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IMPRECISE PROBABILITIES IN ENGINEERING ANALYSES

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Abstract

Probabilistic uncertainty and imprecision in structural parameters and in environmental conditions and loads are challenging phenomena in engineering analyses. They require appropriate mathematical modeling and quantification to obtain realistic results when predicting the behavior and reliability of engineering structures and systems. But the modeling and quantification is complicated by the characteristics of the available information, which involves, for example, sparse data, poor measurements and subjective information. This raises the question whether the available information is sufficient for probabilistic modeling or rather suggests a set-theoretical approach. The framework of imprecise probabilities provides a mathematical basis to deal with these problems which involve both probabilistic and non-probabilistic information. A common feature of the various concepts of imprecise probabilities is the consideration of an entire set of probabilistic models in one analysis. The theoretical differences between the concepts mainly concern the mathematical description of the set of probabilistic models and the connection to the probabilistic models involved. This paper provides an overview on developments which involve imprecise probabilities for the solution of engineering problems. Evidence theory, probability bounds analysis with p-boxes, and fuzzy probabilities are discussed with emphasis on their key features and on their relationships to one another.

Keywords: Uncertainty modeling; Imprecise probabilities; Evidence theory; Probability bounds analysis; Fuzzy probabilities

1. Introduction

The analysis and reliability assessment of engineered structures and systems involves uncertainty and imprecision in parameters and models of different types. In order to derive predictions regarding structural behavior and reliability, it is crucial to represent the uncertainty and imprecision appropriately according to the underlying empirical information which is available. To capture variation of structural parameters, established probabilistic models and powerful simulation techniques are available for engineers, which are widely applicable to real-world problems; for example, see [1, 2, 3]. The required probabilistic modeling can be realized via classical mathematical statistics if sufficient data of a suitable quality are available.

In engineering practice, however, the available data are frequently quite limited and of poor quality. These limitations can sometimes be substantial. Information is often not available in the form of precise models and parameter values; it rather appears as imprecise, diffuse, fluctuating, incomplete, fragmentary, vague, ambiguous, dubious, or linguistic. Moreover, information may variously be objective or subjective, possibly including random sample data and theoretical constraints but also expert opinion or team consensus. Sources of information may vary in nature and trustworthiness and include maps, plans, measurements, observations, professional experience, prior knowledge, and so forth. Changes of boundary conditions and environmental conditions have to be taken into consideration, but are often of a hypothetical nature. Some illustration of this situation can be found in the challenge problems posed in [4]. For an engineering analysis it is then a challenge to formulate suitable numerical models in a quantitative manner, on one hand, without ignoring significant information and, on the other hand, without introducing unwarranted assumptions. If this balance is violated or not achieved, computational results may deviate significantly from reality, and the associated decisions may lead to serious consequences.

Solutions to this problem are discussed in the literature from various perspectives using different mathematical concepts. This includes BAYESian approaches [5, 6, 7, 8], interval probabilities [9, 10], random sets [11], evidence theory [12, 13], fuzzy stochastic concepts [14] and info-gap theory [15]. These concepts are part of the general framework of information theory, which is elucidated in [16] in a rigorous manner. The variety of choices provides the engineer with considerable flexibility in uncertainty modeling, but it creates, at the same time, the question of the most suitable modeling choice. For practical applications this question cannot be answered in general. A realistic mathematical approach can only be formulated by analyzing the nature of the available information in each particular case. To support this analysis and modeling choice, classifications of the available information according to defined criteria are usually employed.

In Section 2, the classification of uncertainty in engineering is reviewed briefly with focus on the facets of epistemic uncertainty. An introduction to imprecise probabilities with an overview of applications in engineering is provided in Section 3. Subsequently, specific features and relationships of several selected concepts from the framework of imprecise probabilities are discussed in Section 4, namely evidence theory in Section 4.2, interval probabilities in Section 4.3, probability bounds analysis in Section 4.4, and fuzzy probabilities in Section 4.5.

Beyond this coarse review, the collection of papers in this Special Issue provides detailed insight into various imprecise probability approaches and highlights their benefits in engineering analyses.

2. Facets of epistemic uncertainty

Aleatory and epistemic uncertainty are often distinguished based on the sources of the uncertainty; see [17, 18, 19, 20, 7, 4]. Initially, such classification appears convenient and straightforward. Irreducible uncertainty is classified as aleatory and refers to a property of the system associated with fluctuations or variability, whereas reducible uncertainty is classified as epistemic and concerns a property of the analyst associated with a lack of knowledge. Aleatory uncertainty is stochastic variation which results from an underlying random experiment and corresponds with the traditional frequentist interpretation of probability theory. Epistemic uncertainty, however, remains as a collection of all problematic cases and does not imply a specific mathematical model.

Commonly, the reason for epistemic uncertainty is subjectivity. In this case a suitable framework for modeling may be provided by subjective probabilities which are consistent with the axioms of probability [21, 22]. In this context it is sometimes argued that expert knowledge can compensate for the paucity of data and limitations of information through the use of BAYESian methods. If a subjective perception regarding a probabilistic model exists and some data for a model update can be made available, a BAYESian approach can be very powerful, and meaningful results using available information can be derived. BAYESian approaches are attracting increasing attention in engineering. Considerable advancements have been reported for the solution of various engineering problems [23, 24, 25, 26, 27, 28, 6, 29, 30, 31] using model updating. Here, one can usually build on a reasonable basis of expert knowledge to specify a suitable model class and to cast prior knowledge into subjective distribution functions. If subjective probabilistic statements can be formulated on rational grounds and some data of suitable quality are available, then BAYESian updating can play its important role. The subjective influence in the model assumption decays quickly with a growing amount of data. When data are available for such updating, a probabilistic model parameter can be estimated with the expected value of a posterior distribution. The result is then a mix of objective and subjective information. Alternatively, the epistemic uncertainty represented by the posterior distribution can be made visible in the result, for example in form of credible intervals, which can be helpful for the communication of the results as explained in [18] in the context of risk assessment. This treatment of subjective information enables the consideration of both aleatory and epistemic uncertainty together in a probabilistic framework.

Epistemic uncertainty, however, is not limited to subjectivity but may also refer to indeterminacy, ambiguity, fragmentary or dubious information and other phenomena, which do not support the analyst in forming a subjective opinion in terms of probabilities. Examples are poor data or linguistic expressions, which indicate a possible value range or bounds rather than a subjective distribution function. In the early design stage, design parameters can be specified only roughly and underlie later changes as the design matures. Further, digital measurements are characterized by a limited precision as no information is available beyond the last digit. Physical inequalities can frequently be utilized to determine bounds for parameters but not to specify characteristics concerning variations, fluctuations, value frequencies, etc. over some value range. The same applies to the numerical description of individual measurements obtained under dubious conditions. Conditional probabilities determined under unknown conditions and marginals of a joint distribution with unknown copula (dependence function) provide bounds for probabilistic models rather than prior probabilistic information for model options. This facet of epistemic uncertainty is associated with several different manifestations of an uncertain variable:

- the variable may take on any value between bounds, but there is no basis to assume probabilities to the options;
- the variable has a particular real value, but that value is unknown except that it is between bounds;
- the variable may take a single value or multiple values in some range, but it is not know which is the case;
- the variable is set-valued.

The characteristics of this type of information can be described most appropriately as *imprecision*. Mathematical models proposed for imprecise variables are set-theoretical and include intervals [32], BAYESian sets [33], rough sets [34], clouds [35] and convex models [36]. Overviews on respective applications in engineering can be found in [37, 38].

The distinction between probabilistic subjectivity and imprecision as different forms of epistemic uncertainty provides a pragmatic criterion to classify non-deterministic phenomena according to the nature of information. From this perspective, aleatory uncertainty and the subjective probabilistic form of epistemic uncertainty can be summarized as probabilistic uncertainty, whereas imprecision refers to the non-probabilistic form of epistemic uncertainty. This classification helps to avoid confusion if uncertainty appears with both probabilistic and non-probabilistic phenomena simultaneously in an analysis. An illustrative example for this situation is a random sample of imprecise perceptions (e.g., intervals due to limited measurement accuracy) of a physical quantity. Whilst the scatter of the realizations of the physical quantity possesses a probabilistic character (frequentist or subjective), each particular realization from the population exhibits, additionally, imprecision—with a non-probabilistic character. If an analysis involves this type of hybrid information, it is imperative to consider imprecision and probabilistic uncertainty simultaneously but to not mix the characteristics so that imprecision does not populate into the probabilistic model and vice versa. This conceptual understanding together with the classification into probabilistic uncertainty and imprecision provides intuitive motivation for imprecise probabilities and their terminology.

