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Abstract

In this paper, we will focus on the application of fuzzy transform (F-transform) in the analysis of time series. We assume that the time series can be decomposed into three constituent components: the trend-cycle, seasonal component and random noise. We will demonstrate that by using F-transform, we can approximate the trend-cycle of a given time series with high accuracy.

Keywords: Fuzzy transform, F-transform, time series, decomposition model.

1. Introduction

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This paper is devoted to analysis of time series using fuzzy transform (in short, F-transform — see [10]). In accordance with the standard approach, we assume that the time series can be decomposed into a trend-cycle TC, a seasonal component S and a random noise R (cf., for example, [2]). Moreover, we will assume that the seasonal constituent component S is a mixture of periodic functions, while the trend-cycle TC can be arbitrary. In this paper, we will show that using the F-transform, we can filter out the seasonal component S and significantly reduce the noise R. What remains is the trend-cycle that is thus approximated with high accuracy. This result is a strong support for the argument that the F-transform is a powerful tool which can be effectively applied in the analysis of time series.

Motivation for this paper comes from the following general characterization of the trend-cycle introduced in OECD Glossary of Statistical Terms:

> *The trend-cycle of a time series is the component that represents variations of low frequency in a time series, the high frequency fluctuations having been filtered out. This constituent component can be viewed as those variations with a period longer than a chosen threshold (usually 1 year is considered as the minimum length of the business cycle).*

This characterization provides the basic idea without formal explication of the

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used terms. A more detailed inspection reveals the assumption that the time series can be decomposed as outlined above. The trend-cycle can be represented by a sufficiently smooth function without clear periodicity or periodic but with periodicity significantly longer than any of the periodicities forming the seasonal constituent component S.

In standard statistical methods used in the analysis of time series, the trend-cycle is assumed to be formed by some, usually simple, function (quite often a linear one) that is usually extracted using regression analysis (cf. [5]). We argue that this approach is dubious since one can hardly suppose that the trend-cycle TC takes a simple course on the whole domain. As will be seen later, the F-transform makes it possible to consider the trend-cycle TC to be a function of arbitrary shape and thus, to extract TC in its true form, whatever it is. At the same time, it provides precise analytic expression for TC.

The problem solved in this paper is similar to signal filtering, where the task is also to remove high frequencies present as noise in a signal. Our approach, however, differs from the classical signal filtering theory, where it is assumed that the function is formed of (infinite) number of sinusoidal functions and filtering is based on removing functions with high frequencies (cf. [14]). Last but not least, the F-transform is computationally simple and so, very fast algorithms are at our disposal.

Results of this paper extend the results from [6, 13]. For the other papers in which the F-transform was applied to the analysis and prediction of time series, see [7, 8, 9].

The structure of this paper is the following. In Section 2, we briefly overview the main notions of the F-transform theory with the focus to the properties needed further. The main part of the paper is Section 3, where we prove that under specific conditions, the F-transform filters out seasonal constituent component consisting of high frequencies and reduces the random noise so that only the trend-cycle TC remains. The section is finished by a simple example demonstrating the theoretical results.

2. Preliminaries

In this section, we will briefly review the main principles of the F-transform. Detailed explanation of the general theory can be found in [10, 11, 12].

Let U be an arbitrary (nonempty) set called a *universe*. By a *fuzzy set* in the universe U we will understand a function $A : U \longrightarrow [0, 1]$. By $\mathscr{F}(U)$ we denote the set of all fuzzy sets on U.

The F-transform is a special technique that can be applied to real continuous functions f, defined on an interval $[a, b] \subset \mathbb{R}$. The essential idea of F-transform is to transfer f into another, simpler space, and then to transfer the respective image back. The latter space consists of finite vectors obtained on the basis of the well formed *fuzzy partitions* of the domain of the given function. Thus, the first step called *direct F-transform* results in the vector of averaged functional values. The second step called *inverse F-transform* converts this vector into another continuous function $\hat{f}$, which approximately reconstructs the original f.

2.1. F^0-transform

Let a function f of a real variable be defined on an interval $[a, b]$. To apply the F-transform, we must first specify a *fuzzy partition* of $[a, b]$.

Definition 1
Let $n \in \mathbb{N}$, $n \geq 2$, be given and $c_0 < \ldots < c_n$ be fixed nodes within $[a, b]$, such that $c_0 = a, c_n = b$. We say that fuzzy sets $A_0, \ldots, A_n \in \mathscr{F}([a, b])$ form a fuzzy partition of $[a, b]$ *if they satisfy the following conditions for $k = 0, \ldots, n$:*

1. *$A_k(c_k) = 1$;*
2. *$A_k(x) = 0$ for $x \notin (c_{k-1}, c_{k+1})$, $k = 0, \ldots, n$ where we formally put $c_{-1} = a$ and $c_{n+1} = b$;*
3. *A_k is continuous;*
4. *A_k strictly increases on $[c_{k-1}, c_k]$ and strictly decreases on $[c_k, c_{k+1}]$;*
5. *for all $x \in [a, b]$,*
$$\sum_{k=0}^{n} A_k(x) = 1. \tag{1}$$

The fuzzy sets $A_0, \ldots, A_n$ are in the theory of F-transform called *basic functions*. The most common is *uniform fuzzy partition*.

Definition 2
Let $A_0, \ldots, A_n$ be a fuzzy partition of $[a, b]$. We say that it is uniform *if the following is satisfied:*

(i) The nodes nodes $c_0 = a, \ldots, c_n = b$ are h-equidistant, i.e. $c_k = a + kh$, $k = 0, \ldots, n$ where $h = \frac{b-a}{n}$.

(ii) The basic functions are symmetric, i.e. $A_k(c_k - x) = A_k(c_k + x)$ for all $x \in [0, h]$ and $k = 1, \ldots, n-1$.

(iii) The following holds for all $k = 1, \ldots, n-1$ and $x \in [x_k, x_{k+1}]$:
$$A_k(x) = A_{k-1}(x - h),$$
$$A_{k+1}(x) = A_k(x - h)$$

Let us remark that shapes of the basic functions are not predetermined and can be chosen on the basis of additional requirements. In the following text, we will always *assume that the considered fuzzy partition is uniform.* By width of a fuzzy set A_k we understand the width of its support. In case of the uniform fuzzy partition, the width of each A_k is $2h$ (except for A_0 and A_n where it is h).

In this paper, we will work with triangular fuzzy sets defined by:
$$A_k(t) = \begin{cases} \frac{t - c_{k-1}}{h}, & t \in [c_{k-1}, c_k], \\ \frac{c_{k+1} - t}{h}, & t \in [c_k, c_{k+1}] \end{cases} \tag{2}$$

for all $k = 0, \dots, n$ (for A_0 and A_n, we consider in (2) only the interval $[c_0, c_1]$ and $[c_{n-1}, c_n]$, respectively). Note also that $\int_{c_{k-1}}^{c_{k+1}} A_k(x)dx = h$ and $\int_{c_k}^{c_{k+1}} A_k(x)dx = \frac{h}{2}$.

It should be noted that the fuzzy transform can also be defined with respect to a more general fuzzy partition, in which some of the conditions of Definition 1 are relaxed, for example condition 5. The details can be found in [4].

