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# Necessary and Sufficient Conditions for Generalized Uniform Fuzzy Partitions ☆

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## Abstract

The fundamental concept in the theory of fuzzy transform (F-transform) is that of fuzzy partition. The original definition assumes that each two fuzzy subsets overlap in such a way that sum of membership degrees in each point is equal to 1. However, this condition can be generalized to obtain a denser fuzzy partition that leads to improvement of approximation properties of F-transform. However, a problem arises how one can effectively construct such type of fuzzy partitions. We use a generating function having special properties and it is not immediately clear whether it really defines a general uniform fuzzy partition. In this paper, we provide necessary and sufficient condition using which we can solve this task so that optimal generalized uniform fuzzy partition can be designed more easily. This is important in various practical applications of the F-transform, for example in image processing, time series analysis, solving differential equations with boundary conditions, and other ones.

*Keywords:* Fuzzy transform, uniform fuzzy partition, F-transform

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## 1. Introduction

Fuzzy transform (F-transform) is a special soft computing technique proposed by Perfilieva in [3] (see also [5]) with a very wide scale of possible applications (see [4, 11]). The F-transform has two phases: direct and inverse. The direct F-transform transforms a bounded real function $f$ to a finite vector of real numbers and the inverse one sends it back. The result is a function $\hat{f}$ approximating $f$. The core of the F-transform technique consists in partitioning of a given continuous interval using fuzzy sets that are in the theory of F-transform usually called *basic functions*. The (finite) system of basic functions is called *fuzzy partition*. It should be noted that the idea of fuzzy partition was initiated in the paper [7] as a natural generalization of the standard concept of a partition where the condition to be mutually disjoint is relaxed a little.

Despite of the result in [5, Corollary 2] saying that the function $\hat{f}$ uniformly converges to the original function $f$, one can recognize a problem with smoothness of $\hat{f}$. For example, one can see in a model of trend of time series using the F-transform that larger spreads of basic functions lead to less smooth trends. A possible solution of this problem is to use a generalization of the concept of fuzzy partition suggested in [8] and investigated in [1, 10]. In contrast with the original definition of fuzzy partition, where only two fuzzy sets can have a non-empty intersection with respect to minimum operation, the generalized fuzzy partition relaxes this condition to an arbitrary number fuzzy sets. In [8, 10], this number is constant, but generally one can omit even this restriction (see [1]). Besides better control of the smoothness of the resulting function, the generalized fuzzy partitions may also better reduce random noise (see [1, Corollary 4.10]).

From the theoretical point of view the most important generalized fuzzy partitions are the uniform ones. They are obtained by means of one fixed fuzzy set $K$ called a generating function, a bandwidth $h$ and a constant shift $r$. Thus, the fuzzy partition is characterized by a triplet $(K, h, r)$.

In this paper, we provide a necessary and sufficient condition for a uniformly defined system of fuzzy sets to form a generalized fuzzy partition. We thus obtain a tool using which we can check effectively if a generating function with a given bandwidth and a shift defines a generalized uniform fuzzy partition.

The paper is structured as follows. In the next section, we investigate necessary and sufficient condition for the uniform fuzzy partitions. Results of this section are generalized in Section 3 where we form the necessary and sufficient condition for the generalized fuzzy partitions. Section 4 illustrates how the necessary and sufficient condition can be used in the analysis of the triangle and raised cosine type fuzzy partitions. Section 5 contains concluding remarks.

## 2. Necessary and sufficient conditions for uniform fuzzy partitions

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the set of natural numbers, integers and reals, respectively. It is well-known that a uniform fuzzy partition is defined using a generating function $K$ which is modified by a parameter $h$ characterizing its spread. Each basic function of the uniform fuzzy partition is then constructed by a suitable shift of the modified generating function $K$.

**Definition 2.1.** A function $K : \mathbb{R} \to [0,1]$ is called a *generating function* if it is an even integrable function that is non-increasing in $[0,\infty)$ such that

$$K(x) \begin{cases} > 0, & \text{if } x \in (-1,1); \\ = 0, & \text{otherwise.} \end{cases} \tag{1}$$

A generating function $K$ is said to be *normal* if $K(0) = 1$.

Note that the previous definition is more general than the analogous definition of a generating function in [6], because the continuity of $K$ is replaced by its integrability and the normality of $K$ is considered as an additional extra condition.[1] For our investigation of necessary and sufficient condition of uniform fuzzy partitions, we will consider uniform fuzzy partitions of the real line defined as follows (cf. [2]).

**Definition 2.2.** Let $K$ be a normal generating function, $h$ be a positive real number and $x_0 \in \mathbb{R}$. A system of fuzzy sets defined by

$$A_i(x) = K\left(\frac{x - x_0}{h} - i\right) \tag{2}$$

for any $i \in \mathbb{Z}$ is said to be a *uniform fuzzy partition of the real line determined by the triplet* $(K, h, x_0)$ if the following condition[2] is satisfied:

$$S(x) = \sum_{i \in \mathbb{Z}} A_i(x) = 1 \tag{3}$$

holds for any $x \in \mathbb{R}$.

The parameters $h$ and $x_0$ are called a *spread* and a *central node*, respectively. The fuzzy sets $A_i$ in (2) that form a uniform fuzzy partition of the real line are called *basic functions*. A simple consequence of (2) is the formula $A_i(x) = A_0(x - hi)$ that holds for any $x \in \mathbb{R}$ and $i \in \mathbb{Z}$. Putting $x_i = x_0 + ih$ one can simply check that $A_i(x_i) = 1$ and $A_i$ is centered around the node $x_i$.

**Remark 2.1 (Important).** One can see that a uniform fuzzy partition (UFP) of closed real intervals used in the fuzzy transform can be extended to a uniform fuzzy partition of the whole real line. Therefore, each UFP of a closed real interval can be understood as a UFP of the real line limited to the closed real interval. Hence, investigation of properties of uniform fuzzy partitions can be restricted to investigation of their properties on the real line. For the sake of simplicity, we will omit the clause "real line" when speaking about uniform fuzzy partition.

[1] In [1], a generating function was called a *basal function*.

[2] This conditions is often called Ruspini's condition.

Let us present two most useful examples of the generating function and a uniform fuzzy partition determined by it (see [5]).

**Example 2.2 (Triangle generating function).** Let $K : \mathbb{R} \to [0,1]$ be defined by

$$K_T(x) = \max(1 - |x|, 0). \tag{4}$$

One can see in Figure 1 part of the uniform fuzzy partition of $\mathbb{R}$ determined by $(K_T, 2, 1)$. For example, the basic function $A_2$ is obtained by transforming of $K_T$ to a fuzzy set $K_{T,h}$ having the bandwidth $h = 2$ and shifting the center 0 of $K_{T,h}$ to the new center (node) $x_2 = x_0 + 2h = 1 + 2 \cdot 2 = 5$. The transformed function $K_{T,h}$ is depicted on Figure 1 using dashed line.