When epistemic uncertainty appears as imprecision, a subjective probabilistic model description would be quite arbitrary. Consider a floor beam with a strict requirement for the maximum deflection. Suppose the dependency between load and deflection is known deterministically, but a load parameter is available in the form of bounds only. This information is naturally modeled as an interval. Since no information about any probabilities exists, one could now assign a uniform distribution to the load interval based on the principle of maximum entropy. This approach is perhaps reasonable in the context of information theory, but it is disconnected from the engineering context of the problem itself. It leads to an averaged result for the deflection of the beam using equal weights for all possible load values within the available interval. However, the maximum deflection, which is of interest, is not directly addressed and can only be retrieved from simulation results with tremendous effort. And for another assigned probability distribution over the load interval the result would be different. Thus the character of the available information is changed; the interval input is transformed into a probabilistic result, the meaning of which is purely based on subjective—or really arbitrary—assumptions and justifications, which may even be out of context. In contrast to this, an interval analysis ensures a consistent translation of the input interval into a result interval without asking for any subjective assumptions. The character of the available information is retained in this analysis. And it delivers directly the maximum deflection, which is the quantity of interest, as bounds on the quantity. This simple example shows how inappropriately modeling epistemic uncertainty can undermine the purpose of an analysis, potentially with severe consequences.

The modeling of imprecision is not limited to the use of intervals. An interval is a quite crude expression of imprecision. The specification of an interval for a parameter implies that, although a number's value is not known exactly, exact bounds on the number can be provided. This may be criticized because the chore of specifying precise numbers is just transferred to the bounds. Fuzzy set theory provides a workable basis for relaxing the need for precise values or bounds. It allows the specification of a smooth transition for elements from belonging to a set to not belonging to a set. Fuzzy numbers are a generalization and refinement of intervals for representing imprecise parameters and quantities. The essence of an approach using fuzzy numbers that distinguishes it from more traditional approaches is that it does not require the analyst to circumscribe the imprecision all in one fell swoop with finite characterizations having known bounds. The analyst can now express the available information in the form of a series of plausible intervals, the bounds of which may grow, possibly even to infinite limits. This allows a more nuanced approach compared to interval modeling. Fuzzy sets provide an extension to interval modeling that considers variants of interval models, in a nested fashion, in one analysis; see [39]. This modeling of imprecision is analogous to probability's modeling of uncertainty, and, like the probabilistic approach, it also produces a distributional answer that is more nuanced than what can be achieved by worst case analysis or bounding with a simple interval. Fuzziness arises in cases where there are degrees or gradations admitting arbitrariness in where defining lines are drawn. In other fields, this is sometimes called *vagueness.* Fuzzy numbers can be defined as special fuzzy sets on the real line, and fuzzy arithmetic operating on these fuzzy numbers has been defined in [40]. These ideas underpin generalizations to possibility distributions [41], fuzzy probability [42], and info-gap decision theory [15]. These developments involve the min-max convolution operator and the extension principle [33, 43] as the standard bases for processing fuzzy information, which agrees with the general engineering understanding of processing set-valued information through engineering computations. Other combination rules for fuzzy sets as used in fuzzy logic are not utilized for this purpose.

Imprecision and uncertainty can appear together in the same problem. For example, suppose only bounds on some parameter of a prior distribution are known. Any appropriate distribution whose parameter is limited to these bounds might then be considered an option for modeling. But the selection of any particular distribution would introduce unwarranted information that cannot be justified except by bald assumption. Even assuming a uniform distribution, which is commonly done in such cases, ascribes more information than is actually given by the bounds. This situation may become critical if no or only very limited data are available for a BAYESian model update. The initial subjectivity is then dominant in the posterior distribution and in the final result. If these results, such as failure probabilities, determine critical decisions, one may wish to consider the problem from the following perspective.

When several probabilistic models are plausible for the description of a problem, and insufficient information is available to assess the suitability of the individual models or to relate their suitability with respect to one another, then it may be of interest to identify the range of possible outcomes, including especially the worst possible case, rather than to average over all plausible model options with arbitrary weighting. The probabilistic analysis is carried out conditional on each of many particular probabilistic models out of the set of plausible models. In reliability assessment, this implies the calculation of an upper bound for the failure probability as the worst case. This perspective can be extended to explore the sensitivity of results with respect to the variety of plausible models, that is, with respect to a subjective model choice. A mathematical framework for an analysis of this type has been established with imprecise probabilities. But this intuitive view is by no means the entire motivation for imprecise probabilities [16, 44]. Imprecise probabilities are not limited to a consideration of imprecise distribution parameters. They are also capable of dealing with imprecise conditions, with dependencies between random variables, and with imprecise structural parameters and model descriptions. Respective discussions can be reviewed, for example, in [45, 46]. Multivariate models can be constructed [47]. Imprecise probabilities also allow statistical estimations and tests with imprecise sample elements [48, 49, 50, 51, 52]. Results from robust statistics in the form of solution domains of statistical estimators can also be considered directly [53].

3. Imprecise probabilities

3.1. Emergence in engineering

A key feature of imprecise probabilities is the identification of bounds on probabilities for events of interest; the uncertainty of an event is characterized with two measure values—a lower probability and an upper probability [54]. The distance between the probability bounds reflects the indeterminacy in model specifications expressed as imprecision of the models. This imprecision is the concession for not introducing artificial model assumptions. Such model assumptions based on expert knowledge are often too narrow, which is known as expert overconfidence [55]. In imprecise probabilities, this problem is circumvented by implementing set-valued descriptors in the specification of a probabilistic model. The model description is thereby limited to some domain, and no further specific characteristics are ascribed. This introduces significantly less information in comparison with a specific subjective distribution function as used in the BAYESian approach. Imprecision in the model description expressed in a set-theoretical form does not migrate into probabilities, but it is reflected in the result as a set of probabilities which contains the true probability. This feature is particularly important when the calculated probabilities provide the basis for critical decisions. With imprecise probabilities the analysis may be performed with various relevant models to obtain a set of relevant results and associated decisions. This helps to avoid wrong decisions due to artificial restrictions in the modeling.

In the first systematic discussion of imprecise probabilities [44] their semantics is summarized with the term *indeterminacy* which arises from ignorance about facts, events, or dependencies. This specifies the context in which imprecise probabilities appear in nature and shows a basic distinction with respect to BAYESian and traditional probabilistic analysis. In view of engineering problems imprecise probabilities arise, in particular, when probabilistic elicitation exercises are incomplete, when probabilistic information appears incomplete or dubious, and when observations of sample elements appear imprecise. Further motivations for imprecise probabilities include observations which cannot be separated clearly, conditional probabilities which are observed with unclear conditions, and marginals of a distribution on a joint space which are specified with imperfect information about the accompanying copula function that characterizes the dependence among the variables.

Imprecise probabilities include a large variety of specific theories and mathematical models associated with an entire class of measures. This variety is discussed in [16] in a unifying context; the diversity of model choices is highlighted, and arguments for imprecise probabilities are summarized. Imprecise probabilities have a close relationship to the theory of random sets [56, 57] and cover, for example, the concept of upper and lower probabilities [58], sets of probability measures [59], distribution envelopes [60], probability bounds analysis using p-boxes [61], interval probabilities [62], CHOQUET capacities [63] of various orders, and evidence theory (or DEMPSTER-SHAFER Theory) [64, 65] as a theory of infinitely monotone CHOQUET capacities [66, 67]. Moreover, fuzzy probabilities [68, 69], with their roots in the theory of fuzzy random variables [70, 42], are also covered under the framework of imprecise probabilities and possess strong ties to several of the aforementioned concepts.

Developments in imprecise probabilities appear in an interaction between mathematics, computer science and engineering. An important source of mathematical developments for utilization in engineering is the symposium series ISIPTA (International Symposium on Imprecise Probability: Theories and Applications) with its proceedings, e.g. [71], and special issues, e.g. [72]. The transition of mathematical findings into engineering is then supported by multidisciplinary workshops, such as the Workshop on Reliable Engineering Computing, which focus on practical utilization of the theories. The respective proceedings, e.g. [73], and journal special issues, e.g. [74, 75], provide a current overview on the developments. In addition to these regular activities, knowledge transfer is organized in individual workshops and mini-symposia at engineering conferences, from which special issues are produced, such as [76, 77, 46, 78, 79]. Also, paper compilations and summarized developments are published in books; see, for example, [45, 80, 81, 82].

The adoption of imprecise probabilities and related theories for the solution of engineering problems can be traced, in its early stage, with the publications [83, 84, 85, 86, 87]. Numerical methods for quantifying and processing imprecision and uncertainty by means of fuzzy random variables in conjunction with a nonlinear analysis are proposed in [82] in order to assess the response and reliability of civil engineering structures. And an entry of imprecise probabilities into standard engineering literature is recorded with [13] with a consideration of evidence theory to analyze complex engineering systems under uncertainty and imprecision in view of a quantitative risk assessment. Along this way and beyond a variety of specific theoretical developments and applications have been published using quite diverse terminology for very similar or equivalent facts, situations and phenomena. This becomes particularly obvious in [46], where various solutions are proposed for the solution of the same simple academic and engineering problems defined in [4]. The solution summary [88] shows, on one hand, how "different" approaches lead to virtually the same results, and on the other hand, how different subjective decisions in the initial modeling can lead to deviations in the results. The proposed solution concepts include a combination of probability theory, evidence theory, possibility theory and interval analysis [20], probability bounds analysis [61], distribution envelope determination [60], sets of probability measures [59], coherent lower and upper probabilities [58], imprecise coherent probabilities [89], coherent lower previsions [90], random set theory [91], probability distribution variable arithmetic [92], and polynomial chaos expansions for equivalence classes of random quantities [93].