Definition 3
Let a fuzzy partition of $[a, b]$ be given by basic functions $A_0, \dots, A_n, n \geq 2$, and let $f : [a, b] \longrightarrow \mathbb{R}$ be an arbitrary continuous function. The $(n + 1)$-tuple of real numbers $\mathbf{F}[f] = (F_0[f], \dots, F_n[f])$ given by

$$F_k[f] = \frac{\int_a^b f(x)A_k(x)dx}{\int_a^b A_k(x)dx}, \qquad k = 0, \dots, n, \tag{3}$$

is called direct fuzzy transform (F-transform) *of f with respect to the given fuzzy partition. The numbers $F_0[f], \dots, F_n[f]$ are called components of the F-transform of f.*

In case that the partition is h-uniform, computation of the components can be simplified as follows:

$$F_0[f] = \frac{2}{h} \int_{c_0}^{c_1} f(x)A_1(x)dx \quad \text{and} \quad F_n[f] = \frac{2}{h} \int_{c_n-1}^{c_n} f(x)A_n(x)dx, \tag{4}$$

$$F_k[f] = \frac{1}{h} \int_{c_{k-1}}^{c_{k+1}} f(x)A_k(x)dx, \qquad k = 1, \dots, n-1. \tag{5}$$

It should be noted that the components of the F-transform are weighted mean values of the original function, where the weights are determined by the basic functions; see, e.g., [10].

In practice, the function f is often given not analytically but in the form of data — results of some measurements. In this case, Definition 3 must be modified in such a way that the definite integrals in (3) are replaced by finite summations. In this case, we speak about *discrete F-transform*.

The F-transform of f with respect to the fuzzy partition $A_0, \dots, A_n$ will be denoted by $\mathbf{F}[f] = (F_0[f], \dots, F_n[f])$. If the function f is clear from the context then the "$[f]$" in $F_i[f]$ can be omitted.

As can be seen from (4), the boundary components $F_0[f]$ and $F_n[f]$ are computed using only halves of the basic functions, which leads to less precision on the boundary of the interval $[a, b]$. Therefore, we will often consider the F-transform only as the vector of inner components $\overline{\mathbf{F}}[f] = [F_1[f], \dots, F_{n-1}[f]]$.

Important property of the F-transform is its linearity; namely, if $h = \alpha f + \beta g \in C[a, b]$ then

$$\mathbf{F}[h] = \alpha[f] + \gamma\mathbf{F}[g].$$

The original function f can be approximately reconstructed from $\mathbf{F}[f]$ using the following inversion formula.

Definition 4
Let $\mathbf{F}[f]$ *be the direct F-transform of* f *with respect to the fuzzy partition* $A_0, \ldots, A_n \in \mathscr{F}([a,b])$*. Then the function* $\hat{f}$ *given on* $[a,b]$ *by*

$$\hat{f}(x) = \sum_{k=0}^{n} F_k[f] \cdot A_k(x), \tag{6}$$

is called the inverse F-transform[1] *of* f*.*

The inverse F-transform $\hat{f}$ is a continuous function on $[a,b]$. Moreover, the linearity property holds for it too, i.e., if $h = \alpha f + \beta g \in C[a,b]$ then

$$\hat{h} = \alpha \hat{f} + \gamma \hat{g}$$

provided that the fuzzy partition is fixed. For various properties of the F-transform and detailed proofs — see [10].

In our analysis below we will also consider the case when $f : [a,b] \longrightarrow \mathbb{C}$ being a complex-valued function of a real variable. Due to the linearity property of the F-transform, we can write the direct F-transform of f as

$$\mathbf{F}[f] = \mathbf{F}[Re(f)] + i\,\mathbf{F}[Im(f)].$$

Hence, the inverse F-transform of f becomes

$$\hat{f} = Re(\hat{f}) + i\,Im(\hat{f}),$$

or in more detail

$$\hat{f}(x) = \sum_{k=0}^{n} F_k[Re(f)] \cdot A_k(x) + i \sum_{k=0}^{n} F_k[Im(f)] \cdot A_k(x).$$

2.2. F^m-transform

The F-transform presented above can be further generalized. Namely, we can introduce higher-degree F-transform (F^m-transform, $m \geq 0$). Its components are polynomials of degree m. Thus, the original F-transform described above coincides with the F^0-transform. Below, we will overview the main points of the F^m-transform. Detailed definitions, theorems and their proofs can be found in [12] where the F^m-transform was introduced.

Let us fix an interval $[a,b]$ of reals. The fuzzy partition will be taken over "full" fuzzy sets, i.e. we will consider only $A_1, \ldots, A_{n-1} \in \mathscr{F}([a,b])$, $n \geq 2$. Let k be a fixed integer from $\{1, \ldots, n-1\}$, and $L_2(A_k)$ be a Hilbert space of square-integrable functions $f : [x_{k-1}, x_{k+1}] \longrightarrow \mathbb{R}$ with the scalar product

$$\langle f, g \rangle_k = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x) g(x) A_k(x) dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x) dx}.$$

[1] We will use the term "inverse F-transform" in two meanings: (a) as the procedure for obtaining the estimation of f and (b), as the resulting function (6) approximating f.

By $L_2(A_1, \dots, A_{n-1})$ we denote a set of functions $f : [a, b] \longrightarrow \mathbb{R}$ such that $f|_{[c_{k-1}, c_{k+1}]} \in L_2(A_k)$ for all $k = 1, \dots, n-1$.

Definition 5
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function from $L_2(A_1, \dots, A_{n-1})$ and $m \geq 0$ be a fixed integer. We say that the n-tuple of polynomials $\mathbf{F}^m[f] = (F_1^m[f], \dots, F_{n-1}^m[f])$ is an F^m-transform *of f with respect to the fuzzy partition $A_0, \dots, A_n$ if*

$$F_k^m[f] = \beta_k^0 P_k^0 + \beta_k^1 P_k^1 + \cdots + \beta_k^m P_k^m,$$

where

$$\beta_k^i = \frac{\int_a^b f(x) P_k^i(x) A_k(x) dx}{\int_a^b P_k^i(x) P_k^i(x) A_k(x) dx}, \quad i = 0, \dots, m$$

and P_k^i are orthogonal polynomials of i-th order. The polynomial $F_k^m[f]$ is called k-th F^m-transform component *of f.*

Each component approximates f in a certain area. Thus, the quality of approximation increases with the increase of the degree of the polynomials. The following can be proved:

(a) Each F^m-transform component $F_k^m[f]$, $k = 0, \dots, n$, minimizes the functional

$$\Phi_k(g) = \int_a^b (f(x) - g(x))^2 A_k(x) dx, \tag{7}$$

defined on elements $g \in L_2^m(A_k)$.

(b) If f is a polynomial of degree $l \leq m$ then each F^m-transform component $F_k^m[f]$, $k = 1, \dots, n-1$ restricted to $[c_{k-1}, c_{k+1}]$ coincides with f on $[c_{k-1}, c_{k+1}]$.

(c) Each F^m-transform component $F_k^m[f]$, $k = 1, \dots, n-1$, satisfies the following recurrent equation:

$$F_k^m[f] = F_k^{m-1}[f] + \beta_k^m P_k^m, \quad m = 1, 2, \dots. \tag{8}$$

(d) The F^m-transform of f is linear, i.e.

$$\mathbf{F}^m[\alpha f + \gamma g] = \alpha \mathbf{F}^m[f] + \gamma \mathbf{F}^m[g],$$

holds for all $f, g \in L_2(A_1, \dots, A_{n-1})$, and for all $\alpha, \beta \in \mathbb{R}$, where the equality is considered over the respective vectors of components.

The following theorem characterizes the F^1-transform of f with respect to a h-uniform fuzzy partition $A_0, \dots, A_n$. Its proof can be found in [12].