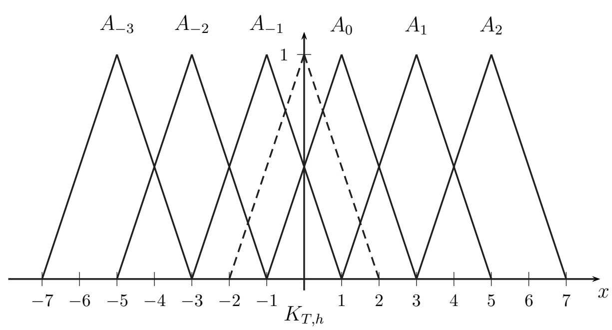


Figure 1: A part of the UFP of the real line determined by $(K_T, 2, 1)$. The transformed triangle generating function $K_{T,h}$ with $h = 2$ centered around 0 is depicted by dashed line.

**Example 2.3 (Raised cosine generating function).** Let $K : \mathbb{R} \to [0,1]$ be defined by

$$K_C(x) = \begin{cases} \frac{1}{2}(1 + \cos(\pi x)), & -1 \le x \le 1; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

On Fig. 2, one can see a part of the UFP of $\mathbb{R}$ determined by $(K_C, 2, 1)$. By dashed line is depicted the transformed raised cosine generating function $K_{C,h}$.

In the sequel, we are interested in conditions under which we can decide if a triplet $(K, h, x_0)$ determines a UFP. One will see that choice of the central node has no influence on verification if a triplet $(K, h, x_0)$ determines a UFP. The following lemma demonstrates that it is sufficient to consider the special case of $(K, h, x_0)$ for $x_0 = 0$.

**Lemma 2.1.** *A triplet $(K, h, x_0)$ determines a UFP iff $(K, h, 0)$ determines a UFP.*

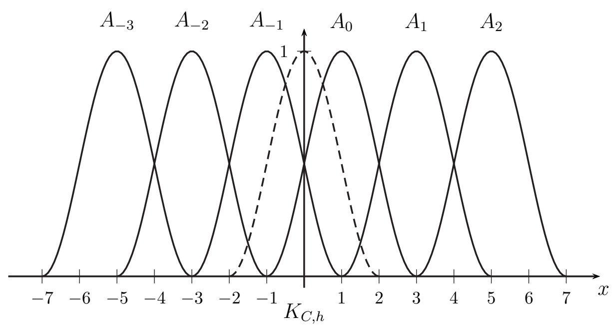


Figure 2: Part of the UFP of the real line determined by $(K_C, 2, 1)$. The transformed raised cosine generating function $K_{C,h}$ with $h = 2$ centered around 0 is depicted by dashed line.

*Proof.* Let $(K, h, x_0)$ determine a UFP and consider the triplet $(K, h, 0)$. By the definition of UFP, the triplet $(K, h, 0)$ determines a UFP if it satisfies (3). Put

$$S_{x_0}(x) = \sum_{i \in \mathbb{Z}} K\left(\frac{x - x_0}{h} - i\right).$$

From the assumption, we know that $S_{x_0}(x) = 1$ for any $x \in \mathbb{R}$, and we have to prove that the same holds for $S_0(x)$. Thus,

$$S_0(x) = \sum_{i \in \mathbb{Z}} K\left(\frac{x}{h} - i\right) = \sum_{i \in \mathbb{Z}} K\left(\frac{(x + x_0) - x_0}{h} - i\right) = S_{x_0}(x + x_0) = 1.$$

The converse implication can be proved by analogous arguments. □

We will below restrict ourselves to uniform fuzzy partitions with $x_0 = 0$ and so, we will write $(K, h)$ instead of $(K, h, x_0)$.

**Example 2.4.** Let $K_T$ be the triangle generating function and $y \in [\frac{1}{2}, 1]$ be an arbitrary element. Then,

$$\int_{1-y}^{y} K_T(x)dx = \int_{1-y}^{y} (1 - x)dx = \left[x - \frac{x^2}{2}\right]_{1-y}^{y} = y - \frac{1}{2}.$$

**Example 2.5.** Let $K_C$ be the raised cosine generating function and $y \in [1/2, 1]$ be an arbitrary element. Then,

$$\int_{1-y}^{y} K_C(x)dx = \int_{1-y}^{y} \frac{1}{2}(1 + \cos(\pi x))dx = \left[\frac{x}{2} + \frac{\sin(\pi x)}{2\pi}\right]_{1-y}^{y} = y - \frac{1}{2}.$$

Note that in both examples, we obtained

$$\int_{1-y}^{y} K(x)dx = y - \frac{1}{2} \tag{6}$$

holds for any $\frac{1}{2} \leq y \leq 1$. Putting $y = 1$ in (6), one can simply obtain from the symmetry of $K$ that $\int_{-1}^{1} K(x)dx = 1$, which implies that $\int_{-\infty}^{\infty} A_i(x)dx = h$ for any $i \in \mathbb{Z}$. Recall that $x_{i+1} = x_i + h$, thus the difference between two consecutive nodes is $h$.These observations motivate us to formulate a necessary and sufficient condition for the uniform fuzzy partitions in the following form.

**Theorem 2.2.** *A triplet $(K, h, x_0)$ determines a uniform fuzzy partition iff $x_1 - x_0 = h \int_{-\infty}^{\infty} K(x)dx$ and $\int_{1-y}^{y} K(x)dx = y - \frac{1}{2}$ holds for any $y \in [\frac{1}{2}, 1]$.*

*Proof.* ($\Rightarrow$) By Definition 2.2, we know that $x_1 - x_0 = h$. Then, it is sufficient to prove that $\int_{-1}^{1} K(x)dx = 1$.

From (3) and Definition 2.2, we have (substituting $u = \frac{x}{h} - i$)

$$2h = \int_{-h}^{h} S(x)dx = \int_{-h}^{h} \left( \sum_{i=0,\pm1} A_i(x) \right) dx = \sum_{i=0,\pm1} \int_{-h}^{h} K\left(\frac{x}{h} - i\right) dx =$$
$$= \sum_{i=0,\pm1} h \int_{-1-i}^{1-i} K(u)du = 2h \int_{-1}^{1} K(u)du.$$

Hence, we obtained that $\int_{-1}^{1} K(u)du = 1$.

Let us now show that $\int_{1-y}^{y} K(x)dx = y - \frac{1}{2}$ holds for any $\frac{1}{2} \leq y \leq 1$. From (3) and Definition 2.2, we have $\int_{-z}^{z} S(x)dx = 2z$ for any positive real number $z$. Consider $z \in [\frac{h}{2}, h]$. Then (substituting $u = \frac{x}{h} - i$)

$$2z = \int_{-z}^{z} S(x)dx = \int_{-z}^{z} \sum_{i\in\mathbb{Z}} K\left(\frac{x}{h} - i\right) dx = \sum_{i=0,\pm1} \int_{-z}^{z} K\left(\frac{x}{h} - i\right) dx$$
$$= \sum_{i=0,\pm1} h \int_{-\frac{z}{h}-i}^{\frac{z}{h}-i} K(u)du =$$
$$= h\Big( \int_{-\frac{z}{h}}^{\frac{z}{h}} K(u)du + \int_{-1}^{\frac{z}{h}-1} K(u)du + \int_{-\frac{z}{h}+1}^{1} K(u)du \Big) =$$
$$= 2h\Big( \int_{0}^{\frac{z}{h}} K(u)du + \int_{-\frac{z}{h}+1}^{1} K(u)du \Big) = 2h\Big(\frac{1}{2} + \int_{-\frac{z}{h}+1}^{\frac{z}{h}} K(u)du\Big),$$

where we used a simple consequence of the symmetry of $K$ (i.e., $K(x) = K(-x)$):

$$\int_{-\frac{z}{h}}^{\frac{z}{h}} K(u)du = 2\int_{0}^{\frac{z}{h}} K(u)du \quad \text{and} \quad \int_{-1}^{\frac{z}{h}-1} K(u)du = \int_{-\frac{z}{h}+1}^{1} K(u)du.$$

By putting $y = \frac{z}{h}$, we obtain the desired statement.