3.2. Engineering application fields

From the initial developments imprecise probabilities have emerged into several application fields in engineering with structured approaches. The largest application field appears as ***reliability assessment***, where imprecise probabilities are implemented to address sensitivities of the failure probability with respect to the probabilistic model choice. As the tails of the distributions are decisive for the failure probability but can only be determined and justified vaguely based on statistical data and expert knowledge, an analysis with an entire set of plausible probabilistic models and the identification of an associated upper bound for the failure probability are beneficial. This reduces the risk of wrong decisions due to unintentionally optimistic modeling. Implementations and applications have been reported on a parametric as well as on a non-parametric basis and with different concepts. For example, evidence theory is used in [54] to address imprecision in the reliability of individual elements in a system and to make this effect visible in the system reliability. In [94] intervals are employed for the description of the imprecision in probabilistic models for a structural reliability assessment. And a reliability analysis with fuzzy distribution parameters is proposed in [95]. The developments in this area have been extended to applicability to larger, realistic and practical problems. An overview in the context of computational intelligence in systems reliability is provided in [96]. In [12] evidence theory is proposed to estimate parameter distributions for structural reliability assessment based on information from previous analyses, expert knowledge and qualitative assessments. This approach is demonstrated in an application to estimate the physical vulnerability of an office building to blast loading.

A comparative study of different modeling options in the framework of evidence theory is presented in [97] and elucidated by means of an example from flood risk analysis. This study is focused on methods for realistic modeling of information typically available in practice and the subsequent integration in industrial risk analysis. Random sets are used in [98] to perform a reliability assessment based on imprecise data and lack of information as part of a real tunnel project. This geotechnical application includes a real case history with model validation by in situ measurements using a random set finite element framework. Another application of random sets in geotechnical engineering is presented in [99]. Measurement data are used to construct random set models in a non-parametric manner using formulations based on the CHEBYSHEV inequality. The models are then used in a finite element based reliability analysis of a sheet pile wall. A systematic development of selected imprecise distribution functions based on imprecise BAYESian inference is presented in [100]. It is shown how limited information can be addressed with a class of priors to eventually bound probabilities of failure. Imprecise BAYESian inference is also known as BAYESian sensitivity analysis or robust BAYES analysis [101]. In [102] probability bounds analysis is compared with BAYESian Markov-Chain Monte-Carlo (MCMC) methods for uncertainty analysis of an environmental engineering problem involving the toxic effects of hypersalinity on an endangered fish species. The comparison reveals good agreement in expected (mean) results, but sometimes strong disagreement in uncertainty characterizations. In [103] and [104] probability bounds analysis is applied to reliability assessment for a dike revetment and a finite-element structural analysis respectively, and the results are compared to traditional probabilistic methods with Monte Carlo simulation. In these examples, the risks can be underestimated with traditional methods whereas probability bounding is able to cover the actual risk range comprehensively, and often with less overall computational effort than Monte Carlo methods.

The conceptual developments are supplemented by the design of numerical methods, which aim at computational efficiency and approximation quality to nurture applicability to real-size engineering problems. In [17] these criteria were used to consider three modeling approaches: interval-valued probability, second-order probability and evidence theory. It was found that a combination of stochastic expansions for probabilistic uncertainty with an optimization approach to determine interval bounds for probabilities provides advantages in terms of accuracy and efficiency. A Monte Carlo approach to estimate interval failure probabilities is presented in [105], which is a combination of stochastic sampling with an efficient interval finite element method. It employs interval parameters to define families of distributions characterized by p-boxes. In a comparison with a BAYESian approach it is shown that interval estimations for the failure probability based on BAYESian results are contained in the interval results, which indicates the influence of subjectivity in the modeling and the potential risk in the case of over-confidence. In [106] the concept of fuzzy probabilities is used for the reliability assessment of an offshore platform. Vagueness and a lack of knowledge in the specification of corrosion effects are made visible in the failure probability, which indicates their sensitivity with respect to assumptions in the corrosion model. Technically, this analysis makes use of the global optimization method from [95, 82] for processing imprecise structural and distribution parameters in combination with importance sampling to calculate failure probabilities. This combination has also been used for time-dependent reliability estimation as shown for textile reinforced structures in [107]. It is easily extendable to other sampling methods.

In the ***analysis of sensitivities*** of model output, imprecise probabilities can provide useful new insights with features for systematic and extended investigations. The consideration of imprecise parameters on a set-theoretical basis enables the investigation of sensitivities with respect to changes of the parameters in the entire set of the input. This reveals sensitivities in a global sense over a finite domain rather than in a differential manner. Tolerances given in absolute terms can thereby be translated directly to bounds for model output without extrapolation as required in differential approaches. The advantages are obvious in cases when the model behavior is strongly nonlinear or discontinuous or when derivatives cannot be determined for some reason. Results from a set-theoretical approach are then more robust and reliable. Another advantage of the exploration of an entire domain for input parameters is the identification of favorable and less favorable parameter adjustments. This information can be used to collect further information or to perform further analyses systematically in order to identify the causes for sensitivities or to exclude sensitivities by parameter restrictions.

In [108] evidence theory is employed to perform a sampling-based sensitivity analysis in different stages in a risk analysis of an engineering system. This includes an exploratory analysis to obtain insight in the model behavior as a basis for further analyses and a subsequent investigation of incremental effects with respect to the parameter specification. Additionally, an investigation is conducted to explore the spectrum of variance-based sensitivity analysis results which corresponds to the evidence theory model used. Probability bounds analysis is used for sensitivity investigations in [109, 110], which can be more informative than traditional probabilistic approaches based on decomposition of variance. Probability bounds analysis is applied to assess the quality of probabilistic models in view of risk assessment by means of result sensitivities with respect to assumptions in the probabilistic model for the input including dependencies. A concept for sensitivity analysis in the framework of coherent lower and upper probabilities is presented in [111]. Three approaches are examined to derive an uncertainty-based sensitivity index, namely, a variance-based approach, a partial expected value of perfect information, and a relative entropy. The proposed interval-valued sensitivity index measures the relative contribution of individual model input variables, in the form of intervals or sets of distribution functions, to the uncertainty in the model output. The examples refer to the challenge problems from [4]. A sensitivity analysis with random sets constructed in a non-parametric manner is discussed in [99]. This makes use of a visualization of random sets in the form of a probability box in order to apply a pinching strategy as explained in [109, 110]. Examples from geotechnical engineering are provided for demonstration. A case study of various approaches for sensitivity analysis by way of an aerospace engineering example is provided in [112]. This includes concepts based on random sets, fuzzy sets, interval spreads, as well as pure probabilistic concepts. The considered performance criteria are computational cost, accuracy, interpretability, ability to incorporate dependencies, and applicability to large scale problems. The findings show that imprecise probabilities provide an extended flexibility in the modeling and competitive features with respect to the criteria.

In the area of ***model validation and verification*** imprecise probabilities provide extended features in two respects. First, they allow the consideration of an entire set of models without prior weighting rather than a single specific one. This refers to model uncertainty. Second, imprecision of data can be taken into account without artificial preconditioning of the data, which refers to data uncertainty. In [113] a validation metric is defined in terms of the area between a predicted distribution from a probabilistic model and the empirical distribution embodied by relevant sample data. A more general discussion and extension of this measure to validate imprecise predictions against imprecise observations in form of intervals, probability distributions, or p-boxes is presented in [114]. A representation of the shortest possible distance between prediction and observation is worked out which takes into account the imprecision of the distributions and their dependencies. In [115] an alternative approach for model validation is proposed that also allows for interval data to be included in the procedure. This parametric approach leads to a family of distributions. A heat transfer problem serves as example. A recursive least-squares estimation with observed interval data in a geodetical context is proposed in [116]. This is motivated by the need to consider uncontrollable external effects and imprecision due to remaining systematic errors in observation data. The approach is demonstrated for a damped harmonic oscillation and for the monitoring of a lock. The validation of complex structural models under a lack of knowledge is also considered in [117]. Starting from internal variables in the model, intervals are determined with stochastic bounds to identify envelopes for the parameter of interest such as a stress or displacement. Like [114], this development includes the consideration of a special distance between the envelopes of the experiments and of the model prediction. Further developments can be expected to emerge in the related fields of model updating and system identification, in which interval methods have attracted attention recently, as documented by [118] and [119]. A combination of these approaches with stochastic developments to combine their advantages as suggested in [114] and [113] seems promising.

