonormal basis for the Hilbert space H of the system. For a

given state $|u\rangle$ define P_U by:

$$P_U\,|v\rangle = \langle u \mid v\rangle\,|u\rangle \quad \text{or}$$

$$P_U = |u\rangle\,\langle u|$$

now:

$$Tr(P_U) = Tr(|u\rangle\,\langle u|) = \sum_i \langle i \mid P_U \mid i\rangle$$

$$= \sum_i \langle i \mid \langle u \mid i\rangle \mid u\rangle$$

$$= \sum_i \langle i \mid u\rangle\,\langle u \mid i\rangle$$

$$= \langle u \mid u\rangle$$

A similar result holds if we let $P_v\,|k\rangle = \langle v \mid k\rangle\,|u\rangle$ then $Tr(|u\rangle\,\langle v|) = \langle v \mid u\rangle$. □

Exercise 6.12. Let H be a positve definite, Hermitian operator. Show that for any $|u\rangle$ and $|v\rangle$:

$$|\langle u \mid H \mid v\rangle|^2 \leq \langle u \mid H \mid u\rangle\,\langle v \mid H \mid v\rangle$$

and that equality $\langle u \mid H \mid v\rangle = 0$ implies $H\,|u\rangle = 0$. Show also that $Tr(H) \geq 0$ and that equality implies $H = 0$.

Proof. Set $H\,|\alpha\rangle = H(|u\rangle + \lambda\,|v\rangle)$ where $\lambda \in \mathbb{C}$. Now $\langle \alpha \mid H \mid \alpha\rangle \geq 0$ since H is positive definite. Furthermore:

$$\langle \alpha \mid H \mid \alpha\rangle = \langle u \mid H \mid u\rangle + \lambda\,\langle v \mid H \mid v\rangle + \lambda^*\,\langle v \mid H \mid u\rangle^* + |\lambda|^2\,\langle v \mid H \mid v\rangle \geq 0$$

This is true for all kets $|u\rangle$ and $|v\rangle$, and all complex numbers λ. Since the inequality is obviously true if $|v\rangle = 0$, consider the case $|v\rangle \neq 0$ and let:

$$\lambda = \frac{-\,\langle v \mid H \mid u\rangle}{\langle v \mid H \mid v\rangle}$$

and then:

$$\langle u \mid H \mid u \rangle - \frac{|\langle v \mid H \mid u \rangle|^2}{\langle v \mid H \mid v \rangle} \geq 0$$

and rearranging the terms:

$$|\langle v \mid H \mid u \rangle|^2 \leq \langle u \mid H \mid u \rangle \langle v \mid H \mid v \rangle .$$

Now clearly if $\langle u \mid H \mid v \rangle = 0$ then $H|u\rangle = 0$. Moreover:

$$Tr(H) = \sum_i \langle i \mid H \mid i \rangle$$

where $\{|i\rangle\}$ is an orthonormal basis for the Hilbert space of the system. It is known that

$$\langle i \mid H \mid i \rangle \geq 0$$

for all $|i\rangle$ since H is positive definite, thus $Tr(H) \geq 0$. If $Tr(H) = 0$ then clearly $H|i\rangle = 0$ and $H = 0$. □

Exercise 6.13. Show that if H and K are two positive definite observables, $Tr(HK) \geq 0$ and that equality implies $HK = 0$.

Proof. Let M be the positive definite square root of H and N be the positive definite square root of K, then:

$$Tr(HK) = Tr(MMNN)$$

by cycling property of the trace, then:

$$Tr(MMNN) = Tr(NMMN) = Tr((MN)^*MN) \geq 0$$

since $\langle MN, MN \rangle \geq 0$. Now if $Tr(HK) = 0$ hence $\langle MN, MN \rangle = 0$ thus $(MN)^2 = 0$ and finally $HK = 0$. □

The following problem closes this section and the chapter.

Exercise 6.14. A being a linear operator, show A^+A is positive definite Hermitian operator and that its trace is equal to the sum of the squares of the modulii of the matrix elements representing A in an arbitrarily chosen representation. Deduce that $Tr(A^+A) \geq 0$ and that $Tr(A^+A) = 0$ implies $A = 0$.

Proof. It is known that $\langle A, B\rangle = Tr(A^+B)$. Let $\{|i\rangle\}$be an orthonormal basis of the representation $\{Q\}$. Let A_Q and A_Q^T be the matrix representation of A and A^+ respectively, in the representation of any observable Q, now

$$Tr\left(A_Q^T A_Q\right) = \sum_i \left(A_Q^T A_Q\right)_{ii} = \sum_i \sum_j \left(A_Q^T\right)_{ij} \left(A_Q\right)_{ji} = \sum_i \sum_j A_{ij}^2$$

thus

$$Tr(A^+A) = \sum_i \sum_j A_{ij}^2$$

clearly then $Tr(A^+A) \geq 0$. Furthermore, given $Tr(A^+A) = 0$ then $A_{ij}^2 = 0$ for all $1 \leq i, j \leq n$ hence $A = 0$, also $Tr(A^+A) = \langle A, A\rangle = 0$ immediately implies $A = 0$. □

Chapter 7

Integrals with Respect to a Complex Gleason Measure

Since one of the main objectives is to develop a functional calculus that permits working with complex Gleason measures, it is natural to follow the steps followed by Lebesgue in his journey of creating a new functional calculus related to Lebesgue measures. Moreover, considering this historic approach, the notion of integral of an operator with respect to a Gleason measure is required. To motivate this notion, following [20] section 3, consider a self-adjoint operator that is a finite linear combination of projectors:

$$A = \sum_{i=1}^{n} \lambda_i P_i,$$

where $P_i \in \mathcal{P}$ and $P_i P_j = 0$ if $i \neq j$.

In analogy with standard Measure Theory, call these operators simple operators. Then it is natural to define the integral of a simple operator with respect to μ by:

$$\int A \, d\mu = \sum_{i=1}^{n} \lambda_i \mu(P_i). \tag{7.0.1}$$

We shall extend this notion of integral to the class of self-adjoint bounded operators. To do that refer to the spectral theorem, which broadly states the conditions under which an operator can be diagonalized (that is represented as a diagonal matrix in some basis, specifically in an orthonormal basis) in the following formulation:

Definition 7.1. A mapping E from $\mathcal{B}(\mathbb{R})$ i.e. the Borel sets of $\mathbb{R}$ into the set $\mathcal{P}$ of all

projector of a Hilbert space H is a *spectral measure* if:

1. $E(\mathbb{R}) = I$

2. $E\left(\bigcup_{n=1}^{\infty} M_n\right) = \sum_{n=1}^{\infty} E(M_n)$ whenever $M_n \cap M_k = \emptyset$ for $n \neq k$, $M_n \in \mathcal{B}(\mathbb{R})$

Furthermore, if M and N are disjoint subsets of $\mathcal{B}(\mathbb{R})$, then $E(M)$ and $E(N)$ are mutually orthogoanl projectors. Given this it is clear that if $M \subseteq N$, $M, N \in \mathcal{B}(\mathbb{R})$, then $E(M) \leq E(N)$, and $E(N \setminus M) = E(N) - E(M)$. Moreover, for all M and N in $\mathcal{B}(\mathbb{R})$ we have $E(M) \leftrightarrow E(N)$ (i.e. $E(M)E(N) = E(N)E(M)$), $E(M \cap N) = E(M) \cdot E(N)$, and $E(M \cup N) + E(M \cap N) = E(M) + E(N)$. In addition recall that for all vectors x and y in H the mapping $\mu_{xy}(M) = \langle E(M)x, y\rangle$, $M \in \mathcal{B}(\mathbb{R})$, is a real-valued (or complex-valued) countably additive signed measure on $\mathcal{B}(\mathbb{R})$.

Theorem 7.2. *To each (possibly unbounded) self-adjoint operator A in a Hilbert space H corresponds a unique spectral measure $E = E_A$ (defined on the Borel sets of $\mathbb{R}$) such that:*

1. .
$$A = \int_{-\infty}^{\infty} \lambda d\mu$$
for $\lambda \in \mathbb{R}^1$, in the sense that
$$Ax = \lim_{n\to\infty} \int_{-n}^{n} \lambda \, dE(\lambda)x$$
and the domain $D(A)$ of A consists of all the vectors x for which
$$D(A) = \left\{x \in H : \int_{-\infty}^{\infty} \lambda^2 d\langle E(\lambda)x, x\rangle < \infty\right\}$$

2. *For each Borel set $U \subset \mathbb{R}$, $E(U)$ commutes with any bounded operator that commutes with A, and:*
$$E(U)A = \int_U \lambda \, dE$$

3. *For any real Borel measurable function* $f(\lambda)$*:*

$$f(A) = \int_{-\infty}^{\infty} f(\lambda)\, dE$$

with

$$D(f(A)) = \{x \in H : \int_{-\infty}^{\infty} |f(\lambda)|^2 \langle E(d\lambda)x, x\rangle < \infty\}$$

4. *The spectral measure* E *is supported in the spectrum* $\sigma(A)$ *of* A*, i.e., for every Borel set* $U \subset \mathbb{R}$*,* $E(U) = E(U \cap \sigma(A))$*.*

Proof. Due to its lengthiness, the proof for the latter theorem will not be included but the reader may refer to [13] (Section 6, page 250). □

In the case of a simple operator, the spectral measure E_A is given by:

$$E_A(U) = \sum_{i:\lambda_i \in U} P_i$$

Consider the measure $\mu \circ E_A$ on the Borel sets of $\mathbb{R}$, then if A is a simple operator according to (7.0.1):

$$\int A\, d\mu = \int_{-\infty}^{\infty} \lambda\, d(\mu \circ E_A) \tag{7.0.2}$$

So for any self-adjoint operator A define $\int A\, d\mu$ using equation (7.0.2). In a similar way, using more general versions of the spectral theorem, it is possible to define the integral $\int A\, d\mu$ when A is a normal operator.