($\Leftarrow$) It easy to see that $S(x)$ is an even periodic function with the period $h$, i.e., $S(x) = S(-x)$ and $S(x) = S(x + h)$. Indeed, we have (recall that

$K(x) = K(-x))$

$$S(x) = \sum_{i\in Z} K\left(\frac{x}{h} - i\right) = \sum_{i\in Z} K\left(-\frac{x}{h} + i\right) = \sum_{i\in Z} K\left(-\frac{x}{h} - i\right) = S(-x),$$

$$S(x) = \sum_{i\in Z} K\left(\frac{x}{h} - i\right) = \sum_{i\in Z} K\left(\frac{x+h}{h} - (i+1)\right) =$$

$$= \sum_{i\in Z} K\left(\frac{x+h}{h} - i\right) = S(x+h).$$

Hence, it is sufficient to prove that $\int_0^z S(x)dx = z$ for $z \in [0,h]$, because this statement is equivalent to $S(x) = 1$ for any $x \in [0,h]$.[3] Since $S(x)$ is periodic with the period $h$, we simply obtain that $S(x) = 1$ for any $x \in \mathbb{R}$. Analogously to the proof of the necessary condition, we obtain

$$\int_{-z}^{z} S(x)dx = 2h\big(\int_0^{\frac{z}{h}} K(u)du + \int_{-\frac{z}{h}+1}^{1} K(u)du\big).$$

If $\frac{z}{h} \in [\frac{1}{2}, 1]$, then (by the assumption on $K$)

$$\int_{-z}^{z} S(x)dx = 2h\big(\frac{1}{2} + \int_{-\frac{z}{h}+1}^{\frac{z}{h}} K(u)du\big) = 2h\big(\frac{1}{2} + \frac{z}{h} - \frac{1}{2}\big) = 2z.$$

If $\frac{z}{h} \in [0, \frac{1}{2})$, then

$$\int_{-z}^{z} S(x)dx = 2h\big(\frac{1}{2} - \int_{\frac{z}{h}}^{-\frac{z}{h}+1} K(u)du\big) = 2h\big(\frac{1}{2} - \int_{1-(-\frac{z}{h}+1)}^{-\frac{z}{h}+1} K(u)du\big) =$$

$$2h\big(\frac{1}{2} - (-\frac{z}{h} + 1 - \frac{1}{2})\big) = 2h\big(\frac{1}{2} + \frac{z}{h} - \frac{1}{2}\big) = 2z.$$

Since $S(x)$ is an even function, we obtain $\int_0^z K(x)dx = z$ for any $z \in [0,h]$, which completes the proof. □

## 3. Necessary and sufficient conditions for generalized uniform fuzzy partitions

As mentioned in Introduction, generalization of uniform fuzzy partitions with more active basic functions can be used to obtain smoother result of (inverse) fuzzy transform and to reduce better random noise. In contrast with generalization of uniform fuzzy partitions considered in [1] and [10], we suppose that generating functions need not take normal form.

[3] Indeed, if $\int_0^z S(x)dx = z$ for $z \in [0,h]$, then putting $H(z) = \int_0^z S(x)dx$ we obtain $\frac{dH(z)}{dz} = S(z) = 1$.

**Definition 3.1.** Let $K$ be a generating function, $h$ and $r$ be positive real numbers and $x_0 \in \mathbb{R}$. A system of fuzzy sets defined by

$$A_i(x) = K\left(\frac{x - x_0 - i\,r}{h}\right) \tag{7}$$

for any $i \in \mathbb{Z}$ is called a *generalized uniform fuzzy partition (GUFP) of the real line determined by the quadruplet* $(K, h, r, x_0)$ if the Ruspini's condition is satisfied.

The parameters $h$ and $x_0$ have the same meaning as in the case of uniform fuzzy partitions, and $r$ is called a *shift*. Let $K$ be a generating function and $h$ a bandwidth. By $K_h(x) = K(\frac{x}{h})$ we denote a generating function modified by the bandwidth $h$.

**Remark 3.1.** It is easy to see that if one requires the normality of $K$, an equivalent definition of a generalized uniform fuzzy partition of the real line is obtained if we require $S(x)$ to be a constant function on $\mathbb{R}$ (cf., [1, 9, 10]).

Clearly, a generalized uniform fuzzy partition determined by $(K, h, h, x_0)$ is a uniform fuzzy partition (by Definition 2.2), where the normality of $K$ immediately follows from the Ruspini's condition. Analogously as in the case of uniform fuzzy partitions (see Lemma 2.1), the central node does not play a significant role in the investigation below.

**Lemma 3.1.** *A quadruplet* $(K, h, r, x_0)$ *determines a generalized uniform fuzzy partition iff* $(K, h, r, 0)$ *determines it.*

*Proof.* Obvious. □

By this lemma, we can restrict ourselves to quadruplets in the form $(K, h, r, 0)$. For simplicity, we will write only $(K, h, r)$ instead of $(K, h, r, 0)$.

**Lemma 3.2.** *If* $(K, h, r)$ *determines a generalized uniform fuzzy partition then* $r = h \int_{-1}^{1} K(x)dx$.

*Proof.* Recall that $K_h(x) = K(\frac{x}{h})$. One can see that $\int_{-h}^{h} K_h(x)dx = h\int_{-1}^{1} K(x)dx$. We will prove that $r = \int_{-h}^{h} K_h(x)dx$.

Let $k$ be the greatest natural number for which $-h + kr < h$ holds true. means that $-h + (k+1)r \geq h$. Put

$$\begin{aligned} R = \int_{h-kr}^{-h+r} K_h(x)dx + \int_{h-(k-1)r}^{-h+2r} K_h(x)dx + \cdots + \int_{h-r}^{-h+kr} K_h(x)dx = \\ = \sum_{i=1}^{k} \int_{h-(k-i+1)r}^{-h+ir} K_h(x)dx. \end{aligned} \tag{8}$$

We will show that $R$ is a remainder in the computation of two special integrals.[4]

Put $r' = (k+1)r - 2h$ and consider the following two integrals

$$\int_{-h-r}^{h+r} S(x)dx = 2(h+r) \quad \text{and} \quad \int_{-h-r'}^{h+r'} S(x)dx = 2(h+r').$$