Benefits of imprecise probabilities have also been reported in the field of ***design under uncertainty***. Through an implementation of imprecision in the numerical algorithms it becomes feasible to consider coarse specifications in early design stages. The models then allow a stepwise reduction of imprecision as the available information grows over the design process, that is, when design details are specified and implemented. Further, results from a sensitivity analysis can be utilized to identify a robust design. A comprehensive study on the implementation of evidence theory in mechatronic design processes is provided in [81]. The proposed coherent methodology enables a quantitative analysis in early design stages based on a limited amount of data and including expert estimates. In [120] it is discussed how the issue of robustness can be addressed directly in the design procedure. Time-dependent structural behavior is analyzed with fuzzy random variables in order to implement input imprecision in a quantitative assessment of robustness. This approach is related to the concept of robust design presented in [121] in a non-probabilistic context. Such design approaches can significantly contribute to achieving economic benefits by reducing design and warranty costs while improving quality and have, thus, already found access to secondary literature such as [122].

The developments discussed above are closely related to ***decision making*** and contain substantial elements for this purpose. Their features for a realistic modeling of imprecision and uncertainty ensure that the available information is properly reflected in computational results; and the evaluation of these results is the basis for deriving engineering decisions [4]. For further reading about elements for deriving decisions in an imprecise probabilities framework we refer to [9, 11, 123, 124, 125, 13, 126]. In the mathematical literature an increasing number of promising developments with imprecise probabilities towards decision making can be observed. This includes, for example, the identification of robust decisions when trade-offs between various attributes in utility hierarchies are not defined precisely. Three methods for this purpose are discussed in [127]. The classical decision rule of maximizing expected utility can be generalized to account for imprecision among the probabilities and payoffs that define the expectation, and traditional non-probabilistic decision rules such as maximin can likewise be generalized for the imprecise case. These generalized decision rules are compared in [128].

For complex decision problems in engineering, which involve both uncertainty and imprecision, credal networks provide attractive features. Credal networks represent an extension of BAYESian networks to deal with imprecision in probabilities. Within the framework of imprecise probabilities, they can be understood as sets of BAYESian networks [129]. As BAYESian networks are currently developing their usefulness in engineering, for example in the assessment of reliability and risk in structural and infrastructure engineering [130, 131], it can be expected that credal networks will also emerge in engineering to deal with cases involving imprecision. Development can already be seen in [132] which presents a case study implementing evidence theory for a BAYESian network to assess the reliability of complex systems. Another future development is seen in the broader use of computational tools from computer science for the implementation of imprecise probabilities in engineering analyses. These tools are already well developed and widely available, for example, as packages in R and in MATLAB. They provide features such as statistical estimations and tests on the basis of imprecise data, the empirical construction of imprecise cumulative distributions, and simulation schemes for imprecise variables. The implemented algorithms are described in various publications such as [133, 97, 134, 135, 136].

Although the advancements in engineering achieved with imprecise probabilities are obvious, some reservation has remained in their adoption so far. Two reasons can be recognized for this reservation. First, imprecise probabilities are frequently misperceived as competitors against established probabilistic methods. But actually, imprecise probabilities are not competitors in this sense; they represent supplementary elements which can complement probability in many cases. Imprecise probabilities enrich the variety of models and can be combined with traditional probabilistic analysis in various manners yielding an improved flexibility and adaptability with respect to the particular situation and providing extended features for engineering analyses. Second, models of imprecise probabilities are perceived as unnecessarily complicated. This argument is, however, only typical for a first view and is not supported by the relatively simple conceptual set up and mechanisms of imprecise probabilities. A discussion of this criticism is provided in [137] against the engineering need for advanced concepts, in particular, in risk assessment. Another sensitive issue is the diversity of concepts covered under the framework of imprecise probabilities. Although there are very close relationships between the concepts which can be brought together in a unified understanding, they are frequently perceived as basically different. In the following section we try to resolve these critical issues in principle.

4. Selected concepts

4.1. Conceptual categories

The ideas of imprecise probabilities may be categorized into three basic groups of concepts associated with three different technical approaches to construct imprecise probabilistic models.

1. Events, which may be complex, are observed phenomenologically and are recorded with coarse specifications. Such a specification might be, for example, "severe shear cracks in a wall". In general, these coarse specifications may be the best information available, or they may arise from limitations in measurement feasibility. The latter applies, for example, to damping coefficients. There is typically no probabilistic information

available to specify distribution functions for these coarse specifications, so that modeling as sets is most appropriate. And an expert may then assign probabilities to entire sets, which represent the observations. Starting from this model, bounds for a set of distribution functions can be constructed. We shall see below that evidence theory can represent these concepts.

2. Parameters of a probabilistic model, the distribution type or, in a non-parametric description, the curve of the cumulative distribution function may only be specified within some bounds. This imprecision may arise, for example, when conflicting information regarding the distribution type is obtained from statistical tests, that is, when the test results for different distributions as well as for compound distributions thereof with any mixing ratio are similar. These test results do not provide grounds for assigning probabilities to the model options. If no additional information is available in such situations, the most suitable approach for modeling the cumulative distribution function is as a set of distributions. In the simplest form, this implies the use of intervals for the distribution parameters. We shall see below that interval probabilities can be used to represent this group of concepts.
3. Outcomes from a random experiment may appear as blurred, for example, due to limitations in the measurement feasibility or due to the manner of characterization of the outcomes. This characterization can emerge, for example, in form of linguistic variables such as when asking a group of people for their perception of the temperature in a room, the results appear as "warm", "comfortable", "slightly warm" etc. This type of information is typically described by fuzzy sets, which provide the additional feature of a membership function in contrast to traditional sets. The membership function for an individual observation, in this context, does not represent any probabilistic information; it expresses a degree of truth with which certain numerical values represent the characterization of the observation, for example, the statement "warm". It also provides a tool for a more nuanced investigation with respect to the magnitude of imprecision. The imprecise perception of a random variable can be translated into a traditional set or fuzzy set of distribution functions. We shall see below that fuzzy random variables can be used to model this group of concepts.

Although some concepts of imprecise probabilities do not completely fall into one of these groups, they usually show clear relationships to them and can be constructed out of them or as combinations thereof. There are also strong relationships between the groups. As we shall see below, probability boxes and fuzzy probabilities possess features to cover all three groups of concepts, and fuzzy probabilities can be considered as nested probability boxes and vice versa. A categorization may so seem to be not necessary. But from a practical point of view, this categorization and the associated features of the concepts as elucidated in the subsequent sections can provide the engineer with a good sense for the modeling of a problem. In any case, the choice of the concept should be driven by both the nature of the *available information* and the *purpose of the analysis*.

In the following sections the three representative concepts and combinations thereof are briefly elucidated, their relationships to one another are examined, and their features for applications in engineering are highlighted.

4.2. Evidence theory

If the information available possesses some probabilistic or probability-related background, but does not meet the preconditions to be specified as a random variable, evidence theory often provides a suitable basis for an appropriate quantification and subsequent processing. Some analysts use subjective assignment of weights—as a degree of confidence—to events that may overlap and do not exclude one another, which represents relaxed restrictions with regard to traditional probability theory. Specifically, the requirement on the probability measure P of countable additivity

$$P\left(\bigcup_i A_i\right) = \sum_i P\left(A_i\right) \tag{1}$$

whenever A_i are pairwise disjoint events is replaced by the less rigorous conditions of sub-additivity and super-additivity; see [64, 67, 138]. The associated generalized uncertainty measures $M(A_i)$ comply with the monotonicity feature

$$M(A_i) \leq M(A_k)\,, \tag{2}$$

whenever $A_i \subseteq A_k$ as a basic requirement according to the theory of monotone measures. A mathematical description with details and background information is provided in [65]. On this basis various specific uncertainty measures may be derived within a range from plausibility to belief which include traditional probability as a special case. Evidence theory, which is also called Dempster-Shafer Theory, may therefore be understood as a generalization of traditional probability theory; it represents a theory of infinitely monotone Choquet capacities [66, 67].

The basic idea, in terms of measure theory, is to distribute the "weight" $w = 1$ over the sets A_i, which are subsets of the fundamental set, so that the sum of the weights is

$$\sum_i w(A_i) = 1\,. \tag{3}$$

The empty set has weight $w(\varnothing) = 0$ by definition. The sets A_i with positive weights $w(A_i) > 0$ are called the focal subsets. The weight assignment reflects the overall evidence that particular events behind the A_i are realized. The focal subsets A_i together with their weights $w(A_i)$ contain the entirety of available information and thus constitute the body of evidence. With reference to the requirement Eq. 3, the weight assignment is called the basic probability assignment in which the weights $w(A_i)$ represent probability masses. The compliance with the traditional definition of probability is, however, not complete because the focal subsets are not required to be disjoint. The probability mass is directly assigned to (imprecise) events represented by the focal subsets; the elementary events behind A_i and their probabilities remain hidden.

Rigorous consideration of the evidence for arbitrary events B leads to different uncertainty measures that characterize either the evidence specifically in support of B, or the evidence that is merely consistent with B. These are the plausibility measure

$$Pls(B) = \sum_{A_i \cap B \neq \varnothing} w\,(A_i), \tag{4}$$

and the belief measure

$$Bel(B) = \sum_{A_i \subseteq B} w\,(A_i), \tag{5}$$

which have the complementary property $Pls(B) + Bel(B^{\mathrm{C}}) = 1$. All other uncertainty measures $M(B)$ that may be derived within this framework are bounded by $Pls(B)$ which is an upper bound on the probability of B, and $Bel(B)$ which is a lower bound on its probability.