Theorem 1
The vector of linear functions

$$\mathbf{F}^1[f] = (\beta_1^0 + \beta_1^1(x - c_0), \ldots, \beta_{n-1}^0 + \beta_{n-1}^1(x - c_{n-1})) \tag{9}$$

is the F^1*-transform of* f *with respect to the* h*-uniform fuzzy partition* $A_1, \ldots, A_{n-1}$*, where*

$$\beta_k^0 = \frac{\int_{c_{k-1}}^{c_{k+1}} f(x)A_k(x)dx}{h}, \tag{10}$$

$$\beta_k^1 = \frac{\int_{c_{k-1}}^{c_{k+1}} f(x)(x - c_k)A_k(x)dx}{\int_{c_{k-1}}^{c_{k+1}} (x - c_k)^2 A_k(x)dx}, \tag{11}$$

for every $k = 1, \ldots, n-1$*.*

Note that $\beta_k^0 = F_k^0[f]$ where $F_k^0[f]$ is the component (5) (we added the superscript 0 to emphasize that Definition 3, in fact, introduces zero degree F-transform).

Corollary 1
If the partition $A_0, \ldots, A_n$ *is uniform and the basic functions* A_k *have triangular shape then (11) becomes*

$$\beta_k^1 = \frac{12 \int_{c_{k-1}}^{c_{k+1}} f(x)(x - c_k)A_k(x)dx}{h^3} \tag{12}$$

Similarly to the original F-transform, the inverse F^m-transform of a function f is defined as a linear combination of basic functions with "coefficients" given by the F^m-transform components.

Definition 6
Let $f : [a, b] \longrightarrow \mathbb{R}$ *be a given function,* $m \geq 0$ *and* $\mathbf{F}^m[f] = (F_1^m[f], \ldots, F_{n-1}^m[f])$ *be the* F^m*-transform of* f *with respect to the fuzzy partition* $A_1, \ldots, A_{n-1}$*. Then the function* $\hat{f}^m : [a, b] \longrightarrow \mathbb{R}$ *defined by*

$$\hat{f}^m(x) = \sum_{k=1}^{n-1} F_k^m[f](x)A_k(x) \tag{13}$$

is called inverse F^m-transform *of* f *with respect to* $\mathbf{F}^m[f]$ *and* $A_0, \ldots, A_n$*.*

The following recurrent formula can be applied:

$$\hat{f}^m(x) = \hat{f}^{m-1}(x) + \sum_{k=1}^{n-1} \beta_k^m P_k^m(x)A_k(x) \tag{14}$$

where $x \in [a, b]$, $m \geq 1$.

Nice approximation properties of the F^m-transform can be proved. Similar to the case of the basic (zero degree) F-transform, the following can be proved:

if $n \to \infty$, where n is the number of basic functions $A_1, \dots, A_{n-1}$, then the obtained sequence of inverse F^m-transforms $\hat{f}^m_{(n)}$ of f, where $m \geq 1$, uniformly converges to f (see [12] for the details).

The following lemmas will be useful later.

Lemma 1
Let $F^0_0[f], \dots, F^0_n[f]$ be components of the F^0-transform w.r.t an h-uniform fuzzy partition $A_0, \dots, A_n \in \mathscr{F}([a, b])$. Let $\hat{f}^o$ be the inverse F-transform of f. Then

(a) $\int_a^b f(x)dx = h\left(\sum_{k=1}^{n-1} F^0_k[f] + \frac{1}{2}(F^0_0[f] + F^0_n[f])\right)$.

(b) $\int_a^b \hat{f}^0(x)\,dx = \int_a^b f(x)\,dx$.

PROOF: (a) was proved in [11].

(b) Under the assumptions, we obtain

$$\int_a^b \hat{f}^0(x)dx = \int_a^b \sum_{k=0}^{n} F^0_k[f] \cdot A_k(x)dx = \sum_{k=0}^{n} F^0_k[f] \int_a^b A_k(x)dx =$$

$$= h\left(\frac{F^0_0[f]}{2} + \sum_{k=1}^{n-1} F^0_k[f] + \frac{F^0_n[f]}{2}\right) = \int_a^b f(x)dx$$

by (a).

□

Lemma 2 ([10])
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function and $\hat{f}$ its inverse F-transform (both zero or first degree). Then

$$\max_{x \in [a,b]} |f(x) - \hat{f}(x)| \leq 2\lambda(h, f) \tag{15}$$

where

$$\lambda(h, f) = \max_{\substack{|x-y|<h \\ x,y \in [a,b]}} |f(x) - f(y)|$$

is the modulus of continuity of f.

Let us remark that for first degree F-transform, this lemma follows from [12, Lemma 2]. Note also that if the function f is smooth “without abrupt changes in its course” then $\lambda(h, f)$ is small.

3. Analysis of time series using F-transform

In this section we will first provide precise definition of the time series taken as a special case of the more general concept of complex-valued stochastic process. Then, assuming additive decomposition of the time series into trend cycle,

periodic seasonal constituent component and noise we will show that the F-transform can be effectively used for extraction of its trend-cycle. By proper setting of the fuzzy partition we can either completely remove or significantly reduce the functions forming the seasonal constituent component, as well as significantly reduce the noise.

3.1. Decomposition of time series

Let us consider a complex-valued stochastic process (see [1, 3])

$$X : [a, b] \times \Omega \longrightarrow \mathbb{C} \tag{16}$$

where $[a, b] \subset \mathbb{R}$ is an interval of reals and $\langle \Omega, \mathscr{A}, P \rangle$ is a probabilistic space. For simplicity, we will usually suppose that $a = 0$.

By a *time series* we understand a stochastic process where $[a, b]$ is replaced by a finite set $Q = \{a = 0, \ldots, b = p\} \subset \mathbb{N}$. Since all the results below derived for a stochastic process (16) can be easily transformed also for the time series, we will always consider (16), unless stated otherwise.

It follows from (16) that for each $t \in [a, b]$ the mapping $X(t, \omega)$, $\omega \in \Omega$ is a random variable. If we fix $\omega \in \Omega$ then we obtain one realization of (16) and in this case, we will write simply $X(t)$.

Our basic assumption is that $X(t, \omega)$ can be decomposed into three constituent components, namely

$$X(t, \omega) = TC(t) + S(t) + R(t, \omega), \qquad t \in [a, b], \omega \in \Omega, \tag{17}$$

where TC is a *trend-cycle* and S is a seasonal constituent component. Both TC and S are usual (i.e. non-random) real or complex-valued functions of a real variable.

The component $R(t, \omega)$ is a random noise. It is assumed to be a real stationary stochastic process, where each $R(t)$ for $t \in \mathbb{R}$ is a random variable whose joint distribution function

$$F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = P(\{R(t_1) \le x_1, \ldots, R(t_n) \le x_n\}), \quad t_1, \ldots, t_n \in \mathbb{R} \tag{18}$$

does not change with time shift, i.e.,

$$F_{t_1+\tau,\ldots,t_n+\tau}(x_1, \ldots, x_n) = F_{t_1,\ldots,t_n}(x_1, \ldots, x_n)$$

for arbitrary $\tau \in \mathbb{R}$. We will, moreover, assume that R has a zero mean and finite variance:

$$\mathbf{E}(R(t)) = \mu = 0 \quad \text{and} \quad \mathbf{D}(R(t)) = \sigma > 0. \tag{19}$$

In this paper, we consider the noise to be represented as the simplest possible type of a stationary stochastic process (see, e.g.,[15, Example 1]), namely, as the process of the type

$$R(t) = \xi e^{i\lambda t+\varphi} \tag{20}$$

where ξ is a random variable with zero mean value and λ a real number. It is known (see, e.g., [15]) that, under reasonable conditions, every stationary random process with 0 mean can be represented as a linear combination of processes of type (20).