Note that $-h-r'+(k+1)r = h$ and $h+r'-(k+1)r = -h$. Now, let us expand the first integral to the integrals containing $K_h$ (substituting $u = x - ir$):

$$\int_{-h-r}^{h+r} S(x)dx = \int_{-h-r}^{h+r} \left( \sum_{i\in\mathbb{Z}} K_h(x-ir) \right) dx = \sum_{i\in\mathbb{Z}} \int_{-h-r-ir}^{h+r-ir} K_h(u)du =$$

$$= \int_{-h-r}^{h+r} K_h(u)du + \int_{-h-2r}^{h} K_h(u)du + \int_{-h}^{h+2r} K_h(u)du +$$

$$+ \sum_{i=1}^{k} \int_{-h-2r-ir}^{h-ir} K_h(u)du + \sum_{i=1}^{k} \int_{-h+ir}^{h+2r+ir} K_h(u)du =$$

$$= 3\int_{-h}^{h} K_h(u)du + \sum_{i=1}^{k} \int_{-h}^{h-ir} K_h(u)du + \sum_{i=1}^{k} \int_{-h+ir}^{h} K_h(u)du =$$

$$= 3\int_{-h}^{h} K_h(u)du + \left( \int_{-h}^{h-r} K_h(u)du + \int_{-h+kr}^{h} K_h(u)du \right) +$$

$$+ \left( \int_{-h}^{h-2r} K_h(u)du + \int_{-h+(k-1)r}^{h} K_h(u)du \right) + \cdots +$$

$$+ \left( \int_{-h}^{h-kr} K_h(u)du + \int_{-h+r}^{h} K_h(u)du \right).$$

Note that by the assumption on $k$, the integrals

$$\int_{-h-2r-ir}^{h-ir} K_h(u)du \quad \text{and} \quad \int_{-h+ir}^{h+2r+ir} K_h(u)du$$

are equal to 0 for any $i = k+1, k+2, \ldots$ Therefore, the infinite sum considered above can be replaced by finite one.

Since $k$ is the greatest natural number with $-h+kr < h$, we obtain that $h-r \leq -h+kr$, $h-2r \leq -h+(k-1)r$, etc. Then,

$$\int_{-h}^{h-r} K_h(u)du + \int_{-h+kr}^{h} K_h(u)du =$$

$$= \int_{-h}^{h-r} K_h(u)du + \int_{h-r}^{-h+kr} K(u)du + \int_{-h+kr}^{h} K_h(u)du - \int_{h-r}^{-h+kr} K(u)du =$$

$$= \int_{-h}^{h} K_h(u)du - \int_{h-r}^{-h+kr} K_h(u)du,$$

[4]In Figure 3, example of the integral $R$ in (8) is presented.

where $\int_{h-r}^{-h+kr} K_h(u)du$ is the last integral in the expression of $R$. Analogously, one could express the remaining brackets which implies

$$2(h+r)=\int_{-h-r}^{h+r} S(x)dx=(3+k)\int_{-h}^{h} K_h(u)du-R. \tag{9}$$

Analogously, let us expand the second integral (we omit the first two steps):

$$\int_{-h-r'}^{h+r'} S(x)dx=\sum_{i\in\mathbb{Z}}\int_{-h-r'-ir}^{h+r'-ir} K_h(u)du=$$

$$=\int_{-h-r'}^{h+r'} K_h(u)du+\sum_{i=1}^{k}\int_{-h-'r-ir}^{h+r'-ir} K_h(u)du+\sum_{i=1}^{k}\int_{-h-'r+ir}^{h+r'+ir} K(u)du=$$

$$=\int_{-h}^{h} K_h(u)du+\sum_{i=1}^{k}\int_{-h}^{h-r'-ir} K_h(u)du+\sum_{i=1}^{k}\int_{-h-'r+ir}^{h} K_h(u)du=$$

$$=\int_{-h}^{h} K_h(u)du+\left(\int_{-h-r'+r}^{h} K_h(u)du+\int_{-h}^{h+r'-kr} K_h(u)du\right)+$$

$$\left(\int_{-h-r'+2r}^{h} K_h(u)du+\int_{-h}^{h+r'-(k-1)r} K_h(u)du\right)+\cdots+$$

$$+\left(\int_{-h-r'+kr}^{h} K_h(u)du+\int_{-h}^{h+r'-r} K_h(u)du\right).$$

In this case, however, we have $-h-r'+r\leq h+r'-kr$, $-h-r'+2k\leq h+r'-(k-1)r$, etc. Since $-h-r'+r=h-kr$ and $h+r'-kr=-h+r$, the declared inequality may be rewritten as $h-kr\leq -h+r$. Then

$$\int_{-h-r'+r}^{h} K_h(u)du+\int_{-h}^{h+r'-kr} K_h(u)du=\int_{h-kr}^{h} K_h(u)du+\int_{-h}^{-h+r} K_h(u)du=$$

$$=\int_{-h}^{h-kr} K_h(u)du+\int_{h-kr}^{h} K_h(u)du+\int_{h-kr}^{-h+r} K_h(u)du=$$

$$=\int_{-h}^{h} K_h(u)du+\int_{h-kr}^{-h+r} K_h(u)du,$$

where $\int_{h-kr}^{-h+r} K_h(u)du$ is the first integral in the expression of $R$.

Analogously, we can express the formula in the remaining brackets which implies

$$2(h+r')=\int_{-h-r'}^{h+r'} S(x)dx=(1+k)\int_{-h}^{h} K_h(u)du+R. \tag{10}$$

Putting $\mu=\int_{-h}^{h} K_h(u)du=h\int_{-1}^{1} K(x)dx$ and adding (9) and (10), we obtain

$$2(2h+r+r')=2(2+k)\mu.$$

Substituting $r' = (k+1)r - 2h$ into the previous equality, we obtain

$$(k+2)r = (k+2)\mu.$$

Hence, we obtain $r = \mu = h \int_{-1}^{1} K(x)dx$, which concludes the proof. □

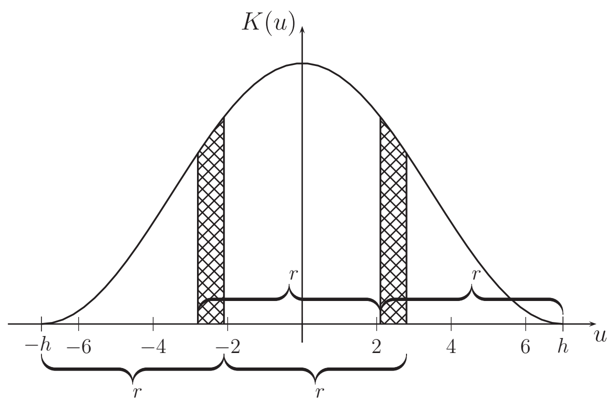


Figure 3: The crosshatched surface expresses the value of $R$ ($h = 7$ and $r = 4.9$).

It is easy to see that this lemma generalizes the first part of the necessary condition for uniform fuzzy partitions. As we know, the uniform fuzzy partitions deal with normal generating functions for which $\int_{-1}^{1} K(x)dx = 1$. Hence, we obtain as the result $r = h$.