Depending on the selection of the focal subsets, the measures of plausibility and belief and the measures between plausibility and belief possess specific characteristics. If, for example, the focal subsets are disjoint singletons (dissonant case) representing elementary events, the special case of traditional probability is obtained with $Pls(B) = Bel(B) = P(B)$. For a nested sequence of focal subsets $A_1 \subset A_2 \subset ... \subset A_i \subset ... \subset A_n$ (consonant case), a possibility measure $\Pi(B) = Pls(B)$ and a necessity measure $N(B) = Bel(B)$ are obtained. In this context a plausibility measure (and, thus, the special case of possibility) represents an upper bound and a belief measure (and, thus, the special case of necessity) represents a lower bound of the probability measure. Furthermore, the family of λ-uncertainty measures $M_\lambda(B)$ introduced in [139] is included in the evidence theory framework. These measures are characterized by the weakened additivity property

$$M_\lambda(B_1 \cup B_2) = M_\lambda(B_1) + M_\lambda(B_2) + \lambda \cdot M_\lambda(B_1) \cdot M_\lambda(B_2)\,, \quad \lambda > -1\,, \tag{6}$$

whenever $B_1 \cap B_2 = \varnothing$. The adjustability of λ according to requirements in each particular case provides considerable flexibility for $M_\lambda(B_j)$, including the special case of traditional probability which is obtained with $\lambda = 0$.

Evidence theory may be used as a basis for an axiomatic characterization of random sets [45]. This is prompted by the interpretation of the focal subsets as random sets. Discussions and engineering applications are provided in the context of sets of probability measures to derive lower and upper probabilities of events [140]. The fields of application include, for instance, structural optimization [141], reliability assessment [99], and geotechnical stability investigations [142].

A variety of theoretical developments and applications of evidence theory in an engineering context are provided in the paper compilation [46] with an overview of the approaches in [88]. The subjects included in the discussion concern uncertainty quantification, computation of model predictions and system responses, simulation, optimization and design, and decision making. In particular, sampling and discretization methods such as the outer discretization method [91] and the iterative rescaling method [58] are proposed in conjunction with random set approaches for a numerical treatment of the associated uncertain quantities. An application to structural reliability assessment is proposed in [12] in this special issue. And the suitability of evidence theory for sensitivity analysis—a field of increasing importance—is emphasized in [108].

These developments underline the fact that evidence theory provides a useful basis for the treatment of imprecise observations and expert knowledge in engineering within a probabilistic framework. Its less restrictive structure in comparison to traditional probability theory enables a direct modeling of information as it appears in nature. Modifications of the observed phenomena, or an introduction of additional fictitious constraints just to meet traditional model requirements are unnecessary. If the available information allows a clear specification of the basic probability assignment, meaningful results can be generated. These results are, in fact, not as detailed as in a traditional probabilistic analysis but sufficient in many cases in view of engineering practice [13].

A crucial point in the practical application of evidence theory is realizing the basic probability assignment in each particular case. Traditional statistical methods from estimation and test theory are not usually applicable for this purpose. The results of the subsequent uncertainty analyses, however, depend essentially on the quality of the basic probability assignment.

Another critical consideration is the aggregation of information via combination rules for evidence which are reviewed in [138]. The most popular one is DEMPSTER's rule of combination, which may be interpreted as a generalization of BAYES' theorem; see [64]. It is, however, known that this rule can yield counterintuitive results when there is substantial conflict among the estimates because it excludes the conflict in the specification of the measure [143, 54, 138], which might make its use problematic in safety analysis particularly. Some argue that the rule is correct [144] but that it may be largely irrelevant in probabilistic argumentation [145].

4.3. Interval probabilities

JUDEA PEARL [146] argues that it would be helpful to have intervals that "portray the degree of ignorance we have about probabilities—namely, the degree to which the information we lack prevents us from constructing a complete probabilistic model of the domain", adding that such intervals "would indeed have a definite advantage over BAYESian methods, which always provide point probabilities". Although PEARL claims that intervals computed under DEMPSTER-SHAFER evidence theory do not actually have this interpretation, it is possible to construct intervals for probabilities that do serve this purpose.

The idea of bounding probability with intervals has recurred many times throughout the history of probability theory. GEORGE BOOLE [147, 148] developed the notion of interval bounds on probability, which he called minor and major limits of probability. He asked what can be said about the probability of an event A given specifications, possibly in the form of bounds, on the probabilities of related events B_1, B_2, $\ldots$, B_n. FRÉCHET [149] derived the best-possible ranges of logical functions of event probabilities irrespective of the stochastic dependence between the event probabilities. These ranges make it possible to make bounding calculations with probabilities that make no dependence assumptions. Bounding probabilities has continued to the present day, e.g., [150, 151, 152, 153]. KYBURG [154] reviews the history of interval probabilities and traces the development of the critical ideas from the middle of the previous century. Probabilistic analyses using bounding arguments of one kind or another are common throughout engineering today.

Developing a complete traditional probability model implies that one can precisely specify probabilities for an often huge collection of subsets from the sample space. In practice, sometimes one may not be able to specify a precise probability for every possible event. This often happens, for instance, when only few data or little information is available, or when we wish to model probabilities that a group agrees with, rather than those of a single individual. In such cases, it may be reasonable to characterize probabilities not as real values, but rather as intervals [148, 154, 62, 145]. An event A (a subset of the fundamental set) might be characterized by some range of probability $[P_1(A),\, P_2(A)] \subseteq [0,\, 1]$ considered reasonable given available information, instead of specifying a crisp probability $P(A)$ with considerable but unstated uncertainty. This uncertainty may have arisen from limited, vague, or dubious information, or from doubt about which of disagreeing experts might be right. An interval probability model may be defined mathematically as a mapping from the space of events (sigma algebra) to the space of intervals on $[0,\, 1]$, which is $\mathbb{I} = \{[a, b],\, a,\, b \in \mathbb{R} \mid 0 \le a \le b \le 1\}$;see also [155]. Interval probabilities specify bounds on probability for an uncertain quantity with underlying randomness that is not known in detail and, thus, they represent a special kind of imprecise probabilities [44].

Calculating with interval probabilities can be straightforward. If $P(A) = [a_1, a_2]$ and $P(B) = [b_1, b_2]$, then sure bounds on the logical conjunction (AND, intersection) and the logical disjunction (OR, union) can be computed with the FRÉCHET inequalities [149] which say

$$P(A\&B) = P(A \cap B) = [\max(0,\, a_1 + b_1 \,-\, 1),\, \min(a_2,\, b_2)], \tag{7}$$

$$P(A \vee B) = P(A \cup B) = [\max(a_1,\, b_1),\, \min(1,\, a_2 + b_2)]. \tag{8}$$

These complement the analogous but tighter rules for conjunction and disjunction that assume that events A and B are independent

$$P(A\&B) = P(A \cap B) = [a_1 \times b_1,\, a_2 \times b_2], \tag{9}$$

$$P(A \vee B) = P(A \cup B) = [1 \,-\, (1 \,-\, a_1)(1 \,-\, b_1),\, 1 \,-\, (1 \,-\, a_2)(1 \,-\, b_2)], \tag{10}$$

and the rule for logical negation (NOT, complementation)

$$P(A^{\mathrm{c}}) = [1 - a_2,\, 1 - a_1]. \tag{11}$$

Other operations such as exclusive disjunction (XOR, set difference) can be computed with similar formulations.

These rules permit the evaluation of probabilistic logical expressions such as event or fault trees or their cutsets [156], failure risk calculations, reliability or unavailability models, and other Boolean expressions with probability inputs. As noted in [145], probabilistic logic can be applied to solve a range of problems, but in practice it has rarely been employed in engineering. One reason may be that early attempts to deploy probabilistic logic required real-valued probabilities and did not admit interval probabilities. They followed a scheme that conflated structural uncertainty with epistemic and aleatory uncertainties and improperly overused independence assumptions, which led to several deficiencies. A modern imprecise approach that distinguishes what we know and do not know about chance events is likely to be more transparently useful. Interval calculations implicitly analyze infinitely many traditional probabilistic models, each specified by sets of point probability values from the respective intervals. Using interval probabilities in engineering is necessary when the information available is not sufficient to formulate clear probabilistic models with substantial confidence. All tenable probabilistic models can thus be implicitly included in an analysis. The effects of imprecision in the probabilistic model specification are clearly reflected in the results so that worrisome prognoses can be detected immediately. Initial applications, although few, already indicate the usefulness of interval probabilities and demonstrate their capabilities. Examples include computation of structural reliability [54, 94], system responses [89], and failure risks [157, 158].

In these applications it is always possible to obtain rigorous bounds on the probabilities of interest, but when the expressions to be evaluated are complex because of cross linkages or subtle dependencies, calculation of the narrowest such bounds, i.e., the best-possible bounds, may require mathematical programming techniques [148]. However, there are many cases which may appear complex in which the calculation of exact bounds is easy because the expression has certain monotonicity properties [159].