Finally, we will also assume that for each $\omega \in \Omega$, the corresponding realization $X(t)$ (and consequently, all its constituent components) is a continuous function.

The seasonal constituent component $S(t)$ is assumed to be a mixture of complex-valued periodic functions

$$S(t) = \sum_{j=1}^{r} P_j \, e^{i(\lambda_j t + \varphi_j)} \tag{21}$$

for some finite r where λ_j are frequencies, φ_j phase shifts and P_j are amplitudes. Thus, (17) becomes

$$X(t) = \, TC(t) + \sum_{j=1}^{r} P_j \, e^{i(\lambda_j t + \varphi_j)} + R(t), \qquad t \in [a, b]. \tag{22}$$

In particular, $S(t)$ can also be considered as a real function being a mixture of real periodic functions

$$S(t) = \sum_{j=1}^{r} P_j \sin(\lambda_j t + \varphi_j). \tag{23}$$

In the following text, we will apply the F-transform to a realization $X(t)$ of the given stochastic process (taken as a real- or complex-valued function) in (22) using an h-uniform fuzzy partition consisting of *n triangular fuzzy sets A_k* of the width $2h$ over equidistant nodes $c_0, \ldots, c_n$ where $c_{k+1} = c_k + h$, $k = 0, \ldots, n-1$ with the distance $h = \frac{b-a}{n}$.

3.2. Removing seasonal constituent component

3.2.1. Basic idea

Note that by the linearity of the F-transform, the components of $\mathbf{F}[X]$ become

$$\mathbf{F}[X] = \mathbf{F}[TC] + \sum_{j=1}^{r} P_j \mathbf{F}[e^{i(\lambda_j t + \varphi_j)}] + \mathbf{F}[R]. \tag{24}$$

In this and the subsequent subsections, we will demonstrate that components of the F-transform of each summand in (24) as well $\mathbf{F}[R]$ either vanish or are significantly reduced. This enables us to use F-transform for separation of the trend-cycle TC only.

Let us consider one periodic function $Pe^{i(\lambda t + \varphi)}$ with the periodicity T and frequency λ where $\lambda = \frac{2\pi}{T}$. Because of the latter equality, we will, in case of need, speak freely either about λ or about T.

For simplicity we will assume that $a = 0$, i.e. we will consider an interval $[0, b]$ and construct a h-uniform fuzzy partition where the distance h between the nodes is related to the periodicity T by simple equality

$$h = d\,T \tag{25}$$

where $d > 0$ is a suitable real number. Thus, we obtain a set of $n \geq 2$ equidistant nodes

$$\{c_0 = 0,\ c_1 = d\,T, \ldots, c_{k-1} = (k-1)d\,T,\ c_k = k\,d\,T,\ c_{k+1} = (k+1)d\,T, \\ \ldots, c_n = nd\,T = b\}. \tag{26}$$

One can immediately see that $h = \frac{2\pi d}{\lambda}$, i.e. $\lambda = \frac{2\pi d}{h}$. This means that when h is fixed then higher d corresponds to higher frequency λ. We will argue below that higher frequencies are filtered out after applying the F-transform.

3.2.2. Removing seasonal constituent component using F^0-transform

In the following technical lemma we will compute components of the F-transform of $Pe^{i(\lambda t+\varphi)}$ with respect to the fuzzy partition over (26).

Lemma 3
Let $Pe^{i(\lambda t+\varphi)}$ be a function with the frequency λ and phase shift φ. Let us fix h and choose $d \in \mathbb{R}$ so that $h = dT$where $T = \frac{2\pi}{\lambda}$. Finally, let (26) be a set of $(n+1)$ h-equidistant nodes and $A_0, \ldots, A_n$ be a fuzzy partition over (26) such that each A_k is the triangular fuzzy set (2) having the width $2h$. Then the components of the F-transform of $Pe^{i(\lambda t+\varphi)}$ are the following:

$$F_k[Pe^{i(\lambda t+\varphi)}] = -\frac{Pe^{i\varphi}}{4d^2\pi^2}\, e^{i2(k-1)d\pi}(e^{i2d\pi} - 1)^2, \tag{27}$$

$$F_0[Pe^{i(\lambda t+\varphi)}] = \frac{Pe^{i\varphi}}{2d^2\pi^2}\left(1 - e^{i2d\pi} + i2d\pi\right), \tag{28}$$

$$F_n[Pe^{i(\lambda t+\varphi)}] = \frac{Pe^{i\varphi}}{2d^2\pi^2}\left(-e^{i2(n-1)d\pi} + e^{i2nd\pi}(1 - i2d\pi)\right). \tag{29}$$

PROOF: First, note that $x_k = \frac{2\pi kd}{\lambda}$. Then we proceed by straightforward computation: using (2) and (5), we obtain k-th component of the F-transform of $Pe^{i(\lambda t+\varphi)}$ as follows:

$$F_k[Pe^{i(\lambda t+\varphi)}] = \frac{P\lambda(k+1)}{2\pi d}\int_{\frac{2\pi kd}{\lambda}}^{\frac{2\pi(k+1)d}{\lambda}} e^{i(\lambda t+\varphi)}dt \\ - \frac{P\lambda(k-1)}{2\pi d}\int_{\frac{2\pi(k-1)d}{\lambda}}^{\frac{2\pi kd}{\lambda}} e^{i(\lambda t+\varphi)}dt+ \\ + \frac{P\lambda^2}{4\pi^2 d^2}\left(\int_{\frac{2\pi(k-1)d}{\lambda}}^{\frac{2\pi kd}{\lambda}} te^{i(\lambda t+\varphi)}dt - \int_{\frac{P2\pi kd}{\lambda}}^{\frac{2\pi k+1d}{\lambda}} te^{i(\lambda t+\varphi)}dt\right) \tag{30}$$

where $k \in \{1, n-1\}$. After integration we obtain formula (27) from (30). For $k = 1$ and $k = n$, we use formulas (4), which give (28) and (29).

□

As for the boundary components F_0, F_n, recall that they are computed using only halves of the corresponding basic functions A_0 and A_n. Consequently, the inverse F-transform approximates the original function in the intervals $[c_0, c_1]$ and $[c_{n-1}, c_n]$ with less precision.

In the following corollary and below, we will denote by $[d]$ the maximal integer part of $d \in \mathbb{R}$.

Corollary 2

Let the conditions of Lemma 3 be satisfied and $d = \frac{h}{T}$. Then for all $k = 1, \ldots, n-1$:

(a)

$$\left|F_k[Pe^{i(\lambda t+\varphi)}]\right| = \frac{P \sin^2(d\pi)}{d^2\pi^2}. \tag{31}$$

(b) Let $d' := d - [d]$. Then

$$\left|F_k[Pe^{i(\lambda t+\varphi)}]\right| = \frac{P \sin^2(d'\pi)}{d^2\pi^2}. \tag{32}$$

PROOF: This follows from (27) by expanding to trigonometric form and realizing that $|e^{i\varphi}| = 1$ and $e^{i2k\pi} = 1$ for any $k \in \mathbb{N}$. □

Corollary 3

Let the conditions of Lemma 3 and Corollary 2 be satisfied. Then the following holds for any $k = 1, \ldots, n-1$:

(a) $\left|F_k[Pe^{i(\lambda t+\varphi)}]\right|$ *does not depend on the phase shift* φ.

(b) $\left|F_k[Pe^{i(\lambda t+\varphi)}]\right| = \left|F_k[Pe^{i(\frac{2\pi d}{h}t+\varphi)}]\right| = 0$ *whenever* $d \in \mathbb{N}$.

