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Towards a Physically Meaningful Definition of Computable Discontinuous and Multi-Valued Functions (Constraints)

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Abstract. In computable mathematics, there are known definitions of computable numbers, computable metric spaces, computable compact sets, and computable functions. A traditional definition of a computable function, however, covers only continuous functions. In many applications (e.g., in phase transitions), physical phenomena are described by discontinuous or multi-valued functions (a.k.a. constraints). In this paper, we provide a physics-motivated definition of computable discontinuous and multi-valued functions, and we analyze properties of this definition.

1 Formulation of the Problem

Need to define computable discontinuous functions. One of the main objectives of physics it to predict physical phenomena, i.e., use the observations to compute the predicted values of the corresponding physical quantities. Many physical phenomena such as phase transitions and quantum transitions include discontinuous dependencies $y = f(x)$ ("jumps"); see, e.g., [2].

In other physical situations, for some values x, we may have several possible values y. From the purely mathematical viewpoint, this means that the relation between x and y is no longer a function, it is a *relation* of a *constraint* $R \subseteq X \times Y$; following the terminology widely used in applications, we will also call them *multi-valued functions.*

To analyze which models of discontinuous or multi-valued behavior are computable and which are not, we need to have a precise definition of what is means for a discontinuous and/or multi-valued function to be computable. Alas, the current definitions of computable functions are mostly limited to continuous case.

What we plan to do. Our main goal is to define what it means for a discontinuous and/or multi-valued function to be computable.

For that purpose, we first explain the current definitions of computable numbers, objects, and functions. Then, we use physical motivations to come up with a new definition of computable discontinuous and multi-valued functions. Finally, we provide a few preliminary results about the new definition.

Computable numbers: reminder. Intuitively, a real number is *computable* if we can compute it with any desired accuracy. In more precise terms, a real number x is called *computable* if there exists an algorithm that, given a natural number n, returns a rational number r_n which is 2^{-n}-close to x: $|x - r_n| \le 2^{-n}$; [1, 3].

Computable metric spaces: motivation. A similar notion of computable elements can be defined for general metric spaces. In general, a element x is computable if there is an algorithm which generates better and better approximation to x. At each moment of time, we only have a finite amount of information about x; based on this information, we produce an approximation corresponding to this information. Any information can be represented, in the computer, as a sequence of 0s and 1s; any such sequence can be, in turn, interpreted as a binary integer n. Let $\widetilde{x}_n$ denote an approximation corresponding to an integer n. Then, it makes sense to require that in a computable metric space, there is a sequence of such approximating elements $\{\widetilde{x}_n\}$.

Computable means, in particular, that the distance $d_X(\widetilde{x}_n, \widetilde{x}_m)$ between such elements should be computable. Thus, we arrive at the following definition.

Computable metric spaces: definition. By a computable metric space, we mean a metric space X with a sequence $\{x_n\}$ of elements such that there is an algorithm that, given two natural numbers m and n, returns the distance $d_X(x_m, x_n)$ (i.e., for every natural number k, returns a rational number r_k which is 2^{-k}-close to $d_X(\widetilde{x}_m, x_n)$).

We say that an element x of a computable metric space X is *computable* if there exists an algorithm that, given a natural number n, returns an integer k_n for which $\widetilde{x}_{k_n}$ is 2^{-n}-close to x: $d_X(\widetilde{x}_{k_n}, x) \le 2^{-n}$.

Computable functions: definition. A function $f : X \to Y$ from a computable metric space X to a computable metric space Y is called *computable* if there exists an algorithm which uses x as an input and computes, for each integer n, a 2^{-n}-approximation y_k to $f(x)$. By "uses x as an input", we mean that to compute y_k, this algorithm can request, for each integer m, a 2^{-m}-approximation x_ℓ to x (and to use the index ℓ of this 2^{-m}-approximation in computing y_k).

Computable functions are continuous. The problem with the above definition is that all the functions computable according to this definition are continuous; see, e.g., [1, 3]. Thus, we cannot use this definition to check how well we can compute a discontinuous function.

This continuity is easy to understand. For example, if we have a function $f(x)$ form real numbers to real numbers which is equal to 0 for $x \le 0$ an to 1 for $x > 0$, then, if we could compute $f(x)$ for a given x with accuracy 2^{-2}, then we would be able, given a computable real number x, to tell whether this number is positive or not, and this is known to be algorithmically impossible.