The notion of interval probability can be extended to other kinds of engineering applications such as artificial intelligence, systems for general reasoning, syllogistic analysis, and related problems in knowledge engineering. In these contexts, when logical inferences are made about propositions characterized by interval probabilities rather than simple binary truth values [160], there can be subtleties in the definition, use, and interpretation of these characterizations [146]. For instance, the probability of A given B is entirely different from the probability of the implication $B \to A$ [161], even though they might seem to represent the same thing. The former is called the conditional probability while the latter might be called the probability of the conditional. Given $P(A) = a$ and $P(B) = b$, the conditional probability $P(A \,|\, B)$ is bounded by $[\max(0,\, (a-1)/b + 1),\, \min(a/b,\, 1)]$, unless $0 \in b$ in which case the bounds on the conditional probability are vacuous. The interval constrains the probability of event A occurring given that event B occurs, assuming nothing about the dependence between the two events. In contrast, the bounds on the probability of the conditional are $[\max(1 - a, b), \min(1, 1 - a + b)]$, which constrain the probability that event A implies the occurrence of event B, assuming nothing about the dependence between the two events. Numerically, conditional probability is quite different from the probability of the conditional. For example, if $a = 0.2$ and $b = 0.5$, $P(A \mid B) \in [0, 0.4]$, whereas $P(B \to A) \in [0.8, 1]$. These bounds are the best possible given no information except the marginal probabilities for each event separately. The bounds can be tightened by information about the dependence between A and B.

Conversely, following [161] one can make an inference about the probability of an event or proposition H from the probability of E and either the conditional probability $P(H \mid E)$ or the probability of the conditional $P(E \to H)$, either of which might represent available evidence or argument about the relationship between H and E, although, clearly, an analyst must distinguish which form this information takes. In many cases, the different inferences yield numerically similar results. For instance, if $P(E) = 0.8$ and $P(H \mid E) = 0.9$, the inference yields the probability interval $[0.72, 0.92]$. If instead we combine $P(E) = 0.8$ with $P(E \to H) = 0.9$, then inference yields $[0.7, 0.9]$. The differences become much greater for rare events and weak evidence. For example, if $P(E) = 0.2$, then $P(H \mid E) = 0.1$ implies $P(H)$ is somewhere in the wide interval $[0.02, 0.82]$, but the related inference using $P(E) = 0.2$ with $P(E \to H) = 0.1$ yields the much narrower interval $[0, 0.1]$ for the probability of the rare event H. Ancillary knowledge about the stochastic dependence between E and H can tighten such inferences.

4.4. Probability bounds analysis with p-boxes

Probability bounds analysis [162, 163, 164, 61] is another of the uncertainty quantification approaches that are considered part of the theory of imprecise probability [44]. If evidence theory described in Section 4.2 is based on the idea that x-values can be bounded rather than specified as points, and interval probability described in Section 4.3 is based on the idea that probabilities can be bounded rather than necessarily given as point values, then probability bounds analysis is based on the combination of these dual ideas. It is a numerical approach that allows the calculation of bounds on arithmetic combinations of probability distributions when perhaps only bounds on the input distributions are known. These bounds are called probability boxes, or p-boxes, and constrain cumulative probability distributions (rather than densities or mass functions). This bounding approach permits analysts to make calculations without requiring overly precise assumptions about parameter values, dependence among variables, or distribution shapes. In principle, the approach allows the analyst to decide what assumptions are reasonable and what are not. When the information about a distribution is very good, the bounds on the distribution will be very tight, approximating the precise distribution that is used in a Monte Carlo simulation. When the information is very poor, the bounds will tend to be much wider, representing weaker confidence about the specification of this distribution.

Probability bounds analysis is essentially a unification of standard interval analysis [165, 166, 32, 167] with traditional probability theory [21, 22, 168, 44]. It gives the same answer as interval analysis does when only range information is available. It also gives the same answers as Monte Carlo simulation does when information is abundant enough to precisely specify input distributions and their dependencies. Thus, it is faithful to both theories and generalizes them to solve problems neither could solve alone. Probability theory has facilities for modeling correlations and dependencies, but cannot easily distinguish between variability and ignorance [146, 169]. Interval analysis expresses ignorance, but it has no useful notions of central tendency or moments and it cannot easily handle dependence among variables. Probability bounds analysis incorporates facilities from probability theory for modeling correlations and dependencies and projecting distribution moments through mathematical expressions. From interval analysis, it inherits its fundamental conception of epistemic uncertainty, as well as important ancillary computational techniques described below.

The diverse methods comprising probability bounds analysis provide algorithms to evaluate mathematical expressions when there is uncertainty about the input values, their dependencies, or even the form of mathematical expression itself. The calculations yield results that are guaranteed to enclose all possible distributions of the output variable so long as the input p-boxes were all sure to enclose their respective distributions. In some cases, a calculated p-box will also be best-possible in the sense that the bounds could be no tighter without excluding some of the possible distributions. As a bounding approach, probability bounds analysis can effectively propagate some kinds of uncertainties that cannot be comprehensively addressed by any Monte Carlo or other sampling approach, even in theory with infinitely many samples. For instance, if an analyst does not know the distribution family for some input, a distribution-free p-box can be used to bound all possible distribution families consistent with the other information available about that variable. Likewise, if the nature of the stochastic dependence between two distributions is unknown, probability bounds analysis can be used to bound all possible distributions that might arise as a function of the inputs whatever their interdependence might be. Such calculations are not possible with a Monte Carlo assessment, or even a sensitivity study involving multiple Monte Carlo simulations, because such problems are intrinsically infinite-dimensional.

P-boxes are defined by left and right bounds on the cumulative probability distribution function of a quantity and, optionally, additional information about the quantity's mean, variance and distributional shape (family, unimodality, symmetry, etc.). A p-box represents a class of probability distributions consistent with these constraints. Let $\mathbb{D}$ denote the space of distribution functions on the real numbers $\mathbb{R}$, i.e., $\mathbb{D} = \{D \,|\, D : \mathbb{R} \to [0,1],\ D(x) \leq D(y)$ whenever $x < y$, for all $x, y \in \mathbb{R}\}$, and let $\mathbb{I}$ denote the space of real intervals, i.e., $\mathbb{I} = \{[i_1, i_2] \,|\, i_1 \leq i_2,\ i_1, i_2 \in \mathbb{R}\}$. Then a p-box is a quintuple $\langle \overline{F}, \underline{F}, m, v, \mathcal{F} \rangle$, where $\overline{F}, \underline{F} \in \mathbb{D}$, while $m, v \in \mathbb{I}$, and $\mathcal{F} \subseteq \mathbb{D}$. This quintuple denotes the set of distribution functions $F \in \mathbb{D}$ matching the following constraints:

$$\underline{F}(x) \leq F(x) \leq \overline{F}(x), \tag{12}$$

$$\int_{-\infty}^{\infty} x \,\mathrm{d}F(x) \in m, \tag{13}$$

$$\left(\int_{-\infty}^{\infty} x^2 \mathrm{d}F(x)\right) - \left(\int_{-\infty}^{\infty} x \,\mathrm{d}F(x)\right)^2 \in v, \tag{14}$$

and

$$F \in \mathcal{F}. \tag{15}$$

The constraints mean that the distribution function F falls within prescribed bounds, the mean of the distribution (given by the RIEMANN-STIELTJES integral) is in the interval m, the variance of the distribution is in the interval v, and the distribution is within some admissible class of distributions $\mathcal{F}$. A p-box is minimally specified by its left and right bounds, in which case the other constraints are understood to be vacuous as $\langle \overline{F},\, \underline{F},\, [-\infty, \infty],\, [0, \infty],\, \mathbb{D}\rangle$.

An arbitrary collection of distribution functions, i.e., a subset of $\mathbb{D}$, is called a credal set. In principle, specifying and computing with credal sets would suffice as the most general theory of imprecise probabilities [44]. A p-box is just a crude but computationally convenient characterization of a credal set. Whereas a credal set might be defined solely in terms of the constraint $\mathcal{F}$ (which would imply $\overline{F}$, $\underline{F}$, m, and v), such a specification is often very difficult to compute with [111]. A p-box usually has a loosely constraining specification of $\mathcal{F}$, or even no constraint so that $\mathcal{F} = \mathbb{D}$. Calculations with p-boxes, unlike with credal sets, are often quite efficient, and workable algorithms for all standard mathematical functions are known [164].