**Lemma 3.3.** *If $(K, h, r)$ determines a generalized uniform fuzzy partition then*

$$y = \frac{r}{2} + \sum_{i=1}^{\infty} \int_{ir-y}^{y+(i-1)r} K_h(x)dx. \tag{11}$$

*holds for any $y \in [\frac{r}{2}, r]$.*

*Proof.* Let $(K, h, r)$ determine a GUFP and $y \in [\frac{r}{2}, r]$ be arbitrary. Analogously to the proof of Lemma 3.2, we consider the integral

$$2y = \int_{-y}^{y} S(x)dx,$$

which may be expanded as follows (substituting $u = x - ir$):

$$\int_{-y}^{y} S(x)dx = \int_{-y}^{y} \left( \sum_{i\in\mathbb{Z}} K_h(x - ir) \right) dx = \sum_{i\in\mathbb{Z}} \int_{-y-ir}^{y-ir} K_h(u)du =$$
$$= \int_{-y}^{y} K_h(u)du + \sum_{i=1}^{\infty} \int_{-y-ir}^{y-ir} K_h(u)du + \sum_{i=1}^{\infty} \int_{-y+ir}^{y+ir} K_h(u)du.$$

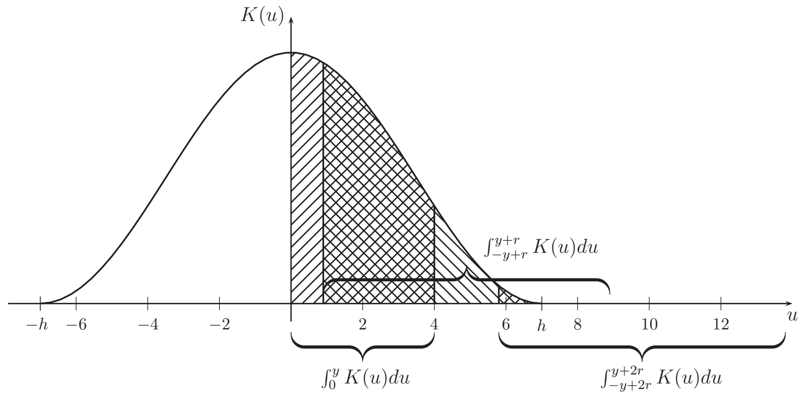


Figure 4: The crosshatched surfaces denote the intersections of consecutive integrals ($y = 3$, $h = 7$, $r = 4.9$).

In Figure 4, one can see three surfaces expressing the right symmetric side of the integral $\int_{-y}^{y} K_h(u)du$, the first integral $\int_{-y+r}^{y+r} K_h(u)du$, and the second integral $\int_{-y+2r}^{y+2r} K_h(u)du$ from the sum $\sum_{i=1}^{\infty} \int_{-y+ir}^{y+ir} K_h(u)du$.[5] Clearly, to calculate the integral $\int_0^h K_h(u)du$ in this special example, the integrals from the sum $\sum_{i=3}^{\infty} \int_{-y+ir}^{y+ir} K_h(u)du$ may be ignored, because they are equal to 0. It is easy to see (cf. Figure 4) that

$$\int_0^y K_h(u)du + \int_{-y+r}^{y+r} K_h(u)du + \int_{-y+2r}^{y+2r} K_h(u)du + \cdots =$$
$$= \int_0^{-y+r} K_h(u)du + 2\int_{-y+r}^{y} K_h(u)du + \int_{y}^{-y+2r} K_h(u)du +$$
$$2\int_{-y+2r}^{y+r} K_h(u) + \int_{y+r}^{-y+3r} K_h(u)du + \cdots .$$

In general, this result can be rewritten as

$$\int_0^y K_h(u)du + \sum_{i=1}^{\infty} \int_{-y+ir}^{y+ir} K_h(u)du =$$
$$= \int_0^{-y+r} K_h(u)du + \sum_{i=1}^{\infty} \left( 2\int_{-y+ir}^{y+(i-1)r} K_h(u)du + \int_{y+(i-1)r}^{-y+(i+1)r} K_h(u)du \right) =$$
$$= \int_0^h K_h(u)du + \sum_{i=1}^{\infty} \int_{-y+ir}^{y+(i-1)r} K_h(u)du = \frac{r}{2} + \sum_{i=1}^{\infty} \int_{ir-y}^{y+(i-1)r} K_h(u)du,$$

[5]In this figure, we took $K_h(u) = 0.35(1 + \cos(\frac{\pi u}{7}))$, $h = 7$, $r = 4.9$, and $y = 3$.

where we used $r = \int_{-h}^{h} K_h(u)du$ from Lemma 3.2. From the symmetry of $K_h$, we obtain that

$$2y = \int_{-y}^{y} S(x)dx = 2\left(\frac{r}{2} + \sum_{i=1}^{\infty} \int_{ir-y}^{y+(i-1)r} K_h(u)du\right), \tag{12}$$

and the proof is finished. □

**Remark 3.2.** Put $z = y - \frac{r}{2}$. Then, we can rewrite the condition (11) as

$$z = \sum_{i=1}^{\infty} \int_{-z+\beta_i}^{z+\beta_i} K_h(x)dx, \tag{13}$$

where $z \in [0, \frac{r}{2}]$ and $\beta_i = (2i-1)\frac{r}{2}$. Since the domain of $K_h$ is $[-h, h]$, one can simply check that

$$z = \sum_{i=1}^{k} \int_{-z+\beta_i}^{z+\beta_i} K_h(x)dx, \tag{14}$$

where $k$ is the least natural number for which it holds $-\frac{r}{2} + \beta_k \geq h$. Of course, we may have $\int_{-z+\beta_k}^{z+\beta_k} K_h(x)dx = 0$ for some choices of $z$.

It is easy to see that if we consider a uniform fuzzy partition determined by $(K, h, h)$, then (11) can be rewritten as

$$y = \frac{h}{2} + \sum_{i=1}^{\infty} \int_{ih-y}^{y+(i-1)h} K_h(x)dx = \frac{h}{2} + \int_{h-y}^{y} K_h(x)dx,$$

which holds for any $y \in [\frac{h}{2}, h]$. Putting $y' = \frac{y}{h}$, we obtain $y' \in [\frac{1}{2}, 1]$. Then, the previous equality may be expressed as

$$hy' - \frac{h}{2} = \int_{h-hy'}^{hy'} K_h(x)dx = h\int_{1-y'}^{y'} K(x)dx.$$

Now, one can see that the necessary condition for the uniform fuzzy partition provided in Theorem 2.2 is a special case of the conditions in Lemmas 3.2 and 3.3. We will show that these conditions are also sufficient.

**Lemma 3.4.** *If $(K, h, r)$ is a triplet such that $r = h\int_{-1}^{1} K(u)du$ and (11) is satisfied for any $y \in [\frac{r}{2}, r]$ then $(K, h, r)$ determines a generalized uniform fuzzy partition.*

*Proof.* Analogously to the proof of Theorem 2.2, one can simply check that $S(x)$ determined by $(K, h, r)$ is an even periodic function with the period $r$. Hence, it is sufficient to prove that $\int_0^y S(x)dx = y$ for any $y \in [0, r]$, because this implies $S(x) = 1$ for any $x \in \mathbb{R}$ (see the sufficiency part of the proof of Theorem 2.2).