(c) $\lim_{d\to\infty} |F_k[Pe^{i(\frac{2\pi d}{h}t+\varphi)}]| = 0$,

$$\lim_{d'\to 0} \left|F_k[Pe^{i(\frac{2\pi([d]+d')}{h}t+\varphi)}]\right| = 0 \quad \textit{and} \quad \lim_{d'\to 1} \left|F_k[Pe^{i(\frac{2\pi([d]+d')}{h}t+\varphi)}]\right| = 0.$$

PROOF: Immediately from Corollary 2. □

We conclude from Corollary 3 that if h is fixed and the frequency λ is increasing then, under the conditions of Lemma 3, the absolute values of F-transform components of $Pe^{i(\lambda t+\varphi)}$ are either equal to zero for $d \in \mathbb{N}$, or they converge to zero otherwise. A graph depicting behavior of $|F_k[Pe^{i(\frac{2\pi d}{h}t+\varphi)}]|$ with respect to d is in Figure 1.

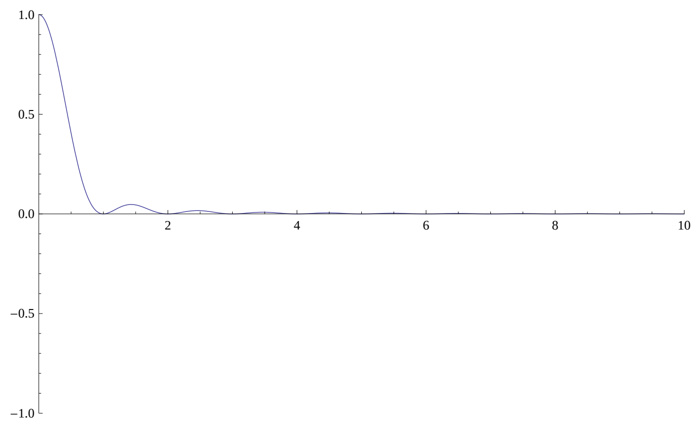

Figure 1: Graph of $|F_k[Pe^{i(\frac{2\pi d}{h}t+\varphi)}]|$ for $k \in \{1, \ldots, n-1\}$ considered as a function of d ($d \in [0, 10]$) where $P = 1$. One can see that absolute values of each k-th component of the F-transform practically vanish for $d \geq 2$.

It is, of course, not surprising that similar results also hold if we confine to real valued seasonal constituent component (23) only. For comparison, we present lemma characterizing inverse F-transform of $\sin(\lambda t + \varphi)$. In correspondence with Definition 4 we will denote inverse F-transform of sin by $\widehat{\sin}$.

Lemma 4
Let $\sin(\lambda t + \varphi)$ be a function with the frequency λ and phase shift φ. Let us fix h and choose $d \in \mathbb{R}$ so that $h = dT$ where $T = \frac{2\pi}{\lambda}$. Finally, let (26) be a set of $(n+1)$ h-equidistant nodes and $A_0, \ldots, A_n$ be a fuzzy partition over (26) such that each A_k is the triangular fuzzy set (2) having the width $2h$. If $d \in \mathbb{N}$ then $\widehat{\sin}(\frac{2\pi d}{h}t + \varphi) = 0$. Otherwise, let $d' = d - [d]$. Then the following inequality holds for the inverse F-transform of $\sin(\lambda t + \varphi)$:

$$|F_k[\sin(\lambda t + \varphi)]| \leq \frac{1}{d^2\pi^2} \sin\varphi, \tag{33}$$

as well as

$$|\widehat{\sin}(\lambda t + \varphi)| \leq \left| \frac{1}{d^2\pi^2} \sin\varphi \right|, \qquad t \in [c_1, c_{n-1}] \tag{34}$$

and $\lim_{d' \to 0} |\widehat{\sin}(\frac{2\pi([d]+d')}{h}t + \varphi)| = 0$ *as well as* $\lim_{d' \to 1} |\widehat{\sin}(\frac{2\pi([d]+d')}{h}t + \varphi)| = 0$.

PROOF: After some computation, we obtain using (2) and (5):

$$F_k[\sin(\lambda t + \varphi)] = \frac{1}{4d^2\pi^2}(2\sin(2kd'\pi)\cos\varphi + 2\cos(2kd'\pi)\sin\varphi)(1 - \cos 2d'\pi) \tag{35}$$

where $d' \in [0,1]$. If $d \in \mathbb{N}$, i.e. $d' = 0$, then (35) is equal to 0. Otherwise

$$\lim_{d' \to 0} F_k[\sin(\lambda t + \varphi)] = 0 \quad \text{and} \quad \lim_{d' \to 1} F_k[\sin(\lambda t + \varphi)] = 0.$$

For $d' = 0.5$ we obtain

$$F_k[\sin(\lambda t + \varphi)] = \pm \frac{1}{d^2\pi^2} \sin\varphi \tag{36}$$

where the sign depends on k, i.e. (36) is positive for k even and negative for k odd. Hence, (33) holds true.

Let $t \in [c_{k-1}, c_k]$, $k \in \{1, \dots, n-1\}$. Then

$$\begin{aligned} |\widehat{\sin}(\lambda t + \varphi)| &= |A_{k-1}(t)F_{k-1}[\sin(\lambda t + \varphi)] + A_k(t)F_k[\sin(\lambda t + \varphi)]| \leq \\ &\leq \underbrace{(A_{k-1}(t) + A_k(t))}_{=1} \left| \frac{1}{d^2\pi^2} \sin\varphi \right| . \end{aligned} \tag{37}$$

Since the right-hand side does not depend on k, we obtain (34).

□

3.2.3. Removing seasonal constituent component using F^1-transform

In [7], we suggested that the F^1-transform may improve estimation of the trend-cycle. Therefore, with respect to the above results we must ask whether the F^1-transform removes seasonal periodic components as well. The answer is positive as follows from the results below.

Recall that we consider nodes of the h-uniform fuzzy partition in the form $c_k = \frac{2\pi k d}{\lambda}$, $k = 0, \dots, n$. Furthermore, unlike the basic (i.e. F^0-) F-transform, components of the F^1-transform are linear functions

$$F_k^1[Pe^{i(\lambda t + \varphi)}](t) = F_k^0[Pe^{i(\lambda t + \varphi)}] + \beta_k^1 \, (t - \frac{2\pi k d}{\lambda}) \tag{38}$$

where the coefficients β_k^1 are computed using formula (11). The variable $t \in (-\infty, \infty)$. However, since in (13) we multiply each k-th component by $A_k(t)$ which equals zero for all $t \notin (c_{k-1}, c_{k+1})$, it makes sense to consider the k-th component only for $t \in [c_{k-1}, c_{k+1}]$.