Remark 7.3. Following F. Riesz and Sz. Nagy ([14], section 130), if A and B are two permutable self-adjoint operators, it follows that A and B are permutable if $E_A(U)$ and $E_B(V)$ are permutable for any measurable sets $U, V \subset \mathbb{R}$, with E_A, E_B the spectral measures associated to A, B. In that case there exists a spectral measure $E_{A,B}$ defined for the Borel subsets of $\mathbb{R}^2$, such that:

$$E_{A,B}(U \times V) = E_A(U)E_B(V)$$

for all "measurable rectangles". Furthermore:

$$A = \int\int_{\mathbb{R}^2} \lambda_1 \, dE_{A,B}(\lambda_1, \lambda_2)$$

$$B = \int\int_{\mathbb{R}^2} \lambda_2 \, dE_{A,B}(\lambda_1, \lambda_2)$$

Proposition 7.4. *If A and B are permutable self-adjoint operators:*

$$\int (A+B) d\mu = \int A \, d\mu + \int B \, d\mu \tag{7.0.3}$$

Proof. Let $E_{A,B}$ be the spectral measure on $\mathbb{R}^2$ associated to the pair (A, B). Then:

$$\int (A+B) \, d\mu = \int\int (\lambda_1 + \lambda_2) \, d(\mu \circ E_{A,B})(\lambda_1, \lambda_2) =$$

$$= \int\int \lambda_1 \, d(\mu \circ E_{A,B})(\lambda_1, \lambda_2) + \int\int \lambda_2 \, d(\mu \circ E_{A,B})(\lambda_1, \lambda_2)$$

$$= \int \lambda_1 d(\mu \circ E_A)(\lambda_1) + \int \lambda_2 d(\mu \circ E_B)(\lambda_2) = \int A \, d\mu + \int B \, d\mu$$

□

Remark 7.5. This property does not hold if A and B are not permutable, as can be seen from the following example: We consider the Hilbert space $H = \mathbb{R}^2$, and denote by S_θ the 1-dimensional subspace generated by the vector $(\cos\theta, \sin\theta)$. Given a function $f : [0, \pi/2) \to [0, 1]$ we can define a Gleason measure μ in H by:

$$\mu(S_\theta) = \begin{cases} f(\theta) & \text{if} \quad 0 \leq \theta < \pi/2 \\ 1 - f(\theta - \frac{\pi}{2}) & \text{if} \quad \pi/2 \leq \theta < \pi \end{cases}$$

and $\mu(0) = 0$, $\mu(H) = 1$. If we take A to be the projection onto S_0 and B to be the projection onto $S_{\pi/4}$, it can be easily seen that (7.0.3) does not hold for general f.

Remark 7.6. Let $(X, \mathcal{M})$ be a measurable space. If $E : \mathcal{M} \to P(H)$ is a spectral measure, and:

$$A = \int_X \lambda dE$$

it follows that:

$$\langle Ax, y \rangle = \int \lambda dE[x, y] \tag{7.0.4}$$

where:

$$E[x, y](U) = \langle E(U)x, y \rangle$$

for all $x, y \in H$, $\lambda \in \mathbb{R}$.

Proposition 7.7. *Let $\mu : \mathcal{S} \to \mathbb{R}$ be a Gleason measure, and assume that μ is represented by the trace-class operator ρ, i.e:*

$$\mu(S) = Tr(\rho P_S) \;\; \forall S \in \mathcal{S}$$

Then for any (not necessarily bounded) μ-integrable self-adjoint operator A in H we have that:

$$\int A \, d\mu = Tr(\rho A)$$

Proof. Let $(e_i)_{i \in I}$ be an orthogonal basis of H such that $e_i \in D(A) \, \forall i \in I$ (this can be done since $D(A)$ is dense in H). Then $\square$

$$Tr(\rho A) = \sum_{i \in I} \langle \rho A e_i, e_i \rangle = \sum_{i \in I} \langle A e_i, \rho e_i \rangle .$$

From (7.0.4), we have that:

$$\langle Ae_i, \rho e_i \rangle = \int \lambda dE_A[e_i, \rho e_i]$$

where E_A is the spectral measure associated with A. Hence:

$$Tr(\rho A) = \int \lambda \, d\left(\sum_{i \in I} E_A[e_i, \rho e_i]\right).$$

On the other hand:

$$\left(\sum_{i \in I} E_A[e_i, \rho e_i]\right)(U) = \sum_{i \in I} E_A[e_i, \rho e_i](U) = \sum_{i \in I} \langle E_A(U) e_i, \rho e_i \rangle =$$

$$= \sum_{i \in I} \langle \rho E_A(U) e_i, e_i \rangle = Tr(\rho E_A(U)) = \mu(E_A(U)).$$

It follows that:

$$Tr(\rho A) = \int \lambda \, d(\mu \circ E_A) = \int A \, d\mu$$

In order to justify these formal computations, we may assume first that A is a positive operator, and then for the general case, we use the decomposition $A = A^+ - A^-$.

Remark 7.8. It follows that when the Gleason measure μ is represented by a trace class operator, the linearity property (7.0.3) holds for any operators A, B (even if they are not permutable).

Proof. This is a direct consequence of the properties of the trace. □

Lemma 7.9. *Let μ be a finite non-negative Gleason measure and A a bounded self-adjoint operator. Then if:*

$$m = m(A) = \inf_{\|x\|=1} \langle Ax, x \rangle$$

$$M = M(A) = \sup_{\|x\|=1} \langle Ax, x \rangle$$

are the lower and upper bounds of A, thus the following result is obtained:

$$m(A)\mu(I) \leq \int A\, d\mu \leq M(A)\mu(I).$$

In particular:

$$\left| \int A d\mu \right| \leq \|A\|\, \mu(I)$$

Proof. Let E_A the spectral measure associated with A. Then E_A is concentrated in the interval $[m, M]$. It follows that:

$$\int A\, d\mu = \int_m^M \lambda\, d(\mu \circ E_A)$$

and hence: □

$$\mu(E_A([m, M]))m \leq \int_m^M \lambda d(\mu \circ E_A) \leq \mu(E_A([m, M]))M$$

since $\mu \circ E_A$ is a non negative Lebesgue-Stieltjes measure. We observe that:

$$\mu(E_A([m, M])) = \mu(I)$$

and the result follows.

Definition 7.10. Let μ be a (non negative) Gleason measure and A a self-adjoint operator. We say that $A = 0$ a.e. with respect to μ if $\mu(\mathrm{Ker}(A)^{\perp}) = 0$.

Lemma 7.11. *If $A = 0$ a.e. with respect to μ, then $\int A\, d\mu = 0$*

Proof. Let E_A the spectral measure associated with A, then

$$\mathrm{Ker}(A) = \{x \in H : Ax = 0\} = E_A(\{0\})$$

Hence

$$\mathrm{Ker}(A)^{\perp} = E_A(\{x \in \mathbb{R} : x \neq 0\})$$

It follows that:

$$\int A \, d\mu = \int_{-\infty}^{\infty} \lambda \, d(\mu \circ E_A) = \int_{\{x \neq 0\}} \lambda \, d(\mu \circ E_A) = 0$$

since $\mu \circ E_A(\{x \neq 0\}) = 0$. $\square$

The generalization of all these results for complex Gleason measure can be done as follows. Consider operators A_1 and A_2, then all the above results can be written for both A_1 and A_2. Furthermore, without loss of generality, assume that both A_1 and A_2 are positive. Next consider the operator $A = A_1 + iA_2$. Then all the above results can be written for A.

Chapter 8

Some Results on Complex Gleason Measures

This chapter contains some of the major theorems of Measure Theory, such as the so-called Lebesgue Decomposition and Radon-Nykodym Theorem, rewritten in the context of the previously mentioned Gleason measures in particular complex Gleason measures.

8.1 Lebesgue Decomposition with respect to a Representable Measure

In this section, we present a different approach for obtaining a Lebesgue decomposition for complex Gleason measures, that applies when the measure μ is a representable measure. Following [20, Sec. 5] we get:

Theorem 8.1. *Let μ, λ be two Gleason measures defined on a Hilbert space H and assume that μ is represented by a positive trace class operator ρ_1. Then, there exist two Gleason measures λ_a and λ_s such that*

$$\lambda(P) = \lambda_a(P) + \lambda_s(P) \tag{8.1.1}$$

for any projector that commutes with the projector P_R onto the range $R(\rho_1)$ of ρ_1, $\lambda_a \perp \lambda_s$, $\lambda_a \ll \mu$ and λ_s is mutually singular with respect to μ. Moreover, if λ is also a representable measure, this decomposition holds for any $P \in \mathcal{P}(H)$.

Proof. Define the required measures by:

$$\lambda_a(P) = \lambda(P_R P)$$

$$\lambda_s(P) = \lambda((I - P_R)P)$$

Since P commutes with P_R, $P_R P$ is a projector. Moreover $P_R P$ and $(I - P_R)P$ are orthogonal, so that (8.1.1) holds. If a subspace S can be written as an orthogonal direct sum:

$$S = \sum_{n \in \mathbb{N}} S_n$$

then if P_S is the projector onto S, and P_{S_n} the projector onto S_n, then $P_R P_S$ may be represented by the orthogonal decomposition:

$$P_R P_S = \sum_{n \in \mathbb{N}} P_R P_{S_n}$$

hence:

$$\lambda_a(S) = \lambda(P_R P_S) = \sum_{n \in \mathbb{N}} \lambda(P_R P_{S_n}) = \sum_{n \in \mathbb{N}} \lambda_a(S_n).$$

It is clear from the definition of λ_a and λ_s that $\lambda_a \subset P_R$, whereas $\lambda_s \subset P_R^{\perp}$. It follows from Definition 6.4, that $\lambda_a \perp \lambda_s$. The proof utilizes the claim that if $\mu(P) = 0$, then $P_R P = 0$, hence $\lambda_a \ll \mu$:

Proof. (of claim) If $P_R P \neq 0$ then, there exists $x \in R(P)$ such that $P_R x \neq 0$, hence $\rho_1 P_R x = \rho_1 x \neq 0$, since $H = R(\rho_1) \oplus Ker(\rho_1)$. From $\langle \rho_1 x, x \rangle = \|\rho_1^{1/2} x\|^2$, with $\rho_1^{1/2}$ the positive square root of ρ_1, it is deduced then that $\langle \rho_1 x, x \rangle \neq 0$. Therefore, $\mu(P) = Tr(\rho_1 P) \neq 0$. □

It remains to show that λ_s is singular with respect to μ. But is clear from the definitions that $\lambda_s \subset I - P_R$, whereas $\mu \subset P_R$, and since P_R and $I - P_R$ give an orthogonal decomposition of the identity, this shows that $\mu \perp \lambda_s$. □

In the case in which λ is also a representable measure, represented by a trace class operator ρ_2, we can define:

$$\lambda_a(P) = Tr(\rho_2 P_R P)$$

$$\lambda_s(P) = Tr(\rho_2 (I - P_R) P)$$

and since the trace is a linear operator, in that case (8.1.1) holds for any projector P. This result can be generalized for complex case. Observe that any complex Gleason measure can be written as $\lambda = \lambda_1 + i\lambda_2$, where λ_1 and λ_2 are real Gleason measures. Thus $\lambda_a = \lambda_{a_1} + i\lambda_{a_2}$ and $\lambda_s = \lambda_{s_1} + i\lambda_{s_2}$. So the above theorem holds for a complex Gleason measure as well.