In the proof of Lemma 3.3, we have shown (see (12)) that

$$\int_{-y}^{y} S(x)dx = 2\left(\frac{r}{2} + \sum_{i=1}^{\infty} \int_{ir-y}^{y+(i-1)r} K_h(u)du\right).$$

As a consequence of the symmetry of $S(x)$, we obtain

$$\int_{0}^{y} S(x)dx = \frac{r}{2} + \sum_{i=1}^{\infty} \int_{ir-y}^{y+(i-1)r} K_h(u)du. \qquad (15)$$

If $y \in [\frac{r}{2}, r]$, then a straightforward consequence of (11) is $\int_0^y S(x) = y$. Analogously to the derivation of the previous formula (15), one can simply check that if $y \in [0, \frac{r}{2})$, then

$$\int_{0}^{y} S(x)dx = \frac{r}{2} - \sum_{i=1}^{\infty} \int_{ir-(r-y)}^{ir-y} K_h(u)du. \qquad (16)$$

Hence, we obtain (using assumption (11) and the fact that $r - y \in [\frac{r}{2}, r]$)

$$\int_{0}^{y} S(x)dx = r - \left(\frac{r}{2} + \sum_{i=1}^{\infty} \int_{ir-(r-y)}^{(r-y)+(i-1)r} K_h(u)du\right) = r - (r-y) = y,$$

which concludes the proof. □

The results of the previous three lemmas provide us necessary and sufficient condition for generalized uniform fuzzy partitions.

**Theorem 3.5.** *A triplet $(K, h, r)$ determines a generalized uniform fuzzy partition iff $r = h\int_{-1}^{1} K(x)dx$ and (11) is satisfied for any $y \in [\frac{r}{2}, r]$.*

As a simple consequence of this theorem we obtain that a GUFP determined by $(K, h, r)$ remains a GUFP if we can change bandwidth of the generating function $K$.

**Corollary 3.6.** *If $(K, h, r)$ determines a generalized uniform fuzzy partition and $\alpha > 0$ is a real number then $(K, \alpha h, \alpha r)$ determines it as well.*

*Proof.* We must prove that $(K, \alpha h, \alpha r)$ satisfies the necessary conditions of Theorem 3.5. We have $r = h\int_{-1}^{1} K(x)dx$ by the assumption. Then

$$\alpha r = \alpha h \int_{-1}^{1} K(x)dx,$$

and the first necessary condition is satisfied. Let $y \in [\frac{\alpha r}{2}, \alpha r]$. Then $\frac{y}{\alpha} \in [\frac{r}{2}, r]$ and

$$\frac{y}{\alpha} = \frac{r}{2} + \sum_{i=1}^{\infty} \int_{ir-\frac{y}{\alpha}}^{\frac{y}{\alpha}+(i-1)r} K_h(x)dx,$$

whence

$$y = \frac{\alpha r}{2} + \sum_{i=1}^{\infty} \alpha \int_{ir-\frac{y}{\alpha}}^{\frac{y}{\alpha}+(i-1)r} K_h(x)dx.$$

Putting $\alpha x = u$, we obtain

$$y = \frac{\alpha r}{2} + \sum_{i=1}^{\infty} \alpha \int_{i\alpha r-y}^{y+(i-1)\alpha r} K_h\left(\frac{u}{\alpha}\right) du = \frac{\alpha r}{2} + \sum_{i=1}^{\infty} \int_{i\alpha r-y}^{y+(i-1)\alpha r} K_{\alpha h}(u)du,$$

where $K_{\alpha h}(u) = K(\frac{u}{\alpha h}) = K_h(\frac{u}{\alpha})$. Hence, the second necessary condition of Theorem 3.5 is satisfied and so, $(K, \alpha h, \alpha r)$ determines a GUFP. □

## 4. An application of necessary and sufficient conditions

In this section, we will demonstrate how the results obtained in the previous section can be applied in investigation of the generalized uniform fuzzy partitions. By $\mathbb{R}^+$ we denote the set of positive reals.

Let us define the product of scalars from $\mathbb{R}^+$ and real functions by

$$(\alpha \odot f)(x) = \alpha f(x). \tag{17}$$

where $\alpha f(x)$ is the product of reals. It is easy to see that if $K$ is a generating function, then $\alpha \odot K$ need not be a generating function, because $(\alpha \odot K)(0)$ may be greater than 1, i.e., $\alpha \odot K$ is not a fuzzy set.

Let $K$ be a generating function such that

$$K(0) = 1 \text{ and } \int_{-1}^{1} K(x)dx = 1.$$

For example, $K$ can be the generating function $K_T$ or $K_C$ from Examples 2.2 and 2.3, respectively. Using Theorem 3.5, $(K, h, r)$ can be a generalized uniform fuzzy partition if $h = r$. By Corollary 3.6, it is sufficient to verify that $(K, 1, 1)$ determines a GUFP, which is equivalent to verification that $(K, 1)$ determines a uniform fuzzy partition.

Now one can ask for which $\alpha \in \mathbb{R}$ the triplet $(\alpha \odot K, 1, \alpha)$ determines a generalized uniform fuzzy partition. Note that

$$r = h \int_{-1}^{1} (\alpha \odot K)(x)dx = \alpha\, 1 \int_{-1}^{1} K(x)dx = \alpha.$$

Below we will present a necessary and sufficient condition for $\alpha$ that allows us to determine infinitely many generalized uniform fuzzy partitions based on the triangle and raised cosine generating functions.

*4.1. Generalized uniform fuzzy partition of triangular type*

Let $K_T$ be the triangular generating function defined in Example 2.2. We will say that a generalized uniform fuzzy partition is of *triangle type* if its generating function is in the form $\alpha \odot K_T$.

**Theorem 4.1.** *Let $\alpha \in \mathbb{R}^+$ and $(\alpha \odot K_T)(0) \in (0, 1]$. Then, $(\alpha \odot K_T, 1, \alpha)$ determines a GUFP iff $\frac{1}{\alpha} \in \mathbb{N}$.*

*Proof.* By Theorem 2.2 and using Remark 3.2, $(\alpha \odot K_T, 1, \alpha)$ determines a GUFP iff

$$z = \sum_{i=1}^{\infty} \int_{-z+\beta_i}^{z+\beta_i} (\alpha \odot K_T)(x)dx \tag{18}$$

holds for any $z \in [0, \frac{\alpha}{2}]$, where $\beta_i = (2i-1)\frac{\alpha}{2}$.

($\Rightarrow$) Let us suppose that $(\alpha \odot K_T, 1, \alpha)$ determines a GUFP and let $k$ be the greatest natural number for which $\beta_k \leq 1$. It is easy to see that if $\beta_k < 1$, then there exists $z \in (0, \frac{\alpha}{2}]$ such that $z + \beta_k \leq 1$ and $-z + \beta_{k+1} \geq 1$. Let us consider two cases. First, let us suppose that $\beta_k < 1$ and let $z \neq 0$ satisfy the previous inequalities. Then, (18) can be rewritten as

$$z = \sum_{i=1}^{k} \int_{-z+\beta_i}^{z+\beta_i} (\alpha \odot K_T)(x)dx = \alpha \sum_{i=1}^{k} \int_{-z+\beta_i}^{z+\beta_i} (1-x)dx =$$

$$= \alpha \sum_{i=1}^{k} \big(z + \beta_i - \frac{z^2 + 2z\beta_i + \beta_i^2}{2} - (-z + \beta_i - \frac{z^2 - 2z\beta_i + \beta_i^2}{2})\big) =$$

$$= \alpha \sum_{i=1}^{k} 2z(1-\beta_i) = 2\alpha z\big(k - \sum_{i=1}^{k}(i\alpha - \frac{\alpha}{2})\big) =$$

$$= 2\alpha z\big(k + \frac{k\alpha}{2} - \frac{k(k+1)\alpha}{2}\big) = 2\alpha z\big(k - \frac{\alpha k^2}{2}\big).$$

This implies that $\alpha k = 1$. Since $k$ is a natural number, we obtain $\frac{1}{\alpha} \in \mathbb{N}$.