Lemma 5
Let $Pe^{i(\lambda t+\varphi)}$ be a function with the frequency λ and phase shift φ. Let us fix h and choose $d \in \mathbb{R}$ so that $h = dT$ where $T = \frac{2\pi}{\lambda}$. Finally, let (26) be a set of $(n+1)$ h-equidistant nodes and $A_0, \dots, A_n$ be a fuzzy partition over (26) such that each A_k is the triangular fuzzy set (2) having the width $2h$. Then for each $k = 1, \dots, n-1$ and $t \in [c_{k-1}, c_{k+1}]$ we obtain:

(a)

$$\begin{aligned} &F_k^1[Pe^{i(\lambda t + \varphi)}](t) = \\ &= \frac{Pe^{i\varphi}}{d^4\pi^4} \sin d\pi \cdot (i3d\pi(2dk\pi - \lambda t)\cos d\pi + (d\pi(d\pi - i6k) + i3\lambda t)\sin d\pi) . \end{aligned} \tag{39}$$

(b)

$$\left|F_k^1[Pe^{i(\lambda t+\varphi)}](t)\right| =$$
$$= \frac{P\sin^2 d\pi}{d^4\pi^4}\cdot\sqrt{d^4\pi^4\sin^2 d\pi + 9(\lambda t - 2dk\pi)^2(\sin d\pi - d\pi\cos d\pi)^2}. \quad (40)$$

(c) Expression (40) can be slightly simplified by putting $d = [d] + d'$:

$$\left|F_k^1[Pe^{i(\lambda t+\varphi)}](t)\right| =$$
$$= \frac{P\sin^2 d'\pi}{d^4\pi^4}\cdot\sqrt{d^4\pi^4\sin^2 d'\pi + 9(\lambda t - 2dk\pi)^2(\sin d'\pi - d\pi\cos d'\pi)^2}. \quad (41)$$

PROOF: The proof is similar to the proof of Lemma 3, by straightforward computation using (11). □

Similar to the case of F^0-transform, we obtain the following:

Corollary 4
Let the conditions of Lemma 5 be satisfied. Then the following holds for any $k = 1,\dots,n-1$ and $t \in [c_{k-1}, c_{k+1}]$:

(a) If $d \in \mathbb{N}$ then $F_k^1[Pe^{i(\frac{2\pi d}{h}t+\varphi)}](t) = 0$.

(b) $\left|F_k^1[Pe^{i(\lambda t+\varphi)}](t)\right|$ does not depend on the phase shift φ.

(c) $\lim_{d\to\infty}\left|F_k^1[Pe^{i(\frac{2\pi d}{h}t+\varphi)}](t)\right| = 0$ and

$$\lim_{d'\to 0}\left|F_k^1[Pe^{i(\frac{2\pi([d]+d')}{h}t+\varphi)}](t)\right| = 0 \quad \textit{and} \quad \lim_{d'\to 1}\left|F_k^1[Pe^{i(\frac{2\pi([d]+d')}{h}t+\varphi)}](t)\right| = 0.$$

PROOF: Immediately from Lemma 5 by straightforward computation. □

Let us also investigate the behavior of $F_k^1[Pe^{i(\lambda t+\varphi)}]$ on an interval $[c_{k-1}, c_{k+1}] = \left[\frac{2\pi(k-1)d}{\lambda}, \frac{2\pi(k+1)d}{\lambda}\right]$. It is convenient to transform t in (38) into an auxiliary variable $t' \in [-1,1]$ by setting $t = c_k + h\,t' = \frac{2\pi d}{\lambda}t'$. Then (38) can be written, for the given k, as the function

$$F_k^1[Pe^{i(\lambda t+\varphi)}](t') = F_k^0[Pe^{i(\lambda t+\varphi)}] + \beta_k^1\,\frac{2\pi d}{\lambda}t', \qquad t' \in [-1,1]. \quad (42)$$

Lemma 6
Let the conditions of Lemma 5 be satisfied, Then for each $k = 1,\dots,n-1$ and $t' \in [-1,1]$ we have:

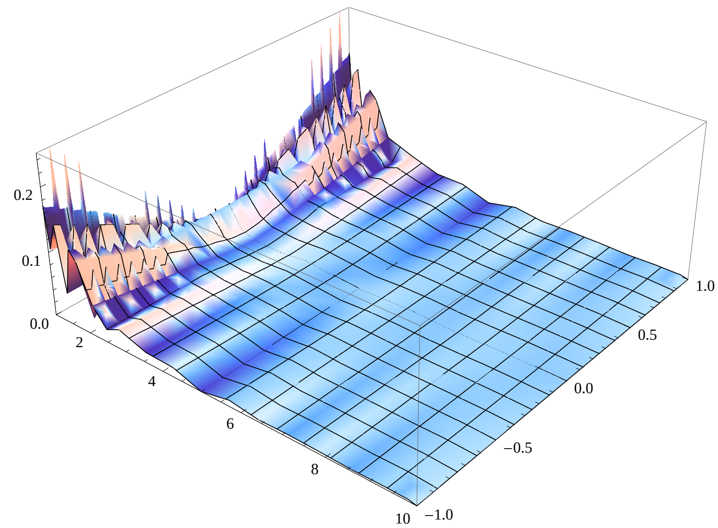

Figure 2: Graph of $|F_k^1[Pe^{i(\lambda t+\varphi)}]|$ where $k \in \{1, \ldots, n-1\}$ and $P = 1$ as a function of d ($d \in [0, 10]$) and t' ($t' \in [-1, 1]$). One can see that absolute values of each k-th component of the F^1-transform also practically vanish for $d \geq 2$.

(a)

$$F_k^1[Pe^{i(\lambda t+\varphi)}](t') = \frac{Pe^{i\varphi}}{d^3\pi^3}\ \sin d\pi\,((d\pi + i6t')\sin d\pi - i6d\pi t'\cos d\pi). \quad (43)$$

(b)

$$\left|F_k^1[Pe^{i(\lambda t+\varphi)}](t')\right| = \left|\frac{Pd\sin(d'\pi)}{d^4\pi^3}\right| \cdot$$
$$\cdot \sqrt{d^2\pi^2\sin^2 d'\pi + 6(t'\sin d'\pi - d\pi t'\cos d'\pi)^2}. \quad (44)$$

PROOF: This follows from (42) by straightforward computation similar to the proof of Lemma 5. □

A graph depicting behavior of the absolute value of F_k^1 with respect to d is in Fig. 2. A graph depicting behavior of the absolute value of F_k^1 for $t' \in [-1, 1]$ with respect to the remainder $d' = d - [d]$ is in Figure 3. One can see that $\left|F_k^1[Pe^{i(\lambda t+\varphi)}](t')\right|$ is a very small number for all $t' \in [-1, 1]$.

3.3. Reducing noise

In this subsection, we will discuss how the F-transform can eliminate the random noise $R(t, \omega)$. We will prove that the use of F-transform drastically

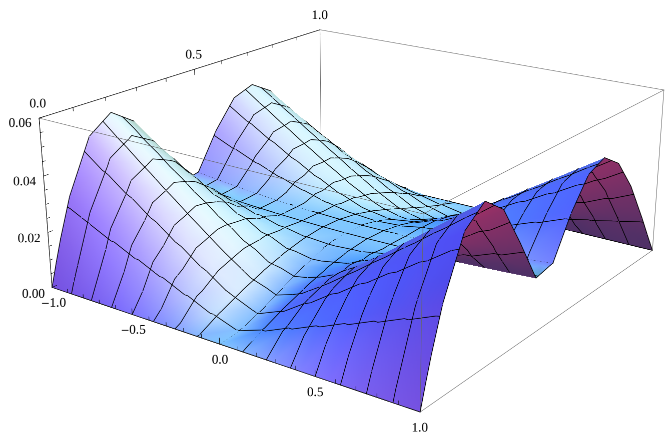

Figure 3: Graph of the detail of $|F_k^1[Pe^{i(\lambda t+\varphi)}]|$ where $P = 1$ and $[d] = 2$ as a function of t' ($t' \in [-1, 1]$) and the remainder values d' ($d' \in [0, 1]$).

decreases the noise of the type (20). As stated above, under reasonable conditions, every stationary random process with 0 mean can be represented as a linear combination of processes of type (20) (see, e.g., [15]). Because of the linearity of F-transform, we can therefore conclude that, in the general case, the use of F-transforms drastically decreases each component (20) of the random noise.