8.2 A version of the Radon-Nikodym Theorem for Complex Gleason measures

This section will proceed in a similar way as in [20], but first will present some results from [20]. Let A be a normal operator, μ a positive Gleason measure and define:

$$\lambda_A(S) = \int A_{|S} d\mu = \int_S A\, d\mu$$

where $A_{|S}$ is the operator AP_S, then λ_A is a Gleason measure on the set of A-invariant subspaces (with the identification of S with P_S we may think it as the set of projectors such that $P_S A \subset AP_S$). In fact, if $S = \bigoplus_{n\in\mathbb{N}} S_n$, then

$$P_S = \sum_{n\in\mathbb{N}} P_{S_n}$$

and using Proposition 7.4 we see that

$$\int_S A\, d\mu = \sum_{n\in\mathbb{N}} \int_{S_n} A\, d\mu$$

since AP_{S_i} and AP_{S_j} commute, since S_i, S_j are A-invariant subspaces.

In the special case that μ is a Gleason measure represented by a trace-class operator, we consider λ_A to be defined for all closed subspaces of H, since as observed before, in that case the linearity property (7.0.3) holds without restrictions.

Lemma 8.2. *λ_A is absolutely continuous with respect to μ.*

Proof. Let S be a closed subspace such that $\mu(S) = 0$. Then it will be shown that $AP_S = 0$ a.e. with respect to μ; in fact if $x \in S^{\perp}$ then $AP_S x = 0$. It follows that

$$S^{\perp} \subset \mathrm{Ker}(AP_S)$$

and then

$$\mathrm{Ker}(AP_S)^{\perp} \subset S$$

Since $\mu(S) = 0$, it follows that $\mu(\mathrm{Ker}(AP_S)^{\perp}) = 0$. This means that $AP_S = 0$ a.e. with respect to μ, then $\int_S A\, d\mu = 0$. □

Now suppose that two Gleason measures λ, μ are given, such that λ is absolutely continuous with respect to μ. It is natural to ask if $\lambda = \lambda_A$ for some self-adjoint operator A. Now we recall the following results in [20].

Lemma 8.3. *Assume that λ and μ are Gleason measures represented by the trace-class operators ρ_1 and ρ_2 respectively, then $\lambda \ll \mu$ if and only if* $\mathrm{Ker}(\rho_2) \subset \mathrm{Ker}(\rho_1)$

Proof. Refer to [20] for the complete proof. □

Theorem 8.4. *(Radon-Nykodym theorem for complex Gleason measures) Let λ, μ be two positive representable Gleason measures, and ρ_1, ρ_2 be their respective density operators, (see [7],[8]) so that*

$$\lambda(S) = Tr(\rho_1 P_S)$$

$$\mu(S) = Tr(\rho_2 P_S)$$

with ρ_1, ρ_2 positive operators.

Assume that $\lambda \ll \mu$. Then there exists a (non necessarily bounded) self-adjoint operator A such that

$$\lambda(T) = \int AP_T d\mu$$

for any closed subspace T of H.

Remark 8.5. Since μ is by hypothesis a Gleason measure represented by a trace-class operator, λ_A is defined for any closed subspace.

Proof. For a complete proof of Theorem 8.4 look at [20]. □

Remark 8.6. Under the assumptions of the lemma, λ and μ are positive Gleason measures. A similar result holds if λ is assumed to be a complex Gleason measure (let $\lambda = \psi + i\nu$ where ψ and ν are positive representable Gleason measures), represented by a normal operator ρ_2. In that case, let B and C be self-adjoint operators such that $\psi(T) = \int BP_T d\mu$ and $\nu(T) = \int CP_T d\mu$ following the same proof the following is obtained:

$$\lambda(T) = \psi(T) + i\nu(T) = \int BP_T d\mu + i \int CP_T d\mu = \int (B + iC)P_T d\mu$$

Moreover if $A = B + iC$, the result is that A is a normal operator (see [24], XII.9.10 for the notion of normal unbounded operator). The following example is directly taken from [10]

Remark 8.7. Gleason theorem can be seen as a version of the Radon-Nikodym theorem. In fact, consider the Gleason measure Δ given by:

$$\Delta(S) = \mathrm{dim(S)}$$

It is clear that Δ is a non-negative Gleason measure (though it may take the value $+\infty$). Then if $A = \sum \lambda_i P_{S_i}$ is a simple self-adjoint operator,

$$\int A\, d\Delta = \sum \lambda_i \mathrm{dim(S_i)} = \mathrm{Tr(A)}.$$

This identity holds also for any operator A of trace class (i.e., Δ- integrable). If μ is another Gleason measure, it is clear that μ is absolutely continuous with respect to Δ since $\mu(\{0\}) = 0$. Gleason theorem says that there exists a self-adjoint operator ρ such that

$$\mu(S) = \int \rho P_S \, d\Delta$$

for any closed subspace S. The condition that H should be separable means that Δ should be a σ-finite Gleason measure (hypothesis of the usual Radon-Nikodym theorem). Furthermore as done previously this result can be extended to a vector-valued measure and, in particular, to a complex Gleason measure. For this particular case it would be enough to represent the complex Gleason measure in binomial form with two non-negative real Gleason measures and continue in a similar fashion as in Remark 6.6 for each measure, then this would yield that the operator ρ is normal.

A Gleason's measure on a family of commuting orthogonal projections behaves as an ordinary measure on σ-algebras, in the sense that it is possible to define a variation.

Chapter 9

The Nemytsky Operator

This chapter intends to introduce the Nemytsky Operator, one of the central concepts of the results that this thesis intends to render. As mentioned in the introduction the so-called Nemytsky operator is a variable-coefficient composition operator of the form $\varphi(x) \to g(x, \varphi(x))$.

9.1 Vector-Valued Nemytsky Operator

This section focuses on presenting some results and definitions from [10]. We take (S, Σ, μ) to be a complete σ-finite measure space. In this chapter X and Y indicate real Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively.

Definition 9.1. A function $g : S \times X \to Y$ is called a *vector-valued N-function* if it satisfies the conditions:

1. For each $u \in X$, the function $x \to g(x, u)$ is Y-measurable.
2. There is a μ-null set A such that for each $x \in S \backslash A$, the function $u \to g(x, u)$ is continuous.

It is therefore clear that, $B^p(X)$ is the space of X-measurable functions $f : S \to X$ for which the function $x \to \|f(x)\|_X \in L^p$.

In the scalar case, conditions 1. and 2. in Definition 9.1 are referred to as the *Caratheodory's conditions* (see [25], p. 20 and references therein). For example:

$$g(x, u) = \sum_{i=1}^{n} a_i(x) \|u - b_i(x)\|_X + T(u) \tag{9.1.1}$$

is a vector valued N-function when $a_i \in B^{\infty}(Y)$, $b_i \in B^1(X)$ and $T \in L(X,Y)$, the space of linear and continuous operators from X into Y. We call the N-function given by (9.1.1) a *vector-valued piecewise linear N-function*, by analogy with the case of scalar piece-wise linear N-functions (see [9], Definition 18).

Let:

$$\mathcal{M}_X = \left\{ \begin{array}{c} m : \Sigma \to X : \text{ vector valued measure satisfying} \\ \text{conditions 2a., 2b. and 2c. in Theorem 3.12} \end{array} \right\}.$$

The map $\Lambda : B^1(X) \to \mathcal{M}_X$ defined as $\Lambda(f) = f d\mu$ is an isomorphism of real vector spaces. Moreover we have the following result:

Proposition 9.2. *([11], p. 174, Proposition 10) For $f \in B^1(X)$:*

$$|f d\mu| = \|f\| \, d\mu,$$

where $\|f\|$ denotes the scalar function $x \to \|f(x)\|_X$. As a consequence, $\mathcal{M}_X$, with the variation norm $\|m\| = |m|(S)$, is a Banach space and Λ becomes an isometric isomorphism.

Proposition 9.3. *Given an N-function $g : S \times X \to Y$ and given an X-measurable function $f : S \to X$, the function $x \to g(x, f(x))$ from S into Y is Y-measurable.*

Proof. The following proof is taken from [10]. By definition of X-measurability, there exists a sequence $\{\varphi_k\}$ of step functions such that $\varphi_k \to f$ in X μ-a.e. as $k \to \infty$. It will be proven first that the function $x \to g(x, \varphi_k(x))$ is Y-measurable for each $k \geq 1$. In other words, it will be shown that given any step function $\varphi = \sum_n e_n \mathcal{X}_{M_n}$, $e_n \in X$, the function $x \to g(x, \varphi(x))$ is Y-measurable. According to the definition of the function φ:

$$g(x, \varphi(x)) = \begin{cases} g(x, 0) & if \quad x \in S \setminus \bigcup_n M_n \\ g(x, e_n) & if \quad\quad x \in M_n \end{cases}$$

Since g is an N-function, the function $x \to g(x, \varphi(x))$ is then Y-measurable on each of the sets $S \setminus \bigcup_n M_n$ and M_n for $n \geq 1$. Lemma 2.9 implies that $g(x, \varphi(x))$ is X-measurable, so

$g(x, \varphi_n(x))$ is X-measurable for each $k \geq 1$. Since $\varphi_k \to f$ in X μ-a.e. as $k \to \infty$, there is a μ-null set A such that for each $x \in S \setminus A$, $\varphi_k(x) \to f(x)$ in X. Moreover, there exists a μ-null set B so that for each $x \in S \setminus B$, the function $u \to g(x, u)$ from X to Y is continuous. So, for $x \in S \setminus (A \cup B)$, the sequence $\{g(x, \varphi_k(x))\}$ converges to $g(x, f(x))$ in Y. Finally, Proposition 2.10 implies that $x \to g(x, f(x))$ is a Y-measurable function, completing the proof. $\square$

Definition 9.4. Given a vector-valued N-function g, the *Nemytsky operator* $N_g(f)$ is defined for an X-measurable function f as:

$$N_g(f)(x) = g(x, f(x)). \tag{9.1.2}$$

It is clear that Proposition 9.3 implies that the Nemytsky operator N_g maps X-measurable functions to Y-measurable functions.