When F is a distribution function and B is a p-box, the notation $F \in B$ means that F is an element of $B = \langle B_1, B_2, [m_1, m_2], [v_1, v_2], \mathcal{B}\rangle$, that is, $B_2(x) \leq F(x) \leq B_1(x)$, for all $x \in \mathbb{R}$, $\mathrm{E}(F) \in [m_1, m_2]$, $\mathrm{Var}(F) \in [v_1, v_2]$, and $F \in \mathcal{B}$. As the notation $X \sim F$ denotes the fact that $X \in \mathbb{R}$ is a random variable governed by the distribution function F, we can likewise write $X \sim B$ to mean that X is a random variable whose distribution function is unknown except that it is in B. And $X \sim F \in B$ can be contracted to $X \sim B$ without mentioning the distribution function explicitly. When there is no additional information about the moments or distribution family beyond what is implied by the two bounding distributions then the quintuple representing the p-box $\langle B_1,\, B_2, [-\infty, \infty], [0, \infty], \mathbb{D}\rangle$ can be denoted more compactly as $[B_1,\, B_2]$. This notation harkens to that of intervals on the real line, except that the endpoints are distributions rather than points. Indeed, p-boxes serve the same role for random variables that interval probabilities serve for events.

P-boxes can be combined together in mathematical calculations yielding results that rigorously contain the uncertainty of the output that is implied by the uncertainties in the input p-boxes. If X and Y are independent random variables with distributions F and G respectively, then $X + Y = Z \sim H$ given by the convolution of F and G,

$$H(z) = \int\limits_{z=x+y} F(x)G(y)\mathrm{d}z = \int\limits_{-\infty}^{\infty} F(x)G(z-x)\mathrm{d}x = F \star G. \tag{16}$$

A similar operation on p-boxes is straightforward for sums. If $X \sim A = [A_1,\, A_2]$ and $Y \sim B = [B_1,\, B_2]$ are stochastically independent, then the distribution of $Z = X + Y$ is in the p-box $[A_1 \star B_1,\ A_2 \star B_2]$. It is often most convenient to effect these two convolutions with a discretization that converts the operation into a series of elementary interval calculations [170, 163, 61]. Finding bounds on the distribution of sums $Z = X + Y$ *without* making any assumption about the dependence between X and Y is actually easier than the problem assuming independence. Makarov [171, 162, 163] showed that

$$Z \sim \left[\sup_{x+y=z} \max(F(x) + G(y) - 1, 0),\ \inf_{x+y=z} \min(F(x) + G(y), 1\right]. \tag{17}$$

These bounds are both rigorous and best-possible in the sense that they could be no narrower without excluding some possible sum distributions, and they are also easy to extend beyond precise input distributions to inputs that are p-boxes [163]. Operations under other assumptions about the dependency between X and Y can also be computed, including cases corresponding to the extreme assumptions of perfect positive or perfect negative dependency, as well as cases where only the sign of the dependence is known. Other operations such as subtraction, multiplication, division, etc., can be derived using transformations. For instance, p-box subtraction $A - B$ can be defined as $A + (-B)$, where the negative of a p-box $B = [B_1, B_2]$ is $[B_2(-x), B_1(-x)]$.

A numerical example illustrates a simple calculation with four imperfectly understood quantitative variables. A is a normally distributed variable whose mean is between 10 and 12 and whose standard deviation is between 1 and 2. B is a positive variable not larger than 9, whose mean is 1. C is known only to be within an interval range between 8 and 15, and D is lognormally distributed with mean 5 and standard deviation 1. P-boxes sure to enclose the probability distributions for A, B, and D depicted in the left three graphs of Fig. 1. We omit the graph for C which would be a simple rectangle between 8 and 15. If these variables are mutually independent, the p-box for the function $AD + B + C$ is depicted as the rightmost graph in the figure. The mean of the output quantity is between 58.9 and 76, and its standard deviation is between 11.2 and 16.4. Its median is between 54 and 79, and p-box's right tail reveals the probability the output quantity is larger than 100 is uncertain, but somewhere between 0 and 12%.

Probability bounds analysis has been used in uncertainty computations in many contexts including series system failure analysis and system reliability [140], quantification of margins of uncertainty [103], finite-element structural

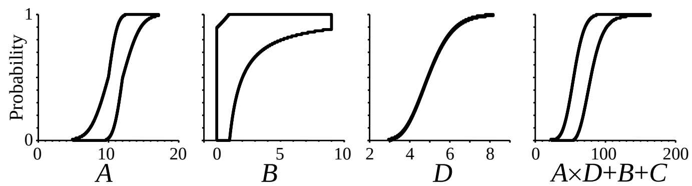

Figure 1: Function of p-boxes

models [172, 112, 173, 104], differential equations of chemical reactions [174], engineering design [123, 175], validation [176, 113], pharmacokinetics [177], human health and ecological risk assessments at Superfund sites [178, 179, 180], and even global circulation models [181]. The Wikipedia page for probability bounds analysis lists over two dozen applications of the method to various engineering problems.

A significant impediment to using probability bounds analysis in common engineering applications may arise in some situations. Although it is always simple to compute bounds that are guaranteed to rigorously enclose the output, the calculation of bounds that are additionally the *best-possible* such bounds can be complicated when there are stochastic or functional dependencies among the inputs. Best-possible bounds on distributions for elementary functions are easily evaluated, but it is not always possible to conveniently compute such bounds for more complex functions. For instance, when a variable or parameter characterized by a p-box appears multiple times in an expression to be evaluated, naive application of the convolution methods can lead to an artifactual inflation or contraction of the uncertainty in the result. For instance, if X is some probability distribution and the evaluation of the expression $X^2 + X$ effectively assumes that the two terms are stochastically independent of one another, the result will likely understate the dispersion of the resulting distribution of sums. The uncertainty in X has in essence been entered into the calculation twice. In other cases, when independence is not counterfactually assumed, the effect can lead to uncertainty in the result that is larger than it should be.

Interestingly, difficulty accompanying repeated uncertain variables appears to a fundamental problem in many other and perhaps all uncertainty calculi. For instance, it is well known in interval analysis [165] where it is called the dependency problem, but it also afflicts fuzzy arithmetic [182], probabilistic logic [183, 159], discrete convolution of probability distributions [184], and even step-wise implementations of Monte Carlo simulations. This problem is part of a more general problem of modeling stochastic and functional dependency in any probabilistic arithmetic for which probability bounds analysis was originally created to address [185, 163].

All of these computational problems can be sidestepped by algebraically rearranging the evaluation expressions to equivalent forms that do not have repetitions of uncertain inputs. For example, X^2+X can be rewritten as $(X+1/2)^2-1/4$ which has no repetitions. When such rearrangement is not possible, a variety of algorithms for special cases can still allow computation of best-possible results. For instance, straightforward methods can handle monotone functions or unate logical expressions [159], or functions that can be decomposed into montone functions. For problems that lack monotonicity but have low dimensionality in the number of uncertain inputs, brute-force computational techniques such as subinterval reconstitution can reduce or eliminate the artifactual inflation of uncertainty, e.g. [61].

There are several other techniques that originated in the field of interval analysis that are likewise fruitfully extended to probability bounds analysis, three of which we mention here. The first technique is TAYLOR arithmetic [186, 187, 188] which is a rigorous method for symbolic evaluation of epistemic uncertainty characterized by intervals. The technique can also be applied to p-box calculations where it can be used to reduce the artifactual inflation of uncertainty from repeated variables. A TAYLOR model for a function f over an input interval X is the TAYLOR polynomial p_n of some order n approximating f and an interval remainder term R_n, which rigorously encloses the approximation error $|f(x) - p_n(x)|$ over $x \in X$. The celebrated TAYLOR theorem is, after all, an equality, not an approximation. The function f can be replaced in calculations by $p_n + R_n$. The interval remainder term is evaluated by interval arithmetic, but the polynomial part of the TAYLOR model is propagated through expressions symbolically wherever possible. The like terms that arise in these symbolic calculations are grouped so that repeated uncertain variables are effectively canceled before they become problems. The order of the TAYLOR polynomial is taken to be high to achieve good fidelity and small uncertainty in the remainder term, into which all truncation and round-off errors in intermediate operations can be folded to obtain a strictly rigorous result. TAYLOR arithmetic has been successfully used to solve nonlinear ordinary differential equations that have epistemic uncertainty expressed as intervals in either parameters or initial conditions [189]. TAYLOR arithmetic can also project uncertainty expressed as p-boxes whether they represent epistemic uncertainty or aleatory uncertainty or both. The approach has been used to project p-boxes characterizing uncertainties in both coefficients and initial conditions through a variety of nonlinear differential equations [174].