Theorem 2
Let $R(t, \omega)$ be a random noise satisfying (19), $A_0, \ldots, A_n$ be a uniform fuzzy partition over $[a, b]$ and $\mathbf{F}^0[R]$ be a direct F^0-transform of R. Then

(a) $\mathbf{E}(F_k^0[R]) = \mu$ *for all* $k = 1, \ldots, n-1$,

(b) $\mathbf{E}(\hat{R}^0(t)) = \mu$ *for all* $t \in [c_1, c_{n-1}]$.

PROOF: (c) Using (5) and the assumption (19) we have

$$\mathbf{E}(F_k^0[R]) = \frac{1}{h}\mathbf{E}\left(\int_{c_{k-1}}^{c_{k+1}} A_k(t)R(t)dt\right) = \frac{1}{h}\int_{c_{k-1}}^{c_{k+1}} A_k(t)\mathbf{E}(R(t))dt = $$
$$= \frac{1}{h}\mu\int_{c_{k-1}}^{c_{k+1}} A_k(t)dt = \mu.$$

(d) Using (b), we obtain $\mathbf{E}(\hat{R}^0(t)) = \sum_{k=0}^{n} A_k(t)\mathbf{E}(R(t)) = \mu\sum_{k=0}^{n} A_k(t) = \mu$ for all $t \in [c_1, c_{n-1}]$. □

Below, we will always work with one realization $R(t)$ for ω fixed. First, we assume that the noise $R(t)$ has the form (20). Then for each $t \in [a, b]$, there exist

one realization of $\xi(t)$. Put $\bar{\xi} = \sup\{\xi(t) \mid t \in [a,b]\}$ and $\underline{\xi} = \inf\{\xi(t) \mid t \in [a,b]\}$. Finally, we put

$$\tilde{\xi} = \begin{cases} \bar{\xi} & \text{if } \bar{\xi} \geq 0, \\ \underline{\xi} & \text{if } \bar{\xi} < 0. \end{cases} \tag{45}$$

Theorem 3
Let $R(t)$ be a noise represented by formula (20), $A_0, \dots, A_n$ be an h-uniform fuzzy partition over $[a,b]$ fulfilling conditions of Lemma 3, $\mathbf{F}^0[R]$ be a direct and $\hat{R}^0$ an inverse F^0-transform of R. Then for each $k = 1, \dots, n-1$

$$|F_k^0[R]| \leq \frac{|\tilde{\xi}| \sin^2(d\pi)}{d^2\pi^2}, \tag{46}$$

as well as

$$|\hat{R}^0| \leq \frac{|\tilde{\xi}| \sin^2(d\pi)}{d^2\pi^2}, \qquad t \in [c_1, c_{n-1}] \tag{47}$$

where d is related to h via (25).

PROOF: By the linearity of the F-transform we have either

$$F_k^0[R(t)] \leq \bar{\xi} \cdot F_k[e^{i\lambda t + \varphi}]$$

if $\bar{\xi} \geq 0$ or

$$F_k^0[R(t)] \geq \underline{\xi} \cdot F_k[e^{i\lambda t + \varphi}]$$

if $\bar{\xi} < 0$. Hence,

$$|F_k^0[R(t)]| \leq |\tilde{\xi}| \cdot |F_k^0[e^{i\lambda t + \varphi}]| \leq \frac{|\tilde{\xi}| \sin^2(d\pi)}{d^2\pi^2} \tag{48}$$

by (31).

Inequality (47) is obtained as follows:

$$|\hat{R}^0(t)| = \left| A_{k-1}(t)\, \tilde{\xi}\, F_{k-1}^0[e^{i\lambda t + \varphi}] + A_k(t)\, \tilde{\xi}\, F_k^0[e^{i\lambda t + \varphi}] \right| \leq$$
$$\leq \underbrace{(A_{k-1}(t) + A_k(t))}_{=1} \frac{|\tilde{\xi}| \sin^2(d\pi)}{d^2\pi^2}.$$

□

Theorem 4
Let $R(t)$ be a noise represented by formula (20), $A_0, \dots, A_n$ be an h-uniform fuzzy partition over $[a,b]$ fulfilling conditions of Lemma 5. Let d be related to h

via (25), $\mathbf{F}^1[R]$ be a direct and $\hat{R}^1$ an inverse F^1-transform of R. Then for each $k = 1, \ldots, n-1$ and $t \in [c_1, c_{n-1}]$

$$|F_k^1[R](t)| \leq \frac{|\tilde{\xi}| \sin^2 d\pi}{d^4\pi^4} \cdot \tag{49}$$

$$\cdot \sqrt{d^4\pi^4 \sin^2 d\pi + 9(\lambda t - 2dk\pi)^2 (\sin d\pi - d\pi \cos d\pi)^2}, \tag{50}$$

as well as

$$|\hat{R}^1| \leq |\tilde{\xi}| \cdot \eta(t), \qquad t \in [c_1, c_{n-1}] \tag{51}$$

where

$$\eta(t) = \frac{\sin^2(d\pi)}{d^4\pi^4} \cdot \max\left\{ \sqrt{d^4\pi^4 \sin^2 d\pi + 9(\lambda t - 2dk\pi)^2 (\sin d\pi - d\pi \cos d\pi)^2} \,\middle|\, k = 1, \ldots, n-1 \right\}. \tag{52}$$

PROOF: This follows from Lemma 5 analogously as in Theorem 3. □

3.4. Estimation of the trend-cycle using F-transform

In this subsection we will show that on the basis of the previous results, the F-transform (both zero and first-degree) is a suitable tool using which we can estimate the trend-cycle with high accuracy.

We will suppose the following:

(i) The stochastic process X can be decomposed as in (22) where the seasonal constituent component S consists of periodic functions having periodicities T_j, $j \in \{1, \ldots, r\}$ (see (21)). The longest of the latter is denoted by $\bar{T}$.

(ii) A number $\bar{d} \in \mathbb{N}$ is chosen and the distance h in (26) between the nodes is set to

$$h = \bar{d}\,\bar{T} \tag{53}$$

so that $n \geq 2$. The corresponding triangular fuzzy partition $A_0, \ldots, A_n$ is fixed

(iii) The trend-cycle TC is a function with no clear periodicity or its periodicity is much longer than $\bar{h}$. Moreover, the modulus of continuity $\lambda(h, TC)$ is small.

Note that (iii) requires that TC is smooth with small changes in its course. By $\hat{S}$ we denote the corresponding inverse F-transform of S.

Theorem 5

Let $S(t)$ be the seasonal constituent component (21) whose members have periodicities T_j, $j = 1, \ldots, r$. Furthermore, let an h-uniform fuzzy partition $A_0, \ldots, A_n$ be formed over the equidistant set of nodes (26) with the distance $h = \bar{d}\,\bar{T}$ where $\bar{T}$ is the longest periodicity among all T_j, $j \in \{1, \ldots, r\}$.