Remark 9.5. It is known ([26], p. 155, [25], p. 20), that (9.1.2) defines in the scalar case a continuous and bounded operator from L^1 into itself if and only if there exist a function $a \in L^1$ and a number $b \geq 0$ such that:

$$|g(x, u)| \leq a(x) + b|u|, \tag{9.1.3}$$

μ-a.e. in S. A similar proof of this result can be established for the vector valued case, if we write (9.1.3) in the form:

$$\|g(x, u)\|_Y \leq a(x) + b\|u\|_X, \tag{9.1.4}$$

μ-a.e. in S, with $a \in L^1$ and $b \geq 0$. Thus, for a vector-valued N-function g satisfying (9.1.4), the Nemytsky operator N_g is a bounded and continuous operator from $B^1(X)$ into $B^1(Y)$.

Proposition 9.6. *If g is a vector valued N-function satisfying (9.1.4), there exists a unique operator $\overline{N}_g : \mathcal{M}_X \to \mathcal{M}_Y$ so that the diagram:*

$$\begin{array}{ccc} B^1(X) & \overset{N_g}{\to} & B^1(Y) \\ \Lambda \downarrow & & \downarrow \Lambda \\ \mathcal{M}_X & \overset{\overline{N}_g}{\to} & \mathcal{M}_Y \end{array}$$

is commutative. That is to say:

$$\Lambda \circ N_g = \overline{N}_g \circ \Lambda$$

on $B^1(X)$.

Proof. As in the scalar case, (see [9], Proposition 15), propose:

$$\overline{N}_g (f d\mu) = g(\cdot, f(\cdot))\, d\mu. \tag{9.1.5}$$

The Radon-Nikodym theorem determines the function f in (9.1.5) μ-almost everywhere. However, if $h = f$ μ-a.e., there is a μ-null set A so that $h(x) = f(x)$ for $x \in S \backslash A$. Then, for any $E \in \Sigma$:

$$\begin{aligned} \overline{N}_g (f d\mu)(E) &= \int_E g(\cdot, f(\cdot))\, d\mu \\ &= \int_{(S\backslash A)\cap E} g(\cdot, h(\cdot))\, d\mu = \overline{N}_g (h d\mu)(E). \end{aligned}$$

This observation and the properties of the Nemytsky operator N_g, imply that the operator $\overline{N}_g$ is well defined. Moreover:

$$\begin{aligned} \Lambda \circ N_g (f) &= g(\cdot, f(\cdot))\, d\mu \\ &= \overline{N}_g (f d\mu) = \overline{N}_g \circ \Lambda (f), \end{aligned}$$

for every $f \in B^1(X)$. Finally, the operator $\overline{N}_g$ is unique, for if there is another operator,

say $\widetilde{N}_g : \mathcal{M}_X \to \mathcal{M}_Y$, that also makes the diagram commutative, then the following is obtained:

$$\begin{aligned}\widetilde{N}_g(fd\mu) &= \widetilde{N}_g \circ \Lambda(f) \\ &= \Lambda \circ N_g(f) = \overline{N}_g \circ \Lambda(f) \\ &= \overline{N}_g(fd\mu),\end{aligned}$$

for every $f \in B^1(X)$, or $\widetilde{N}_g = \overline{N}_g$ on $\mathcal{M}_X$. This completes the proof of the proposition. □

With all these results, now define the map $g \to \overline{N}_g$ as a functional calculus. Denote $\mathcal{N}$ the family of vector valued N-functions $g : S \times X \to X$ satisfying the condition

$$\|g(x,u)\|_X \leq a(x) + b\,\|u\|_X, \tag{9.1.6}$$

μ-a.e. in S, with $a \in L^1$ and $b \geq 0$. Also define:

$$\overline{\mathcal{N}} = \left\{\overline{N}_g : g \in \mathcal{N}\right\}.$$

Proposition 9.7.

1. The spaces $\mathcal{N}$ and $\overline{\mathcal{N}}$ are real vector spaces and the map $g \to \overline{N}_g$ from $\mathcal{N}$ into $\overline{\mathcal{N}}$ is linear.

2. $B^1(X) \subseteq \mathcal{N}$, in the sense that every function $g \in B^1(X)$ defines an N-function $g(x)$ that satisfies (9.1.6). Moreover, given $g \in B^1(X)$:

$$\overline{N}_g(fd\mu) = gd\mu.$$

3. The space $\mathcal{N}$ is closed under the composition operation:

$$(g_1 \circ g_2)(x,u) = g_1(x, g_2(x,u))$$

and:

$$\overline{N}_{g_1 \circ g_2} = \overline{N}_{g_1} \circ \overline{N}_{g_2}. \tag{9.1.7}$$

Proof. Clearly the spaces $\mathcal{N}$ and $\overline{\mathcal{N}}$ are real vector spaces. Moreover, if $\alpha, \beta \in \mathbb{R}$ and $g_1, g_2 \in \mathcal{N}$:

$$\begin{aligned} \overline{N}_{\alpha g_1 + \beta g_2}(f d\mu) &= [\alpha g_1(\cdot, f(\cdot)) + \beta g_2(\cdot, f(\cdot))] \, d\mu \\ &= \alpha \overline{N}_{g_1}(f d\mu) + \beta \overline{N}_{g_2}(f d\mu). \end{aligned}$$

It is clear that the Bochner integrable functions define N-functions in $\mathcal{N}$. Moreover, given $g \in B^1(X)$ and $f \in B^1(X)$:

$$N_g(f) = g$$

and:

$$\overline{N}_g(f d\mu) = g d\mu,$$

according to Proposition 9.6.

To see that the space $\mathcal{N}$ is closed under the composition operation $\circ$, observe that there exist μ-null sets A_1 and A_2 so that for each $x \in S \backslash A_i$, the map $u \to g_i(x, u)$ is continuous from X into itself, for $i = 1, 2$. Thus, if $x \in S \backslash (A_1 \bigcup A_2)$, the map $u \to g_1(x, g_2(x, u))$ is continuous as well. Now fix $u \in X$. Then, the function $x \to g_i(x, u)$ from S into X is X-measurable, so Proposition 9.3 implies that the function $x \to g_1(x, g_2(x, u))$ is also X-measurable. Furthermore:

$$\begin{aligned} \|g_1(x, g_2(x, u))\|_X &\leq a_1(x) + b_1 \|g_2(x, u)\|_X \\ &\leq a_1(x) + b_1 (a_2(x) + b_2 \|u\|_X) \\ &= a_1(x) + b_1 a_2(x) + b_1 b_2 \|u\|_X. \end{aligned}$$

So, $g_1 \circ g_2 \in \mathcal{N}$. Finally, prove (9.1.7):

$$\begin{aligned}
\overline{N}_{g_1 \circ g_2}(f d\mu) &= g_1(\cdot, g_2(\cdot, f(\cdot)))\, d\mu \\
&= \overline{N}_{g_1}\left[\overline{N}_{g_2}(f d\mu)\right] \\
&= \left(\overline{N}_{g_1} \circ \overline{N}_{g_2}\right)(f d\mu).
\end{aligned}$$

This completes the proof of the proposition. □

9.2 The Operator $\overline{N}_g$ for Vector-Valued Piecewise Linear N-functions

The results in [10] on how to extend to vector-valued measures the Nemytsky operator N_g associated to the piece-wise linear N-function g defined by (9.1.1) are presented in this section.

Lemma 9.8. *If $m_1, m_2 : \Sigma \to X$ are vector-valued measures and $m_1 \perp \mu, m_2 \perp \mu$, then $m_1 + m_2 \perp \mu$.*

Proof. Refer to [10] section 5. The proof is as follows: According to Remark 3.6 and Lemma 3.7, there exists partitions $S = X \cup Y = V \cup W$, $X, Y, V, W \in \Sigma$, so that:

$$\begin{aligned}
\mu(Y) &= \mu(W) = 0 \\
m_1(X') &= 0,\ for\ all \quad Y' \subseteq Y,\ Y' \in \Sigma \\
m_2(V') &= 0,\ for\ all \quad V' \subseteq V,\ V' \in \Sigma.
\end{aligned}$$

We consider now the partition $S = (X \cap V) \cup (Y \cup W)$. For this partition:

$$\mu(Y \cup W) \leq \mu(Y) + \mu(W) = 0$$

Fix $R \subseteq X \cap V$, $R \in \Sigma$:

$$(m_1 + m_2)(R) = m_1(R) + m_2(R) = 0,$$

proving that the measures $m_1 + m_2$ and μ are mutually singular. This completes the proof of the lemma $\square$

The following result identifies the Lebesgue Decomposition of $\overline{N}_g(m)$.

Proposition 9.9. *If $m \in \mathcal{M}_X$ and $m = fd\mu + m_s$ is the Lebesgue Decomposition of m,*

$$\overline{N}_g(m) = \overline{N}_g(fd\mu) + \left(\sum_{i=1}^{n} a_i\right) |m_s| + T \circ m_s,$$

with $\left(\sum_{i=1}^{n} a_i\right) |m_s| + T(m_s) \perp \overline{N}_g(fd\mu)$.

Proof. The following proof is contained in [10] section 5, refer to such for more details. Since $m_s \perp \mu$, Lemma 3.7 implies that the measures m_s and $(f - b_i)d\mu$ are also mutually singular. In turn Lemma 3.8 implies that we may write:

$$\overline{N}_g(m) = \sum_{i=1}^{n} a_i |fd\mu + b_i d\mu| + \left(\sum_{i=1}^{n} a_i\right) |m_s| + T(f(\cdot))d\mu + T \circ m_s$$

$$= \overline{N}_g(fd\mu) + \left(\sum_{i=1}^{n} a_i\right) |m_s| + T \circ m_s$$

Since $|m_s|$and μ are mutually singular, Remark 3.6 and Lemma 3.7 imply that there is a partition $S = A \cup B$, $A, B \in \Sigma$, such that:

$$\begin{aligned} |m_s|(A') &= 0 \text{ for all } \quad A' \subseteq A,\ A \in \Sigma, \\ \mu(B') &= 0 \text{ for all } \quad B' \subseteq B,\ B \in \Sigma. \end{aligned}$$

then:

$$\left[\left(\sum_{i=1}^{n} a_i\right) |m_s|\right](A') = \int\limits_{A'} \left(\sum_{i=1}^{n} a_i\right) d\,|m_s|$$
$$= 0$$

So, the measures $(\sum_{i=1}^{n} a_i)\,|m_s|$ and μ are also mutually singular. It remains to see that the measures $T \circ m_s$, and μ are mutually singular as well.