Computable compact set. In analyzing computability, it is often useful to start with *pre-compact* metric spaces, i.e., metric spaces X for which, for every positive real number $\varepsilon > 0$, there exists a finite ε*-net*, i.e., a finite list of elements L such that every element $x \in X$ is ε-close to one of the elements from this list. In a Euclidean space, every bounded set is compact. A pre-compact set is *compact* if every converging sequence has a limit.

A natural idea is to call a compact metric space X *computable compact* if X is a computable metric space and there is an (additional) algorithm that, given an integer n, returns a finite list L_n of elements of X which is a 2^{-n}-net for X.

2 Towards A New Definition of Computable Discontinuous and Multi-Valued Functions

Simplifying comment. Before we start analyzing the problem, let us make one important comment. Functions can not only be discontinuous or multi-valued, they can also be undefined for some inputs x. However, in contrast to discontinuity and multiplicity of values, this is not a serious problem: if a relation is not everywhere defined, we can make it everywhere defined if we consider, instead of the original set X, a projection of R on this set. For example, a function $\sqrt{x}$ is not everywhere defined on the real line, but it is everywhere defined on the set of all non-negative real numbers. Thus, without losing generality, we can assume that our relation is everywhere defined.

Definition 1. *A relation $R \subseteq X \times Y$ is called* everywhere defined *if for every $x \in X$, there exists a $y \in Y$ for which $(x, y) \in R$.*

Analysis of the problem. From the physical viewpoint, what does it mean that the dependence between x and y – as described by a given discontinuous and/or multi-valued function – is computable?

In the ideal case, when we have a continuous single-valued dependence, the value x uniquely determines the value $y = f(x)$. In this case, once we know x, we want to compute $f(x)$ with a given accuracy. This is exactly the idea behind the usual definition of a computable function.

For a multi-valued function, for the same input x, we may get several different values y. In this case, it is desirable to compute the *set* of all possible value y corresponding to a given x. When we limit ourselves to multi-valued mappings from a compact set X to a compact set Y, the set of x-possible values of y is pre-compact, and thus, with any given accuracy, can be described by a finite list L of possible values. In other words:

- first, the list L should represent *all* possible values, i.e., if y is a possible value of $f(x)$ for a given x, then y should be close to one of the values from the finite list L;
- second, all the values from the list L must be possible values; in other words, for every value from the list, there must exist a close possible value of $f(x)$.

Discontinuity provides an additional complexity which can be illustrated on the example of the above discontinuous function $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x > 0$. In particular, for $x = 0$, we get $f(x) = f(0) = 0$. However, at each stage of the computation, we only know an approximate value of x. So, when the actual value of the input is $x = 0$, we will never find out whether x is non-positive (in which case $f(x) = 0$) or positive (in which case $f(x) = 1$). Thus, no matter how accurately we measure x, the only information about y that we can conclude is y is either equal to 0 or equal to 1. In general, we need to take into account not only the values $f(x)$ for a given x, but also the values $f(x')$ corresponding to values x' which are close to x. In view of this, the above properties of the list L must be appropriately modified:

- first, the list L should represent *all* possible values, i.e., if y is a possible value of $f(x')$ for some x' which is close to the given x, then y should be close to one of the values from the finite list L;
- second, all the values from the list L must be possible values; in other words, for every value from the list, there must exist a close value y which is a possible value of $f(x')$ for some x' which is close to x.

In general, the closeness does does not have to be the same in both cases. Thus, we arrive at the following definition.

Definition 2. *Let X and Y be computable compact sets with metrics d_X and d_Y. An everywhere defined relation $R \subseteq X \times Y$ is called* computable *if there exists an algorithm that, given a computable element $x \in X$ and computable positive numbers $0 < \varepsilon < \varepsilon'$ and $0 < \delta$, produced a finite list $\{y_1, \ldots, y_m\} \subseteq Y$ that satisfies the following two properties:*

(1) *if $(x', y) \in R$ for some x' for which $d_X(x', x) \le \varepsilon$, then there exists an i for which $d_Y(y, y_i) \le \delta$;*
(2) *for each element y_i from this list, there exist values x' and y for which $d_X(x, x') \le \varepsilon'$, $d_Y(y_i, y) \le \delta$, and $(x', y) \in R$.*

3 Properties of the New Definition

Main result. If X and Y are metric spaces with metrics d_X and d_Y, then on their Cartesian product $X \times Y$ (i.e., the set of all pairs (x, y), $x \in X$ and $y \in Y$) we can define a metric $d_{X \times Y}((x, y), (x', y')) \stackrel{\text{def}}{=} \max(d_X(x, x'), d_Y(y, y'))$. One can check that if X and Y are both compact sets, then the product $X \times Y$ is also a compact set: to get an ε-net for $X \times Y$, it is sufficient to take ε-nets L_X for X and L_Y for Y; one can then easily check that the set $L_X \times L_y$ of all possible pairs is an ε-net for the Cartesian product $X \times Y$. This construction is computable, so we conclude that the Cartesian product of computable compact sets is also a computable compact set.