(a) *Let* $d_j = \frac{h}{T_j} \in \mathbb{N}$ *for all* $j = 1, \ldots, r$*. Then* $\hat{S}^0(t) = 0$ *as well as* $\hat{S}^1(t) = 0$ *for all* $t \in [c_1, c_{n-1}]$*.*

(b) *Let* $I \subset \{1, \ldots, r\}$ *be a set of subscripts for which* $d'_j = d_j - [d_j] \in (0,1)$*,* $j \in I$*. Then the following holds for all* $t \in [c_1, c_{n-1}]$*:*

(ba)

$$|\hat{S}^0(t)| \leq \sum_{j \in I} \left| \frac{P_j \sin^2(d'_j \pi)}{d_j^2 \pi^2} \right|, \tag{54}$$

(bb)

$$|\hat{S}^1(t)| \leq \sum_{j \in I} P_j \cdot \eta_j(t) \tag{55}$$

where

$$\eta_j(t) = \max \left\{ \frac{\sin^2 d'_j \pi}{d_j^4 \pi^4} \cdot \right.$$

$$\left. \cdot \sqrt{d_j^4 \pi^4 \sin^2 d'_j \pi + 9(\lambda t - 2 d_j k \pi)^2 (\sin d'_j \pi - d_j \pi \cos d'_j \pi)^2} \,\right| k = 1, \ldots, n-1 \right\}.$$

PROOF: (a) Follows immediately from Corollaries 3(b) and 4(b).

(ba) Using Corollary 3(b) we, similarly to (37), obtain for all $t \in [c_{k-1}, c_k]$ and $k = 2, \ldots, n-1$

$$|\hat{S}^0(t)| = \left| \sum_{k=0}^{n} A_k(t) F_k^0[S] \right| = \left| \sum_{k=0}^{n} A_k(t) \sum_{j \in I} F_k^0[P_j e^{i(\lambda_j t + \varphi_j)}] \right| =$$

$$\left| \sum_{j \in I} \sum_{k=0}^{n} A_k(t)\, F_k[P_j e^{i(\lambda_j t + \varphi_j)}] \right| \leq \sum_{j \in I} \sum_{k=0}^{n} A_k(t)\, |F_k^0[P_j e^{i(\lambda_j t + \varphi_j)}]| \leq$$

$$\leq \sum_{j \in I} |A_{k-1}(t) F_{k-1}^0[P_j e^{i(\lambda_j t + \varphi_j)}] + A_k(t) F_k^0[P_j e^{i(\lambda_j t + \varphi_j)}]| \leq$$

$$\leq \sum_{j \in I} \underbrace{(A_{k-1}(t) + A_k(t))}_{=1} \left| \frac{P_j \sin^2(d'_j \pi)}{d_j^2 \pi^2} \right|$$

from which (54) follows.

(bb) is proved in a similar way using Lemma 5(c). □

Let us denote

$$D^0 = \sum_{j \in I} \left| \frac{P_j \sin^2(d'_j \pi)}{d_j^2 \pi^2} \right| + \frac{|\tilde{\xi}| \sin^2(d\pi)}{d^2 \pi^2}, \tag{56}$$

$$D^1 = \sum_{j \in I} \sum_{j \in I} \eta_j(t) + |\tilde{\xi}| \cdot \eta(t) \tag{57}$$

where $d_j = \frac{h}{T_j}$ and $I \subset \{1, \dots, r\}$ is the set of all subscripts, for which $d'_j = d_j - [d_j] \in (0, 1)$ and $\tilde{\xi}$ is determined in (45) and $\eta(t)$ is given by (52). On the basis of the discussion above, we can formulate the following theorem.

Theorem 6
Let $X(t)$ be realization of the stochastic process in (17) considered over the interval $[0, b]$. If we construct a fuzzy partition over the set of equidistant nodes (26) with the distance (53) then the corresponding inverse F-transform of $X(t)$ gives the following estimator $\hat{X}^m$ of the trend-cycle with the error:

$$|\hat{X}^m(t) - TC(t)| \leq 2\lambda(h, TC) + D^m, \qquad t \in [c_1, c_{n-1}] \tag{58}$$

where D^m, $m \in \{0, 1\}$ is the error (56) or (57), respectively depending on the degree of the applied F-transform.

PROOF: The theorem is a consequence of Theorems 3 and 4. □

Let us emphasize that from the analysis above it follows that D^m is a small number (many summands in (56) or (57) are even equal to zero) and, by the assumption, the modulus of continuity $\lambda(h, TC)$ is also small. Hence, we may conclude from (58) that the *F-transform enables us to extract trend-cycle from the time series $X(t)$ with high accuracy.*

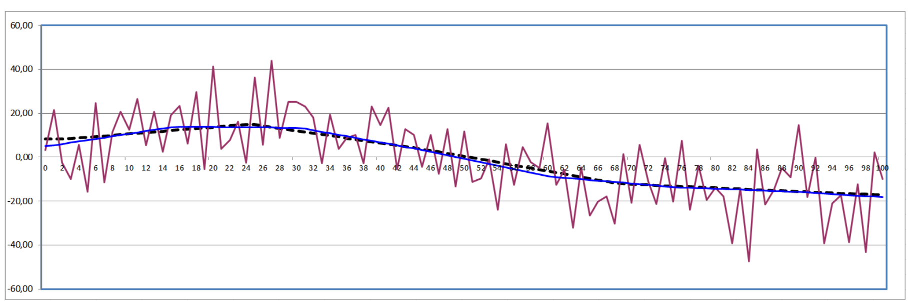

Figure 4: F-transform of the artificial time series $X(t)$, $t \in \{0, 100\}$. The dotted line depicts the original trend-cycle $TC(t)$ given by the data. Using (59), it is completed by values of the four sine members and the noise $R(t)$ to form the artificial time series depicted also in the figure. After application of the F-transform to $X(t)$, we obtained approximation $\hat{X}(t)$ of the trend-cycle. This is depicted by the solid line that is almost identical with the original $TC(t)$.

3.5. Experiment

To verify the above methodology, we prepared a simple experiment with the data. We artificially formed a time series $X(t)$ on the set of integers $\{0, 100\}$ as follows:

$$\begin{aligned} X(t) = TC(t) &+ 5\sin(0.63t + 1.5) + 5\sin(1.26t + 0.35) \\ &+ 15\sin(2.7t + 1.12) + 7\cos(0.41t + 0.79) + R(t). \end{aligned} \tag{59}$$

The function $TC(t)$ in (59) is the trend-cycle given by artificial data without clear periodicity (it is depicted by dotted line in Figure 4; its modulus of continuity is $\lambda(30, TC) = 3.22$). The other four sine members form the seasonal constituent component $S(t)$. Their periodicities are $T_1 = 10$, $T_2 = 5$, $T_3 = 2.3$, $T_4 = 15.4$, respectively. Therefore, we set $\bar{T} = T_4$ and $d = 1$, i.e. the distance (53) is $h = 15$. Consequently, the width of basic functions is $2h = 30$ (the time axis is discrete and so, fractions are neglected). Since all $d_1, \ldots, d_4$ are close to natural numbers, the error D^0 in (56) is practically 0. The $R(t)$ is a random noise with average $\bar{\mu} = -0.24$.

The result of application of the F^0-transform is depicted in Figure 4. One can see from it that the both the whole seasonal constituent component as well as the noise were almost completely removed. Maximal difference $|\, TC(t) - \hat{X}^0(t)| = 3.32$ and so, we conclude that the trend-cycle was estimated with the error corresponding to (58) (cf. also (15)).

4. Conclusion

In this paper, we analyzed application of the F-transform to time series under the assumption that the latter can be additively decomposed into three constituent components: trend-cycle, seasonal component and random noise. We showed that the seasonal component can be practically eliminated and if the random noise is ergodic with zero (or very small) mean value μ then it can be significantly reduced. Therefore, our computation leads to the conclusion that the F-transform is a convenient means using which the trend-cycle, as informally characterized by OECD, can be extracted. Unlike classical statistical approaches where the trend-cycle is assumed in advance to be some specific simple function (quite often linear), the F-transform does not use any predefined shape and still provides exact formula for computation of the trend-cycle. Further investigation will focus on improving methods for prediction both of the trend-cycle as well as of the seasonal component.

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