Since $|m_s| \perp \mu$, there is a partition $S = A \cup B$, $A, B \in \Sigma$, such that $|m_s|(A) = 0$ and $\mu(B) = 0$. Then $|T(m)| \leq \|T\|\,|m|$, implies that $|T(m_s)|(B) = 0$ and Lemma 9.8 implies that $(\sum_{i=1}^{n} a_i)\,|m_s| + T \circ m_s$ and μ are mutually singular. Finally since $\overline{N}_g\,(fd\mu) = g(\cdot, f(\cdot))d\mu$ is absolutely continuous with respect to μ, we conclude that $(\sum_{i=1}^{n} a_i)\,|m_s| + T \circ m_s$ and $\overline{N}_g\,(fd\mu)$ are mutually singular. $\square$

Chapter 10

Results

This chapter focuses on some of the results that were derived from all the previous chapters.

10.1 Main Results

This section first recalls some results from [10] section 6, and then presents the main result of this thesis; Theorem 10.6.

Lemma 10.1. *Let $G : [0, T] \times S \times X \to X$ be a function satisfying the conditions:*

1. $\|G(t, x, u)\|_X \leq a(x) + b\|u\|_X$, *for some* $a \in L^1$ *and* $b \geq 0$, *for* μ*-a.e.* $x \in S$, $0 \leq t \leq T$ *and* $u \in X$.

2. *The function* $x \to G(t, x, u)$ *is* Σ*-measurable for each* $0 \leq t \leq T$ *and* $u \in X$.

3. *There exists* $C > 0$ *such that* $\|G(t, x, u_1) - G(t, x, u_2)\|_X \leq C\|u_1 - u_2\|_X$, *for* $0 \leq t \leq T$, $u_1, u_2 \in X$ *and for* μ*-a.e.* $x \in S$.

4. *There exists* $C > 0$ *such that* $\|G(t_1, x, u) - G(t_2, x, u)\|_X \leq C\|u\|_X |t_1 - t_2|$, *for* $0 \leq t_1, t_2 \leq T$, $u \in X$ *and for* μ*-a.e.* $x \in S$.

Then, the following properties hold:

a) *For each* $0 \leq t \leq T$, *the function* $G_t : S \times X \to X$ *defined as* $G_t(x, u) = G(t, x, u)$ *is an* N*-function.*

b) *For each* $0 \leq t \leq T$, *the Nemystky operator* N_{G_t} *maps* $B^1(X)$ *to itself.*

c) The function $f(t,x) \to N_{G_t}(f(t,\cdot))(x)$ maps $C[0,T;B^1(X)]$ continuously into itself.

Proof. The following proof was extracted from [10] and closely follows the proof of Lemma 21 in [9]. The proof of a) is a direct application of conditions 2 and 3, while, in particular, the proof of b) follows from condition 1 and directly from Remark 9.5. In regards to the proof of c) observe that by substituting L^1 for $B^1(X)$ and modifying the norm to the corresponding space the proof of Lemma 21 in [9] holds and proceeds in a similar manner to obtain the desired result. $\square$

If the function $G : [0,T] \times S \times X \to X$ satisfies the hypotheses of Lemma 10.1 and $m \in C[0,T;\mathcal{M}_X]$, define:

$$\mathcal{A}(m)(t) = \overline{N}_{G_t}(m(t)). \tag{10.1.1}$$

It follows from Lemma 10.1 that the operator $\mathcal{A}$ is continuous from $C[0,T;\mathcal{M}_X]$ into itself.

The next result is a well known extension of the Banach fixed point theorem.

Proposition 10.2. *Let (M,d) be a complete metric space and consider a map $f : M \to M$. If there exists $k \in \{1,2,...\}$ such that the composite map $f^{(k)}$ is a contraction, then the map f has a unique fixed point.*

Theorem 10.3. *If we assume that the operator $\mathcal{A}$ is given by (10.1.1) and the function G satisfies the conditions stated in Lemma 10.1, the initial value problem:*

$$\begin{cases} \frac{dm}{dt} + \mathcal{A}(m)(t) &= 0 \text{ for } 0 < t < T \\ m(0) &= m_0 \end{cases}. \tag{10.1.2}$$

will have one and only one solution in $C^1[0,T;\mathcal{M}_X]$ for each $m_0 \in \mathcal{M}_X$.

Proof. We cite the proof included in [10], this is only a sketch of the proof since it follows closely the proof of Theorem 23 in [9]. We observe that the initial value problem (10.1.2) has the same solutions in $C^1[0,T;\mathcal{M}_X]$ as the integral equation:

$$m(t) = m_0 + \int_0^t \mathcal{A}(m)(s)\, ds. \tag{10.1.3}$$

To prove that (10.1.3) has one and only one solution in $C^1[0,T;\mathcal{M}_X]$ it suffices to show that the operator $\mathcal{T}$ defined on $C[0,T;\mathcal{M}_X]$ as:

$$\mathcal{T}(m) = m_0 + \int_0^t \mathcal{A}(m)(s)\, ds$$

has a unique fixed point. According to Proposition 10.2, the operator $\mathcal{T}$ has a unique fixed point if $\mathcal{T}^{(k)}$ is a contraction in $C[0,T;\mathcal{M}_X]$ for some $k \in \{1,2,...\}$. The operator $\mathcal{T}^{(k)}$ will be a contraction in $C[0,T;\mathcal{M}_X]$ for some $k \in \{1,2,...\}$ if:

$$\|\mathcal{T}^{(k)}(m_1) - \mathcal{T}^{(k)}(m_2)\| \leq \frac{C^k T^k}{k!}\|m_1 - m_2\|, \tag{10.1.4}$$

for $m_1, m_2 \in C[0,T;\mathcal{M}_X]$ and $k \in \{1,2,...\}$. Estimate 10.1.4 can be proved by induction, completing the proof of Theorem 10.3. □

Following a formalization of the argument made in Proposition 9.2 is provided.

Theorem 10.4. *([11], p. 174, Proposition 10) We observe that $C[0,T;\mathcal{M}_b]$ is isometrically isomorphic to $C[0,T;B^1]$ endowed with the norm:*

$$\|f\| = \sup_{0\leq t\leq T} \|f(t)\|_{B^1}.$$

Indeed:

$$\|(f d\mu)(t)\|_{\mathcal{M}_b} = \|f(t)\|_{B^1},$$

for each $0 \leq t \leq T$.

Proof. It follows from the arguments given in Chapter 9 section 9.1. □

Next the definition of a positive operator valued-measure is given before moving on to presenting our main result. We recall that (S, Σ, μ) is a complete σ-finite measure space

and X is a real Banach space with norm $\|\cdot\|$.

Definition 10.5. A positive-operator valued measure (POVM) is a mapping F whose values are bounded non-negative self-adjoint operators on a Hilbert space H, that is, $F : \Sigma \to \mathcal{B}(H)_+$ such that:

1. $F(\emptyset) = 0$, $F(S) = I_H$; and

2. $F(\bigcup_{i=1}^{\infty} M_i) = \sum_{i=1}^{\infty} F(M_i)$ whenever $M_i \bigcap M_j = \emptyset$ for $i \neq j$.

In short such F is a non-negative countably additive measure on the σ-algebra Σ. Next, we present our main result.

Theorem 10.6. *A Complex Gleason measure can be used as an operator measure for the Nemytsky operator.*

Proof. Even though the proof follows from Theorem 10.3 and Theorem 8.4 a detailed proof is provided. Recall that a measure can be used as an operator measure for the Nemytsky operator if given the Nemytsky operator with conditions as in Theorem 10.1 one can construct a complex Gleason Measure that will work as a positive-operator valued measure. Consider the initial value problem given in Theorem 10.3:

$$\begin{cases} \frac{dm}{dt} + \mathcal{A}(m)(t) &= 0 \text{ for } 0 < t < T \\ m(0) &= m_0 \end{cases} \tag{10.1.5}$$

It is known from theorem 10.3 that the initial value problem 10.1.5 has a unique solution, $m \in C^1[0, T; \mathcal{M}_X]$ (where $C^1[0, T; \mathcal{M}_X]$ is the set of vector-valued measures continuous in the interval $[0, T]$ satisfying properties stated in Theorem 3.12). Following Theorem 8.4 let m be a representable complex Gleason measure and let ρ_1 be its density operator, so that:

$$m(S) = Tr(\rho_1 P_S)$$

with ρ_1 positive. Let $P_S = \{P_i;\ P_i$ is a projector such that $P_i\mathcal{A} \subset \mathcal{A}P_i\}$, we define $n_{\mathcal{A}} : P_S \to \mathbb{C}$ by:

$$n_{\mathcal{A}}(S) = \int \mathcal{A}P_S dm.$$

Then by previous results in section 6, $n_{\mathcal{A}}$ is a complex Gleason measure in the space of $\mathcal{A}$-invariant subspaces; now the complex Gleason measure may be written as $m = \mu + i\nu$, where μ, υ are positive Gleason measures and obtain $n_{\mathcal{A}}(S) = \lambda_{\mathcal{A}}(S) + i\sigma_{\mathcal{A}}(S) = \int \mathcal{A}P_S d\mu + i\int \mathcal{A}P_S d\nu$ where $\lambda_{\mathcal{A}}, \sigma_{\mathcal{A}}$ define Gleason measures, respectively, by virtue of the results at the beginning of section 6. Note also that by Lemma 6.1, $n_{\mathcal{A}}$ is absolutely continuous with respect to m. Furthermore by virtue of Theorem 8.4 define $n : P_S \to \mathbb{C}$ by:

$$n(T) = n_{\mathcal{A}}(T) = \int \mathcal{A}P_T dm$$

for any closed subspace T of H. Thus n is a measure, and in particular a complex Gleason measure. To conclude, defining an operator F on $C[0, T; \mathcal{M}_b]$ by:

$$F(m) = \int \mathcal{A}P_T dm$$

and since m is a non-negative countably additive measure on the families of projections in P_S, by Definition 10.5, then F is a positive-operator valued measure. □

10.2 Applications to Quantum Mechanics and Examples

The following examples were entirely taken from [10].