Our first – somewhat surprising – result is that this new definition is equivalent to simply requiring that the set R (describing the graph of the relation) is a computable compact set:

Proposition 1. *Let X and Y be computable compact sets. A relation $R \subseteq X \times Y$ is computable if and only if the set R is a computable compact set.*

Proof. $\Leftarrow$ Let us first prove that if R is a computable compact set, then the relation R is computable in the sense of Definition 2. Indeed, let x be a computable element of X, and let the computable positive values $\varepsilon < \varepsilon'$ be given. Then, according to a known result from [1], we can find a computable value $\varepsilon_0 \in (\varepsilon, \varepsilon')$ for which the set $S \stackrel{\text{def}}{=} \{(x', y) \in R : d_X(x, x') \le \varepsilon_0\}$ is also a computable compact set. Thus, for a given computable number $\delta > 0$, there exists a finite δ-net for this set S. Let us denote the elements of this δ-net L by $(x_1, y_1), \ldots, (x_m, y_m)$. Let us show that, as the desired finite list, we can now take the list $\{y_1, \ldots, y_m\}$. Let us prove that this list satisfies both desired properties.

(1) If $(x', y) \in R$ for some x' for which $d_X(x, x') \le \varepsilon$, then, due to $\varepsilon < \varepsilon_0$, we have $d_X(x, x') < \varepsilon_0$. Thus, $(x', y) \in S$. Since $L = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ is a δ-net for the set S, we conclude that there exists an index i for which $d_{X\times Y}((x', y), (x_i, y_i)) \le \delta$. By definition of $d_{X\times Y}$, this means that $\max(d_X(x', x_i), d_Y(y, y_i)) \le \delta$ and therefore, $d_Y(y, y_i) \le \delta$. The first property from Definition 1 is proven.

(2) Let us now prove the second property. Let y_i be one of the selected elements. By our construction, the corresponding pair (x_i, y_i) belongs to δ-net for the set S. In particular, this means that $(x_i, y_i) \in S$. This means that $(x_i, y_i) \in R$ and that $d_X(x, x_i) \le \varepsilon_0$. Since $\varepsilon_0 < \varepsilon'$, we conclude that $d_X(x, x_i) \le \varepsilon'$. Thus, for each i, there exists $x' = x_i$ and $y = y_i$ for which $d_X(x, x') \le \varepsilon'$, $d_Y(y_i, y) = 0 \le \delta$, and $(x', y) \in R$. The second property is proven as well.

$\Rightarrow$ Let us now prove that if R is a computable relation in the sense of Definition 2, then R is computable compact set. For that, we must show how, given a computable positive real number $\alpha > 0$, we can generate an α-net for this set R. First, we use that fact that X is a computable compact, and generate an $(\alpha/2)$-net $\{x_1, \ldots, x_k\}$. For each point x_i, we then apply Definition 2 for $\delta = \varepsilon = \alpha/2$ and $\varepsilon' = \alpha$ and generate the corresponding list $\{y_{i1}, \ldots, y_{im_i}\}$. Let us show that the pairs (x_i, y_{ij}) form an α-net for the set R.

Indeed, by Definition 2, for each i and j, there exist values x' and y for which $d_X(x_i, x') \le \varepsilon' = \alpha$, $d_Y(y_{ij}, y) \le \delta = \alpha/2$, and $(x', y) \in R$. Thus, the pair (x_i, y_{ij}) is α-close to an element $(x', y) \in R$.

Vice versa, let $(x, y) \in R$. Since x_i form an $(\alpha/2)$-net, there exists an i for which $d(x, x_i) \le \alpha/2 = \varepsilon$. From Property (1) of Definition 2, we can now conclude that there exists a j for which $d_Y(y, y_{ij}) \le \delta = \alpha$. Thus, $d_{X\times Y}((x, y), (x_i, y_{ij})) = \max(d_X(x, x_i), d_Y(y, y_{ij})) \le \max(\alpha/2, \alpha) = \alpha$. The proposition is proven.

Inverse functions: a corollary. If the range of R is the whole set Y, then, from Proposition 1, it follows that a multi-valued function (relation) R is computable if and only if its inverse $R^{-1} = \{(x, y) : (y, x) \in R\}$ is computable.

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