Let us suppose that $\beta_k = 1$, i.e., $k\alpha - \frac{\alpha}{2} = 1$. Since (18) is satisfied for any $z \in [0, \frac{\alpha}{2}]$, let us suppose that $z \in (0, \frac{\alpha}{2})$. Then, we have (applying the previous results and the fact that $(k-1)\alpha = 1 - \frac{\alpha}{2}$)

$$z = \sum_{i=1}^{k-1} \int_{-z+\beta_i}^{z+\beta_i} (\alpha \odot K_T)(x)dx + \int_{-z+\beta_k}^{1} (\alpha \odot K_T)(x)dx =$$

$$= 2\alpha z\big((k-1) - \frac{\alpha(k-1)^2}{2}\big) + \alpha \int_{-z+1}^{1} (1-x)(x)dx =$$

$$= 2\alpha z(k-1)\big(1 - \frac{\alpha(k-1)}{2}\big) + \frac{\alpha z^2}{2} = 2z(1 - \frac{\alpha}{2})(1 - \frac{1-\frac{\alpha}{2}}{2}) + \frac{\alpha z^2}{2} =$$

$$= z(1 - \frac{\alpha}{2})(1 + \frac{\alpha}{2}) + \frac{\alpha z^2}{2} = z(1 - \frac{\alpha^2}{4}) + \frac{\alpha z^2}{2},$$

which implies that $z = \frac{\alpha}{2}$. But contradicts $z \neq \frac{\alpha}{2}$. Hence, $\beta_i$ must be less than 1 and so, $\frac{1}{\alpha} \in \mathbb{N}$.

($\Leftarrow$) Let us consider a triplet $(\alpha \odot K_T, 1, \alpha)$ and $\frac{1}{\alpha} \in \mathbb{N}$. We must prove that (18) is satisfied for an arbitrary $z \in [0, \frac{\alpha}{2}]$. By the assumption on $\alpha$, we have $z + \beta_k \leq 1$ for any $z \in [0, \frac{\alpha}{2}]$ and $k\alpha = 1$, where $k$ was defined above. Therefore, using the previous results and the fact that $k\alpha = 1$, we obtain

$$\sum_{i=1}^{\infty} \int_{-z+\beta_i}^{z+\beta_i} (\alpha \odot K_T)(x)dx = \sum_{i=1}^{k} \int_{-z+\beta_i}^{z+\beta_i} (\alpha \odot K_T)(x)dx =$$
$$= 2\alpha k z\big(1 - \frac{\alpha k}{2}\big) = 2z(1 - \frac{1}{2}) = z,$$

which concludes the proof. □

**Remark 4.1.** It follows from the previous theorem and Corollary 3.6 that each generalized uniform fuzzy partition of triangle type has the form $(\alpha \odot K_T, h, \alpha h, x_0)$ for arbitrary $h \in \mathbb{R}^+$, $1/\alpha \in \mathbb{N}$ and $x_0 \in \mathbb{R}$.

In Figure 5, a part of the generalized uniform fuzzy partition of triangle type for $h = 2$, $\alpha = 1/4$ and $x_0 = 1$ is presented.

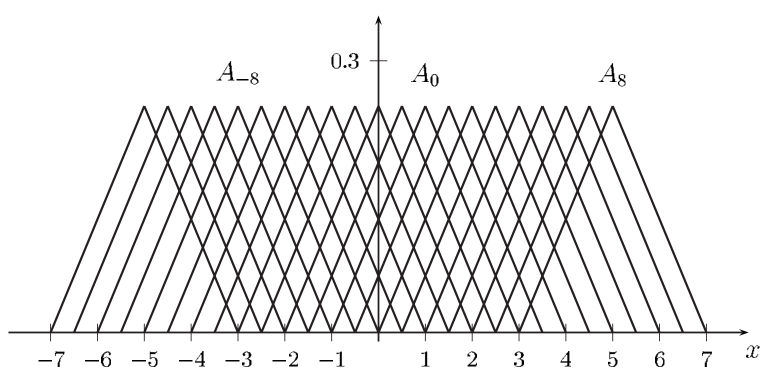


Figure 5: A part of the triangle type GUFP of the real line determined by $(0.25 \odot K_C, 2, 0.5, 1)$.

*4.2. Generalized uniform fuzzy partition of raised cosine type*

Let $K_C$ denote the raised cosine generating function defined in Example 2.3. We will say that a generalized uniform fuzzy partition is of *raised cosine type* if its generating function has in the form $\alpha \odot K_C$.

**Theorem 4.2.** *Let $\alpha \in \mathbb{R}^+$ and $(\alpha \odot K_C)(0) \in (0, 1]$. Then, $(\alpha \odot K_C, 1, \alpha)$ determines a generalized uniform fuzzy partition iff $\frac{1}{\alpha} \in \mathbb{N}$.*

*Proof.* Since the proof is nearly the same as the proof of Theorem 4.1, we omit some of its parts.

($\Rightarrow$) Let $(\alpha \odot K_C, 1, \alpha)$ determine a generalized uniform fuzzy partition and $z \in (0, \frac{\alpha}{2}]$ be such that $z + \beta_k \le 1$ and $-z + \beta_{k+1} \ge 1$. First, let $\beta_k < 1$ and $z \ne 0$ satisfy the previous inequalities. Then, (18) can be rewritten as

$$z = \sum_{i=1}^{k} \int_{-z+\beta_i}^{z+\beta_i} (\alpha \odot K_C)(x)dx = \sum_{i=1}^{k} \int_{-z+\beta_i}^{z+\beta_i} \frac{\alpha}{2}(1 - \cos(\pi x))dx =$$

$$\alpha \sum_{i=1}^{k} \left( \frac{z+\beta_i}{2} + \frac{\sin(\pi(z+\beta_i))}{2\pi} + \frac{z-\beta_i}{2} - \frac{\sin(\pi(-z+\beta_i))}{2\pi} \right) =$$

$$\alpha z k + \frac{1}{2\pi} \sum_{i=1}^{k} (\sin(\pi z)\cos(\pi\beta_i) + \cos(\pi z)\sin(\pi\beta_i) -$$

$$\sin(-\pi z)\cos(\pi\beta_i) - \cos(-\pi z)\sin(\pi\beta_i)) = \alpha z k + \frac{\alpha \sin(\pi z)}{\pi} \sum_{i=1}^{k} \cos(\pi\beta_i).$$

Putting $V = \frac{\alpha}{\pi} \sum_{i=1}^{k} \cos(\pi\beta_i)$, we can simplify the previous equality to

$$z = \alpha z k + V \sin(\pi z).$$

Now, let us suppose that $V \ne 0$. Then, the previous equality can be rewritten as (recall that $z \ne 0$)

$$\frac{\sin(\pi z)}{z} = \frac{1 - \alpha k}{V},$$

but this is a contradiction, because the function $\sin(\pi z)/z$ is not a constant function in $(0, \frac{\alpha}{2}]$. Hence, $V = 0$, which implies that $\alpha k = 1$, and so, $\frac{1}{\alpha} \in \mathbb{N}$.