10.2.1 Example 1. A complex Gleason measure in Quantum Mechanics

Now consider a self-adjoint bounded linear operator A (i.e. observable) with eigenvalues $\{\lambda_i | \lambda_{i+1} > \lambda_i\}$ and corresponding eigenspaces $\{S_i\}$. The eigenspaces $\{S_i\}$ are orthogonal for $i \neq j$, pick $\tilde{\psi}_i \in S_i$, $\tilde{\psi}_j \in S_j$:

$$0 = \left\langle \tilde{\psi}_i \mid A \mid \tilde{\psi}_j \right\rangle - \left\langle \tilde{\psi}_i \mid A \mid \tilde{\psi}_j \right\rangle = \lambda_i \left\langle \tilde{\psi}_i \mid \tilde{\psi}_j \right\rangle - \lambda_j \left\langle \tilde{\psi}_i \mid \tilde{\psi}_j \right\rangle$$

$$= (\lambda_i - \lambda_j) \left\langle \tilde{\psi}_i \mid \tilde{\psi}_j \right\rangle$$

therefore $\left\langle \tilde{\psi}_i \mid \tilde{\psi}_j \right\rangle = 0$, and we conclude that $\{S_i\}$ are orthogonal, for more details refer to one of the previous sections; Chapter 3, section 5.1 (also consult [1],[7],[8],[13])

The Quantum Mechanics notation is used in this section: $< \psi_i | \psi_j >$ denotes the inner product and $< \psi_i | A | \psi_j > = < \psi_i | A \psi_j > = < A \psi_i | \psi_j >$ (A is Hermitian). It is clear that when the operator A is written in the middle of the inner product it is automatically assumed to be self-adjoint (For more details on the Quantum Mechanics notation see for example [1],[7],[8],[13])

Moreover we can restrict ourselves to the Hilbert space H given by the direct sum of $\{S_i\}$. Let us assume a particle in a superposition of pure states $\psi_i \in S_i$, $\|\psi_i\| = 1$:

$$\psi = \sum_i c_i \ \psi_i$$

where $\sum_i |c_i|^2 = 1$ and $|c_i|^2 = c_i^* c_i$ is called the probability amplitude of the state ψ_i. In other words the probability of the state ψ_i. It is possible to define a complex Gleason measure in the following way:

$$\mu_\psi(P_i) := P_i \psi$$

where P_i is a projector on a subspace S_i. μ_ψ clearly defines a complex Gleason measure

since it is additive on any given family of orthogonal projectors $\{P_i\}$, $P_iP_j = 0$ for $i \neq j$:

$$\mu_\psi \left(\sum_i P_i \right) = \sum_i \mu_\psi(P_i)$$

and has values in the complex plane. We can use the complex Gleason measure to recover c_i in the following way:

$$\int P_i d\mu_\psi = \mu_\psi(P_i) = P_i\psi = c_i,$$

where the complex number c_i denotes the particle's amount in a particular state ψ_i.

10.2.2 Example 2. Connection of complex and real Gleason measures

Following the previous example in section 10.2.1, the connection between the complex Gleason measure giving us c_i, and the real Gleason measure giving $|c_i|^2$ *through the Nemytsky operator* will be shown. Let A, ψ_i be defined as in section 10.2.1 and consider a finite superposition of pure states ψ_i:

$$\psi = \sum_{i=1}^{n} c_i \ \psi_i \quad , \quad \sum_{i=1}^{n} |c_i|^2 = 1,$$

and denote S_i as:

$$S_i = \mathrm{span}(\psi_i)$$

and P_i the projector on the linear subspace S_i. Using the Quantum Mechanics notation, the projector P_i will be described as:

$$P_i =< \psi_i|$$
$$P_i\psi =< \psi_i|\psi > .$$

To be consistent with notation introduced in section 9.1, we will take:

$$X = \mathbb{C}$$
$$Y = \mathbb{R}$$

and the measurable space will be the set S of all the orthogonal subspaces S_i with sigma algebra generated by S_i. Consider the function f defined as:

$$f : S \to \mathbb{C}$$
$$f(S_i) = P_i\psi.$$

We can then write μ_ψ from section 10.2.1 as:

$$\mu_\psi(S_i) = (\Lambda f)(S_i) = f(S_i) \cdot \Delta(S_i) = P_i\psi \cdot \dim(S_i) = P_i\psi$$

where $\mu_\psi \in \mathcal{M}_X$, and Δ is the measure given in section 8.2, that gives the dimension of S_i. We will define the following N-function g as:

$$g(S_i, y) = (P_i\psi)^* \cdot y.$$

Then g satisfies the N-function properties given in definition 9.1. For fixed $y \in \mathbb{C}$ the function:

$$S \mapsto g(S, y) = \Big(\sum_i (P_i\psi)^*\Big)y \ , \ S = \bigcup_i S_i$$

is measurable. And for each $S = \bigcup\limits_i S_i$ the function

$$y \mapsto g(S, y) = \Big(\sum_i (P_i\psi)^*\Big)y \ , \ S = \bigcup_i S_i$$

is continuous.

The Nemytsky operator N_g is acting on f in the following way:

$$(N_g f)(S_i) = g(S_i, f(S_i)) = g(S_i, P_i\psi) = (P_i\psi)^*(P_i\psi)$$

Now we can define a real Gleason measure on S as:

$$\tilde{\mu}_\psi(S_i) = (\Lambda N_g f)(S_i) = (P_i\psi)^*(P_i\psi) \cdot \Delta(S_i) = (P_i\psi)^*(P_i\psi) \cdot \dim(S_i) = (P_i\psi)^*(P_i\psi)$$

and we obtain:

$$\mu_\psi(S_i) = (\Lambda f)(S_i) = P_i\psi = c_i$$
$$\tilde{\mu}_\psi(S_i) = (\Lambda N_g f)(S_i) = (P_i\psi)^*(P_i\psi) = |c_i|^2$$

where $|c_i|^2$ denotes the probability of the particle with wave function given by ψ being at the state ψ_i.

10.2.3 Example 3. The electron's spin

This example is included in order to emphasize the importance of complex and real Gleason measures. Following the two previous examples, consider the observable A associated with a electron's spin that is given by the spin quantum number m_l (for details see [7]). Now, such operator A has a discrete spectrum, i.e. the set of eigenvalues is discrete and specifically, is given by two values namely $\lambda_\uparrow$ and $\lambda_\downarrow$, with eigenvectors (which physicists call eigenstates) given by $\psi_\uparrow$, $\psi_\downarrow$ respectively, these represent the two only possible states/values of an electron's spin. Moreover, we let $\psi_\uparrow$ to represent an eigenvector in which the particle has a spin directed upwards and $\psi_\downarrow$ an eigenvector with spin directed downwards. Now, given that a particle can be in a state of spin up or down, consider the particle in a superposition of states with a wave function described by:

$$\psi = c_1\psi_\uparrow + c_2\psi_\downarrow$$

where $|c_1|^2+|c_2|^2 = 1$ and $|c_1|^2 = c_1^* c_1$ (respectively $|c_2|^2$) is called the probability amplitude of the state ψ_1 (respectively ψ_2). Furthermore let a particle be in a state of amount ${}^{12i}/_{13}$ in spin up and an amount of ${}^{5}/_{13}$ in spin down. Tts wave function is then given by:

$$\psi = \frac{12}{13} i \psi_\uparrow + \frac{5}{13} \psi_\downarrow.$$

Now, if we let $S_\uparrow$ (respectively $S_\downarrow$) be the eigenspace corresponding to the eigenvalue $\lambda_\uparrow$ (respectively $\lambda_\downarrow$), and define μ_ψ, $\tilde{\mu}_\psi$ as in example 2 (section 10.2.2), we then get:

$$\mu_\psi(S_\uparrow) = (\Lambda f)(S_\uparrow) = P_\uparrow \psi = c_1 = \frac{12}{13} i$$

$$\tilde{\mu}_\psi(S_\uparrow) = \tilde{\mu}_\psi(\Lambda N_g f)(S_\uparrow) = (P_\uparrow \psi)^* (P_\uparrow \psi) = |c_1|^2 = c_1^* c_1 = \left(-\frac{12}{13} i\right)\left(\frac{12}{13} i\right) = \frac{144}{169} \approx 0.85$$

and similarly:

$$\mu_\psi(S_\downarrow) = (\Lambda f)(S_\downarrow) = P_\downarrow \psi = c_2 = \frac{5}{13}$$

$$\tilde{\mu}_\psi(S_\downarrow) = \frac{25}{169} \approx 0.15$$

where $P_\uparrow$ (respectively $P_\downarrow$) is a projector on the subspace $S_\uparrow$ (respectively $S_\downarrow$), then the probability that the particle described by the wave function ψ is at the pure state $\psi_\uparrow$ (respectively $\psi_\uparrow$) is 0.85 (respectively 0.15). To conclude this example note that the possible values of the spin depend solely on the type of particle given, in this case for simplicity we presented an electron to easily see the relation with Gleason measures.

10.2.4 Example 4. The positive-operator valued measure

To further exemplify the role of a positive-operator valued measure in quantum mechanics, we present the following simple example. Let H be a Hilbert space, recall that a POVM is a mapping that takes an element of the σ-algebra of subsets of H and maps it to a positive operator. Furthermore to every operator of this type, given a state there is a measure associated to the operator via the scalar product. For example let us consider the problem of measuring the orbital angular-momentum direction of an electron. The outcomes of such experiment are determined by the quantum number m_l with possible values $-l, ..., 0, ... + l$, where l is the orbital quantum number. We can now define a positive-operator valued measure considering Ω to be the Borel sigma algebra of the set $\{-l, ..., 0, ... + l\}$, then the POVM is a map

$$\mathcal{E} : \Omega \to \mathcal{B}(H)$$

defined by:

$$\mathcal{E}(\{i\}) = P_i, \;\; \mathcal{E}(\emptyset) = 0, \;\; \mathcal{E}(\bigcup i) = \sum P_i, \;\; -l \leq i \leq +l$$

for each integer number i, where P_i is the positive-operator associated to the orbital angular-momentum direction measurement. Let ρ be the density operator, associated with a given state and define:

$$\mu_\rho(\psi) = Tr(\mathcal{E}(\psi)\rho) \;\; \forall \psi \in \Omega$$

Then μ_ρ defines a measure and in particular a probability measure. Moreover, if we take $\{-l + 2\} \in \Omega$ then $\mu_\rho(\{-l + 2\})$ would be the probability that when measuring ρ the outcome $-l + 2$ would be obtained. Such relation clarifies the necessity of the empty set being mapped to 0, the whole space to 1 and lastly the operators being positive and add up to one.

10.2.5 Example 5. The Nemytsky pperator using vector-valued measures

To further exemplify vector-valued measures and their relation with the Nemystky operator, consider the following example in [10] (Example 3). As an illustration, Let $\mathcal{A}$ be the operator as considered in Theorem 10.3. With this purpose, construct first a function G that satisfies the hypothesis of Lemma 10.1. Fix a function $H : X \to X$ satisfying the two conditions:

1. There exists $C_1 > 0$ such that $\|H(u)\|_X \leq C_1 \|u\|_X$ for all $u \in X$.

2. H is a Lipschitz function; that is to say, there exists $C_2 > 0$ such that $\|H(u_1) - H(u_2)\|_X \leq C_2 \|u_1 - u_2\|_X$ for all $u_1, u_2 \in X$.

Then, given $a \in L^1$ we define $G : [0, T] \times S \times X \to X$ as $G(t, x, u) = H(a(x) + tu)$. It is then claimed that G satisfies conditions 1-4 in Lemma 10.1. In fact, $\|G(t, x, u)\|_X = \|H(a(x) + tu)\|_X \leq C_1 \|a(x) + tu\| X \leq C_1 (\|a(x)\|_X + T \|u\|_X)$, so condition 1. is satisfied. If t is fixed $0 \leq t \leq T$, $u \in X$, the function $x \mapsto H(a(x)+tu)$ is Σ-measurable, because it is the composition, in the necessary order, of the Σ-measurable function $x \mapsto a(x)$ and the continuous function $r \mapsto H(r + tu)$. So condition 2. holds.

Then the following may be written:

$$
\begin{aligned}
\|G(t, x, u_1) - G(t, x, u_2)\|_X &= \\
\|H(a(x) + tu_1) - H(a(x) + tu_2)\|_X &\leq \\
C_2 t \|u_1 - u_2\|_X &\leq C_2 T \|u_1 - u_2\|_X .
\end{aligned}
$$

Therefore, condition 3 is satisfied.

Finally, fixing t_1, t_2 with $0 \leq t_1, t_2 \leq T$, $x \in S$, $u \in X$, hence:

$$
\begin{aligned}
\|G(t_1, x, u) - G(t_2, x, u)\|_X &= \\
\|H(a(x) + t_1 u) - H(a(x) + t_2 u)\|_X &\leq C_2 \|u\|_X |t_1 - t_2| .
\end{aligned}
$$

Thus, condition 4 is satisfied as well.

According to Lemma 10.1, for each $0 \leq t \leq T$, the function $G_t : S \times X \mapsto X$ defined as $G_t(x,u) = H(a(x) + tu)$ is an N-function.

If $\lambda \in C[0,T;\mathcal{M}_b]$, theorem 10.4 implies that $\lambda(t) = f(t,\cdot)d\mu$ for a unique $f \in C[0,T;B^1]$. So, it is possible to define the operator:

$$\mathcal{A} : C[0,T;\mathcal{M}_b] \to C[0,T;\mathcal{M}_b]$$

as:

$$\mathcal{A}(\lambda)(t) = \overline{N_{G_t}}(\lambda(t)) = H(a(\cdot) + tf(t,\cdot))d\mu.$$

Chapter 11

Some Extensions

This chapter is motivated by the previous results obtained. The intention of this chapter is to use the extension of the conditions necessary for the Nemystky operator i.e continuity and measurability, to rewrite theorems and propositions for this new generalized conditions. Taking Remark 9.5 as inspiration, the next Theorem 11.2 is presented, which is a result obtained by Krasnoselskii and Vainberg on the continuity and mesurability of a generalized Nemystky operator.

Let g be a vector valued N-function as defined in Chapter 9 and $\mathcal{N}_g(f)(x) = g(x, f(x))$ be the Nemytsky operator. Recall from Proposition 9.3 and Definition 9.4 that the Nemytsky operator $\mathcal{N}_g$ as defined for an N-function $g : S \times X \to Y$ maps X-measurable functions to Y-measurable functions. Define:

$$F^0 = \{f : S \to X : \ f \; is \; \Sigma - measurable\}$$

in particular if $g : S \times X \to X$ is an N-function then $\mathcal{N}_g$ maps F^0 into F^0.

Definition 11.1. $\mathcal{N}_g$ is an operator from L^p into L^q $(p, q > 0)$ if $\mathcal{N}_g(f)(x) \in L^q$ for every $f \in L^p$.

Theorem 11.2. *Let $\mathcal{N}_g$ be the Nemystky operator as defined in section 9.1, and let $p, q \in [1, \infty)$. Then the following assertions hold for the Nemytsky operator $\mathcal{N}_g(f)(x) = g(x, f(x))$:*

1. *If $\mathcal{N}_g$ maps all of L^p into L^q, then it is a continuous and bounded operator from L^p into L^q, and there is a function $a(x) \in L^q$ and a constant $b \geq 0$ such that*

$$\|g(x, u)\|_Y \leq a(x) + b \, \|u\|_X^r , \quad u \in L^p, \quad r = p/q \tag{11.0.1}$$

2. *If the latter inequality i.e., 11.0.1 is satisfied, then $\mathcal{N}_g$ maps all of L^p into L^q, and is therefore a continuous and bounded operator from L^p into L^q.*

Proof. The proof is omitted due to its lengthiness. The reader should consutl[26], page 155 for a detailed proof. □

A more restrictive form of the above theorem is included next in theorem 11.3.

Theorem 11.3. *Let $p \in [1, \infty)$ and let $a(x) \in L^q$, with*

$$\frac{1}{p} + \frac{1}{q} = 1$$

Suppose g satisfies the conditions stated in Definition 9.5, and for some constant $b \geq 0$ we have

$$\|g(x,u)\|_Y \leq a(x) + b\,\|u\|_X^{p-1}, \quad \|u\|_X \in L^p,$$

then $\mathcal{N}_g$ maps all of L^p into L^q, and is a continuous and bounded operator from L^p into L^q.

Proof. This is obvious since, solving for q on the equality:

$$\frac{1}{p} + \frac{1}{q} = 1$$

yields:

$$q = \frac{p}{p-1}$$

and substituting back on $r = {}^{p}/_{q}$ we obtain $r = p - 1$. The rest follows directly from 11.2. □

Recall now that $B^p(X)$ is the space of X-measurable functions $f : S \to X$ for which the function $x \to \|f(x)\|_X \in L^p$

Proposition 11.4. *For a N-function satisfying (11.0.1) the Nemytsky operator $\mathcal{N}_g$ is bounded and continuous operator from $B^p(X)$ to $B^q(Y)$*

Proof. The first assertion, that $\mathcal{N}_g$ is bounded and continuous is guaranteed by Theorem 11.2. Furthermore the function $u : S \to X$ is X-measurable by definition of the Nemytsky operator, and again by Theorem 11.2 we know $\mathcal{N}_g$ is an operator from L^p to L^q, that is, $\|u(x)\|_X \in L^p$, it is obvious then that $u \in B^p(X)$. On the other hand, by the same token $g(x, u) \in B^q(Y)$. $\square$

Let $\mathcal{N}^p$ be the family of vector valued N-functions $g : S \times X \to X$ satisfying the condition

$$\|g(x, u)\|_X \leq a(x) + b \|u\|_X^r, \quad u \in L^p, \quad r = p/q$$

μ-a.e. in S, with $a \in L^q$ and $b \geq 0$.

Proposition 11.5. *The space $\mathcal{N}^p$ is closed under the composition operation*

$$(g_1 \circ g_2)(x, u) = g_1(x, g_2(x, u))$$

Proof. This proof follows that of Propostion 9.7 part (3) with obvious modifications. To see that the space $\mathcal{N}^p$ is closed under the composition operation $\circ$, we observe that there exist μ-null sets A_1 and A_2 so that for each $x \in S \backslash A_i$, the map $u \to g_i(x, u)$ is continuous from X into itself, for $i = 1, 2$. Thus, if $x \in S \backslash (A_1 \bigcup A_2)$, the map $u \to g_1(x, g_2(x, u))$ is continuous as well by definition of N-function. We now fix $u \in X$. Then, the function $x \to g_i(x, u)$ from S into X is X-measurable, so Proposition 9.3 implies that the function $x \to g_1(x, g_2(x, u))$ is also X-measurable. Furthermore,

$$\begin{aligned}
\|g_1(x, g_2(x, u))\|_X &\leq a_1(x) + b_1 \|g_2(x, u)\|_X^r \\
&\leq a_1(x) + b_1 \left(a_2^r(x) + b_2^r \|u\|_X^r\right) \\
&= a_1(x) + b_1 a_2^r(x) + b_1 b_2^r \|u\|_X^r .
\end{aligned}$$

where $c(x) = a_1(x) + b_1 a_2^r(x) \in L^q$ and we define $d = b_1 b_2^r$ to obtain:

$$\|g_1(x, g_2(x,u))\|_X \leq c(x) + d\|u\|_X^r \tag{11.0.2}$$

So, $g_1 \circ g_2 \in \mathcal{N}$. □

Recall that the map $\Lambda : B^1(X) \to \mathcal{M}_X$ defined as $\Lambda(f) = f d\mu$ is an isomorphism of real vector spaces. Unfortunately the map $\Gamma : B^p(X) \to \mathcal{M}_X$ is not an isomorphism, this is easy to note, since if so that would imply that $B^1(X)$ and $B^p(X)$ are isomorphic which is clearly not the case. Considering the previous issue, a different approach is taken. Next a well-known theorem is presented in order to continue the discussion.

Theorem 11.6. *Let $\mathscr{L}^p$ be the space of all continuous linear functionals on L^p and $f, g : S \to X$ be two vector-valued measurable functions on S. Suppose $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\mathscr{L}^p = B^q(X)$$

i.e., for every bounded linear functional l on $B^p(X)$ there is a unique $h \in B^q$ so that:

$$l(f) = \int_S f(x)h(x)d\mu$$

for all $f \in B^p$. Moreover $\|l\|_{\mathscr{L}^p} = \|f\|_{B^q}$.

Clearly, the linear functional l leads naturally to a signed measure v, furthermore the given continuity of l, v is absolutely continuous with respect to μ. Furthermore let:

$$\mathcal{M}_a = \{\nu : \Sigma \to \mathbb{R}^* \ \textit{signed measure} : \ \nu \ll \mu\}$$

With this previous notions we take motivation on [9] to develop the following results.