Let $\beta_k = 1$. Then, we have (using the previous results)

$$z = \sum_{i=1}^{k-1} \int_{-z+\beta_i}^{z+\beta_i} (\alpha \odot K_C)(x)dx + \int_{-z+\beta_k}^{1} (\alpha \odot K_C)(x)dx =$$

$$\alpha z(k-1) + \frac{\alpha \sin(\pi z)}{\pi} \sum_{i=1}^{k-1} \cos(\pi\beta_i) + \frac{\alpha z}{2} - \frac{\alpha \sin(\pi z)}{2\pi} =$$

$$\alpha z(k - \frac{1}{2}) + \frac{\alpha \sin(\pi z)}{\pi} \left( \sum_{i=1}^{k-1} \cos(\pi\beta_i) - \frac{1}{2} \right) = \alpha z(k - \frac{1}{2}) - \frac{\alpha \sin(\pi z)}{2\pi},$$

where we used $\sum_{i=1}^{k-1} \cos(\pi\beta_i) = 0$. This equality follows from the assumption on $\beta_k$, where we can put $\beta_i = i/k$, and the fact that $\cos(i\pi/k) = -\cos(\pi - i\pi/k) = -\cos((k-i)\pi/k)$. Hence, we obtain

$$\frac{\sin(\pi z)}{z} = \frac{2\pi(\alpha(k - 1/2) - 1)}{\alpha},$$

but this is a contradiction, because the function $\sin(\pi z)/z$ is not a constant function in $(0, \alpha/2]$. Hence, $\beta_i$ is less than 1 and so, $1/\alpha \in \mathbb{N}$.

($\Leftarrow$) Let us consider a triplet $(\alpha \odot K_C, 1, \alpha)$ and $1/\alpha \in \mathbb{N}$. We must prove that (18) is satisfied for an arbitrary $z \in [0, \alpha/2]$. By the assumption on $\alpha$, we have $k\alpha = 1$ and $z + \beta_k \leq 1$ for any $z \in [0, \alpha/2]$. Therefore, using the previous results and the fact that $k\alpha = 1$, we obtain

$$\sum_{i=1}^{\infty} \int_{-z+\beta_i}^{z+\beta_i} (\alpha \odot K_C)(x)dx = \sum_{i=1}^{k} \int_{-z+\beta_i}^{z+\beta_i} (\alpha \odot K_C)(x)dx =$$

$$\alpha z k + \frac{\alpha \sin(\pi z)}{\pi} \sum_{i=1}^{k} \cos(\pi \beta_i) = z + \frac{\alpha \sin(\pi z)}{\pi} \sum_{i=1}^{k} \cos(\pi (2i-1)\frac{\alpha}{2}) =$$

$$z + \frac{\alpha \sin(\pi z)}{\pi} \sum_{i=1}^{k} \cos((2i-1)\frac{\pi}{2k}) = z,$$

where we used $\sum_{i=1}^{k} \cos((2i-1)\frac{\pi}{2k}) = 0$. Again, this equality follows from the assumption on $\alpha$, i.e., $\alpha = \frac{1}{k}$, and the fact that

$$\cos((2i-1)\frac{\pi}{2k}) = -\cos(\pi - (2i-1)\frac{\pi}{2k}) =$$

$$-\cos(\frac{2k\pi}{2k} - (2i-1)\frac{\pi}{2k}) = -\cos(2(k-i)-1)\frac{\pi}{2k}.$$

Hence, $(\alpha \odot K_C, 1, \alpha)$ determines a generalized uniform fuzzy partition. □

Analogously to Remark 4.1, one can characterize the class of all generalized uniform fuzzy partitions of cosine type. In Figure 6, a part of the raised cosine type GUFP for $h = 2$, $\alpha = 1/2$ and $x_0 = 1$ is presented.

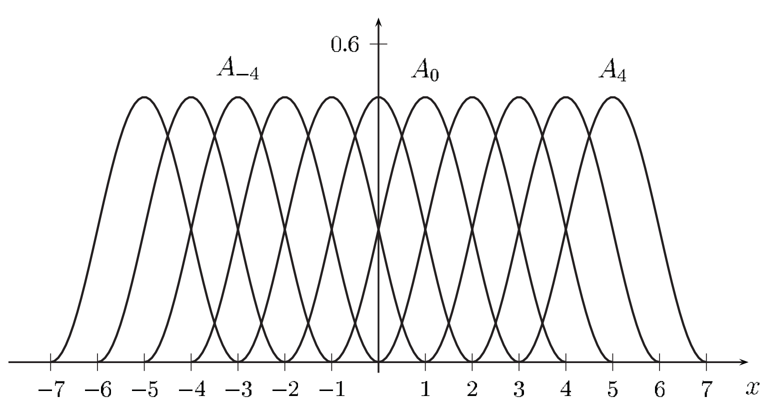


Figure 6: A part of the raised cosine type GUFP of the real line determined by $(0.5 \odot K_C, 2, 1, 1)$.

## 5. Concluding remarks

In this paper, necessary and sufficient conditions for generalized uniform fuzzy partitions were found. We have shown that a quadruplet $(K, h, r, x_0)$ determines a generalized uniform fuzzy partition if $r = h \int_{-1}^{1} K(x)dx$. In practice, this condition may significantly help us to design an optimal GUFP, because $r$ is derived from $K$ and $h$! Now, if $(K, h, r, x_0)$ is such that the previous necessary condition is satisfied, we have two possibilities how to verify that $(K, h, r, x_0)$ determines GUFP:

1. to check for $(K, h, r)$ that the Ruspini's condition (3) is satisfied for all $x \in [0, r]$;
2. to check for $(K, h, r)$ that the equality (11) (or equivalently (13)) is satisfied for all $y \in [\frac{r}{2}, r]$ (or $z \in [0, \frac{r}{2}]$).

It should be noted that it is sufficient to verify the Ruspini's condition for all $x \in [0, r]$, because the function $S(x)$ expressing the sum in (3) is a periodic function with the period $r$. Both verifications can be done theoretically in a similar way as was demonstrated for the generalized uniform fuzzy partitions of triangle and raised cosine type, or using a computer.

Finally, let us remark that generalized *non-uniform* partitions determined by symmetric generating functions can be defined as a *linear combination* of generalized uniform fuzzy partitions:

$$(\mathbf{K}, \mathbf{h}, \mathbf{r}, \mathbf{x_0}, \mathbf{a}) = a_1(K_1, h_1, r_1, x_{10}) + \cdots + a_n(K_n, h_n, r_n, x_{n0}),$$

where $\mathbf{K} = (K_1, \ldots, K_n)$, $\mathbf{h} = (h_1, \ldots, h_n)$, etc., $a_i > 0$ for any $i = 1, \ldots, n$, $a_1 + \cdots + a_n = 1$. Naturally, the j-th basic function $\mathbf{A}_j$ of $(\mathbf{K}, \mathbf{h}, \mathbf{r}, \mathbf{x_0}, \mathbf{a})$ is defined by

$$\mathbf{A}_j(x) = a_1 A_{j1}(x) + \cdots + a_n A_{jn}(x),$$

and it is easy to check that the Ruspini's condition is satisfied for $(\mathbf{K}, \mathbf{h}, \mathbf{r}, \mathbf{x_0}, \mathbf{a})$. Investigation of necessary and sufficient conditions for other types of generalized uniform fuzzy partitions (such as generating functions defined using splines) is a topic for future research.

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