Proposition 11.7. *Let $g : S \times X \to X$ be a nonnegative N-function. Then, there exists a*

unique operator $\overline{\mathcal{N}}_g : \mathcal{M}_a \to \mathcal{M}_a$ *such that:*

$$l \circ \mathcal{N}_g(f) = \overline{\mathcal{N}}_g \circ l(f) \tag{11.0.3}$$

for all $f \in B^p$

Proof. Recall that, according to Proposition 9.3 and Definition 9.4 $\mathcal{N}_g$ maps F^0 into F^0 ,that is, it maps X-measurable functions to X-measurable functions, and then the nonnegative measurable function $g(x, f(x))$ has a μ-integral for every $f \in B^p$. Given $\nu \in \mathcal{M}_a$ with $\nu = fhd\mu$, i.e. the measure given in Theorem 11.6 by:

$$\nu = l(f) = \int_S f(x)h(x)d\mu$$

define the operator $\overline{\mathcal{N}}_g$ as:

$$\overline{\mathcal{N}}_g(\nu) = \int_S g(x, f(x))h(x)d\mu. \tag{11.0.4}$$

Since Theorem 11.6 assures the uniqueness of f up to μ-a.e. it is needed to show that $\overline{\mathcal{N}}_g$ is well defined. If $f = j$ everywhere except for a null set $O \in \Sigma$, and $E \in \Sigma$

$$\int_E g(x, j(x))h(x)d\mu = \int_{E\cap(S\setminus O)} g(x, f(x))h(x)d\mu = \int_E g(x, f(x))h(x)d\mu$$

So $\overline{\mathcal{N}}_g(\nu) = \overline{\mathcal{N}}_g(fhd\mu) = \overline{\mathcal{N}}_g(jhd\mu)$. The definition given by 11.0.4 implies that 11.0.3 holds. Suppose now that $T : \mathcal{M}_a \to \mathcal{M}_a$ is another operator satisfying

$$l \circ \mathcal{N}_g(f) = T \circ l(f)$$

then

$$T(fhd\mu) = T \circ l(f) = l \circ \mathcal{N}_g(f) = \overline{\mathcal{N}}_g \circ l(f) = \overline{\mathcal{N}}_g(fhd\mu)$$

hence $T = \overline{\mathcal{N}}_g$. This completes the proof of the proposition. $\square$

With this notion, define now

$$\overline{\mathcal{N}^p} = \{\overline{N}_g : g \in \mathcal{N}_g\}$$

Proposition 11.8. *The operator defined previously in 11.0.4 satisfies several properties:*

1. The spaces $\mathcal{N}^p$ and $\overline{\mathcal{N}^p}$ are real vector spaces and the map $g \to \overline{N}_g$ from $\mathcal{N}^p$ into $\overline{\mathcal{N}^p}$ is linear.

2. $B^q(X) \subseteq \mathcal{N}$, in the sense that every function $g \in B^q(X)$ defines an N-function $g(x)$ that satisfies (11.0.2). Moreover, given $g \in B^q(X)$,

$$\overline{N}_g (fhd\mu) = ghd\mu.$$

Proof. Clearly the spaces $\mathcal{N}^p$ and $\overline{\mathcal{N}^p}$ are real vector spaces. Moreover, if $\alpha, \beta \in \mathbb{R}$ and $g_1, g_2 \in \mathcal{N}^p$,

$$\begin{aligned} \overline{N}_{\alpha g_1 + \beta g_2} (fhd\mu) &= [\alpha g_1 (\cdot, f(\cdot)) + \beta g_2 (\cdot, f(\cdot))]\, h(\cdot) d\mu \\ &= \alpha \overline{N}_{g_1} (fhd\mu) + \beta \overline{N}_{g_2} (fhd\mu). \end{aligned}$$

It is clear that the Bochner integrable functions define N-functions in $\mathcal{N}^p$. Moreover, given $g \in B^q(X)$ and $f \in B^p(X)$,

$$N_g (f) = g$$

and

$$\overline{N}_g (fhd\mu) = ghd\mu,$$

according to Proposition 11.7. □

Remark 11.9. The operator defined previously in 11.0.4 satisfies several properties:

1. From Propositon 11.7, it can be concluded that $\overline{\mathcal{N}}_g(\mu) = g(x,1)hd\mu$

$$\overline{\mathcal{N}}_g(\mu) = \int_S g(x,1)h(x)d\mu$$

since l maps the measure μ to the identically one function.

2. If g_1 and g_2 are two nonnegative N-functions, their sum g_1+g_2 is also an N-function. Furthermore the operator $\overline{\mathcal{N}}_g$ is additive in g. That is

$$\overline{\mathcal{N}}_{g_1+g_2} = \overline{\mathcal{N}}_{g_1} + \overline{\mathcal{N}}_{g_2}$$

i.e.

$$\begin{aligned} \overline{\mathcal{N}}_{g_1+g_2}(\nu) &= \int_S (g_1+g_2)(x,f(x))h(x)d\mu \\ &= \int_S \left[g_1(x,f(x)) + g_2(x,f(x))\right] h(x)d\mu \\ &= \int_S g_1(x,f(x))h(x)d\mu + \int_S g_2(x,f(x))h(x)d\mu \\ &= \overline{\mathcal{N}}_{g_1}(\nu) + \overline{\mathcal{N}}_{g_2}(\nu) \end{aligned}$$

3. Given a Σ-measurable function $\alpha : S \to [0,\infty]$ and a nonnegative N-function g, the multiplicative product αg is also an N-function and

$$\overline{\mathcal{N}}_{\alpha g} = \alpha \overline{\mathcal{N}}_g$$

Note that the product $\alpha\overline{\mathcal{N}}_g(fhd\mu)$ is the measure defined on $E \in \Sigma$ by:

$$\int_E \alpha(x)g(x,f(x))h(x)d\mu$$

4. The product of two nonnegative N-functions, given two nonnegative N-functions g_1 and g_2 is:

$$\overline{\mathcal{N}}_{g_1g_2} = g_1(x,f(x))g_2(x,f(x))h(x)d\mu.$$

5. Given two N-functions $g_1, g_2 : S \times X \to X$, the function $g_2(x, g_1(x,u))$ is also an

N-function. Furthermore

$$\overline{\mathcal{N}}_{g_2(x,u)} \circ \overline{\mathcal{N}}_{g_1(x,u)} = \overline{\mathcal{N}}_{g_2(x,g_1(x,u))}.$$

11.1 Conclusions

This thesis was able to generalize previous results on Gleason measures to complex Gleason measures, and develop a functional calculus for complex measures in relation to the Nemytsky operator. Furthermore the interpretation of our results in the field of quantum mechanics was given including some concrete examples and further extensions of several theorems.

Bibliography

[1] Edwards, David. *"The Mathematical Foundations of Quantum Mechanics"*. Synthese 42: 1–70. (1979).

[2] Gleason, A. M. *"Measures on the closed subspaces of a Hilbert space"*. Indiana University Mathematics Journal 6: 885–893. (1957)

[3] Pitowsky, I. *"Quantum mechanics as a theory of probability"*: 10095. (2005).

[4] Wilce, A. *"Quantum Logic and Probability Theory"*. In The Stanford Encyclopedia of Philosophy (Spring 2006 Edition), Edward N. Zalta (ed.) (2006).

[5] Dvurecenskij, Anatolij. *Gleason's Theorem and Its Applications. Mathematics and its Applications* , Vol. 60. Dordrecht: Kluwer Acad. Publ. (1992), 348.

[6] Renardy, Michael and Rogers, Robert C. *An introduction to partial differential equations.* Texts in Applied Mathematics 13 (Second ed.). New York: Springer-Verlag. (2004) 356.

[7] A. Messiah, *Quantum Mechanics*, John Wiley (1958).

[8] J. von Neumann, *Mathematical foundations of Quantum Mechanics*, Princenton University Press. (1955).

[9] J. Alvarez and M.C. Mariani, *Extensions of the Nemytsky Operator: Distributional Solutions of Nonlinear Problems*, Journal of Mathematical Analysis and Applications 338 (2008), 588-598.

[10] J. Alvarez, M. Eydenberg and M.C. Mariani, *The Nemytsky Operator on Vector Valued Measures*, preprint.

[11] N. Dinculeanu, *Vector Measures*, Pergamon Press (1967).

[12] W. Rudin, *Real and Complex Analysis, Third Edition*, McGraw-Hill (1987).

[13] E. Prugovecki, *Quantum Mechanics in Hilbert Spaces*, Second Edition, Academic Press (1981).

[14] F. Riesz, B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar Publishing Co. (1955).

[15] M. Rieffel, *The Radon-Nikodym theorem for the Bochner integral*, Trans. Amer. Math. Soc. 131 (1968), 466-487.

[16] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis, Volume I*, American Mathematical Society (2000).

[17] J. R. Ringrose, *Compact non-self-adjoint operators*, Van Nostrand Reinhold Company, (1971).

[18] A. N. Sherstniev, *On representation of measures on orthogonal projections in a Hilbert space by bilinear forms*, Soviet Math., 9 (1970), 90-97.

[19] S. Maeda, *Probability measures on projections in von Neumann algebras*, Rev. Math. Phys., Vol. 1, 2 (1990), 235-290.

[20] P. De Napoli, M. C. Mariani, *Some remarks on Gleason measures*, Studia Math. 179 (2007), 99-115.

[21] G. B. Folland, *Real Analysis: Modern Techniques and their Applications*, Second Edition, Wiley (1984).

[22] J. Mikusinki, *The Bochner Integral*, Academic Press, (1978).

[23] M. Cotlar, R. Cignoli. *Nociones de Espacios Normados*. Eudeba. (1971).

[24] N. Dunford, J. Schwartz, *Linear Operators, General Theory*, Wiley Classics Library, (1998).

[25] A.M. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Macmillan (1964).

[26] M.M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Holden-Day (1964).

Curriculum Vitae

Miguel A Valles Morales was born on May 1, 1990. The first son of Manuel Antonio Valles Faudoa and Blanca Estela Morales Batrez, he graduated from COBACH No.6 (High School),Ciudad Juarez, Chihuahua, Mexico, in the spring of 2008. He inmediatly entered The University of Texas at El Paso. While working on his Bachelor's degree in Mathematics he worked as a Teaching Assitant exclusively for Pre-calculus classes. He was a member of the Noyce Scholars, received the Oustanding Undergraduate student in Mathematics award and graduated Magna Cum Laude with a Bachelor in Science in Mathematics in the fall 2012. He then proceeded to work as a Mathematics Teacher in Burges High School, El Paso, Texas, starting in the fall 2013 and resigning in the summer of 2014 since he decided to pursue a Master's degree.

In the summer 2014, he entered the Graduate School of The University of Texas at El Paso. While pursuing a master's degree in Mathematics he worked as a Teaching Assistant. He was selected as a member of the 21st Century Scholars and received the Outsanding Graduate student in Mathematics award in the spring 2016.

Permanent address: 664 Cancellare Ave
El Paso, Texas 79932