Another technique from interval analysis that enriches probability bounds analysis is BERNSTEIN expansion [157, 158, 190, 191, 192] which is a strategy for bounding the range of an arbitrarily complex finite polynomial function. This technique can be used, for instance, to complete the TAYLOR arithmetic computation which involves a polynomial of the initial variables. An ancillary technique such as BERNSTEIN expansion must be employed to project uncertainty through this polynomial, which generally has many repeated uncertain variables appearing as various powers of a variable across the terms of the polynomial. BERNSTEIN expansion is an outside-in strategy in the sense that it yields conservative bounds that get tighter (but always remain true bounds) with additional computational investment. Given an arbitrary univariate polynomial with coefficients a_k, $k = 1, \ldots, K$, the coefficients for its BERNSTEIN expansion are

$$b_i = \sum_{k=1}^{i} a_k \frac{\binom{i-1}{k-1}}{\binom{K-1}{k-1}} \tag{18}$$

where $i = 1, \ldots, K$. The largest of these coefficients is a guaranteed upper bound on the range of the polynomial over the unit interval, and the smallest is a guaranteed lower bound on it. They are not necessarily best possible, but they improve quadratically by subdividing the problem, so they approach the exact bounds very quickly. These expansions are analytic rather than approximate like the CAUCHY-deviate method [193] or Monte Carlo simulation, so there is no numerical error associated with the calculation. Although limited to polynomial functions, BERNSTEIN expansions are quite general for propagating epistemic uncertainty. They can be used with TAYLOR models as well as any polynomial function. They work for polynomials whose terms involve a single uncertain variable, and generalize straightforwardly for the multivariate case. They are easy to compute for up to perhaps a dozen dimensions, and several computational shortcuts are available for use in challenging problems. CRESPO and KENNY [157] describe how they can be applied to aleatory uncertainty, and to cases where some variables are aleatory and some are epistemic. BERNSTEIN expansion can be extended to probability bounds analysis in a straightforward way by applying the method to each of many interval calculation problems into which a p-box calculation can be decomposed.

The third technique inherited from interval analysis is the CAUCHY-deviate method [193] which projects intervals through a function using an approximate rescaling technique based on Monte Carlo sampling from CAUCHY distributions around (not necessarily within) the input intervals. Monte Carlo simulation is famously efficient for propagating purely probabilistic uncertainty, but Monte Carlo methods applied to the interval propagation problem yield gross underestimates of the true output uncertainty. The rescaling used in the CAUCHY-deviate method is essentially a mathematical trick that recognizes how severely Monte Carlo sampling tends to underestimate epistemic uncertainty and corrects for this underestimation by computing the output range as a function of the statistical breadth of the propagated samples. The method does not need to know the specification of the function to which it is applied, but it must be able to query its value for scalar inputs. The approximation is good when the function is nearly linear or the breadths of the input uncertainties are small relative to the nonlinearity. The approximation accuracy depends on the number of samples employed, but not on the number of uncertain inputs. The CAUCHY-deviate method escapes the curse of dimensionality in the same way as and for the same reason that Monte Carlo simulation does. This insensitivity to the dimensionality of the problem means that the method works just as well for a thousand input variables as it does for ten. About 200 sample function evaluations are needed to obtain 20% relative accuracy of half-width of output range. Fewer samples yield lower accuracy, but scaling the results by the square root of the number of samples (which is reasonable under a linearity assumption) can compensate for small samples. The method is most efficient when the dimensionality of the problem is high. Its results are asymptotically correct, but they are not rigorous, so the method is not one of the techniques in interval analysis that can be used for automatically verifying the computational results.

Although the CAUCHY-deviate method works only for interval uncertainty, because probability bounds analysis can be decomposed into a series of interval computation problems, it can be extended to project p-boxes through nearly linear functions. The degree of nonlinearity for which the method yields good results can in principle be larger than would be tolerated for simple interval inputs because the decomposition of p-boxes into constituent intervals gives narrower ranges. The CAUCHY-deviate method endows probability bounds analysis with some important new capacities. Like Monte Carlo simulation, it sidesteps the problem of repeated uncertain variables to approximate best-possible bounds. The CAUCHY-deviate method also permits uncertainty analysis of black-box functions. Ordinarily, probability bounds analysis is said to be an intrusive method because it requires knowledge of the individual mathematical operations involved in a computation which it then decomposes into sequential unary and binary calculations. Intrusive methods cannot do uncertainty quantification for black-box functions whose internal details are not known. Probability bounds analysis is no longer intrusive once it is enriched with the CAUCHY-deviate method which only requires that the function can be evaluated and does not need to know what is inside the function.

The defining general feature of probability bounds analysis is that its results rigorously enclose the results of a probabilistic analysis. Whether or not the results can be shown to be best possible, in any case the output p-boxes are usually merely bounds on possible distributions. They are enclosures of credal sets, and not always perfect representations of credal sets themselves. This is, after all, what allows them to be so much more computationally convenient than working with credal sets. But it means that p-boxes often also enclose distributions that are not themselves possible. For instance, the p-box specified by knowing the minimum, maximum and mean values of a variable includes distributions that do not obey these constraints, even though the p-box can be shown to be best possible. Likewise, the set of probability distributions that could result from adding random values without specifying any dependence assumption between their distributions is generally a proper subset of all the distributions enclosed by the p-box computed for that sum. That is, there are distributions within the output p-box that could not arise under any dependence between the two input distributions. The output p-box will, however, always contain all distributions that are possible so long as the input p-boxes were sure to enclose their respective underlying distributions, and it is also the tightest such enclosure that does so. This property often suffices for use in risk analysis and other fields requiring calculations under uncertainty [176, 172, 194, 169].

4.5. Fuzzy probabilities

Fuzzy probability theory can be regarded as a marriage between fuzzy set theory and probability theory. It enables the consideration of a fuzzy set of probabilistic models, which are variously plausible according to the available information. Aleatory uncertainty and subjective probabilistic information are captured in probabilistic models, and imprecision in the probabilistic model specification is described with fuzzy sets. This preserves uncertainties as probabilistic information and imprecision as set-theoretical information throughout the entire analysis and does not let them migrate into one another. In the case that only fuzzy information is available, the special case of a pure fuzzy analysis appears. On the other hand, if all information can be captured with precisely specified probabilistic models, the result is equal to the traditional probabilistic result.

With the interpretation of fuzzy modeling as an extension to interval modeling, as mentioned in Section 2, the very close relationship between fuzzy probabilities and probability boxes becomes obvious. In this context, the min-max operator and the extension principle [33, 43] are used as the basis for the processing of fuzzy information. A fuzzy number $\tilde{x}$ on $\mathbf{X} = \mathbb{R}^n$ is defined as the set

$$\tilde{x} = \{(x, \mu(x)) \mid x \in \mathbf{X}\}, \ \mu(x) \geq 0 \,\forall\, x \in \mathbf{X}, \tag{19}$$

where $\mu(x)$ is the *membership function* (also known as the *characteristic function*) of the fuzzy number $\tilde{x}$, which represents the degree with which the elements x belong to $\tilde{x}$; it is assumed to be normalized (in the sense that $\sup \mu(x) = 1$), and has only one element x for which $\mu(x) = 1$; see Fig. 2. The crisp sets

$$x_\alpha = \{x \in \mathbf{X} \mid \mu(x) \geq \alpha\} \tag{20}$$

extracted from $\tilde{x}$ for real numbers $\alpha \in (0, 1]$ are called *α-level sets*. These sets form a sequence of nested sets with the property

$$x_{\alpha_k} \subseteq x_{\alpha_i} \forall \alpha_i, \alpha_k \in (0, 1] \text{ with } \alpha_i \leq \alpha_k \,. \tag{21}$$

On this basis a fuzzy number can be described as a family of α-level sets, via α-discretization [33] as illustrated in Fig. 2, and a fuzzy set of probabilistic models can be regarded as a set of probability boxes. This consideration provides a first access to fuzzy probabilities starting from probability boxes to consider various box sizes in a nested fashion in one analysis. Developments to approach fuzzy probabilities from this direction originated from [195] with the notion of hybrid arithmetic; see [196]. Similarly, a further connection to the conception of info-gap models for probability distributions exists as described in [197]. A fuzzy probabilistic model can so be formulated in the same manner as a probability box, but provides the additional feature of a nuanced description of the imprecision in the probabilistic model. This is discussed in a geotechnical context in [198], starting from an interval perspective. Respective discussions on quantification are provided in [199, 200] and also in [201]. Interval-valued information in the specification of parameters, distribution types, dependencies, or functional values of a distribution can be implemented including a gradual subjective assessment of the interval sizes. For example, the results from interval estimations on various confidence levels and conflicting statistical test results for various thresholds of rejection probabilities can be used as the basis for a modeling with stepwise changing interval sizes. This perspective relates fuzzy probabilities closely to interval probabilities, where the imprecision emerges in the probability measure. But it is also connected to evidence theory in the same way as probability boxes. When the focal sets in evidence theory are set-valued (interval-valued in the one-dimensional case) images of random elementary events so that the basic probability assignment is determined—and not a subjective matter left with the analyst, then p-boxes can be constructed by belief and plausibility distributions. When the focal sets appear as fuzzy-valued images of random elementary events, then p-boxes can be obtained in the same way for each α-level leading to a fuzzy probability distribution in overall. Once a fuzzy probabilistic model is established, the same analysis methods as in p-box approach can be used for processing, applied to each α-level. That is, for any selected α-level, the complete framework of probability bounds analysis is applicable.

In this context, it becomes obvious that the membership function serves only instrumentally to summarize various plausible interval models in one embracing scheme. The interpretation of the membership value μ as epistemic possibility, which is sometimes proposed may be useful for ranking purposes, but not for making critical decisions. The importance of fuzzy modeling lies in the simultaneous consideration of various magnitudes of imprecision at once in the same analysis. As discussed in [202] the nuanced features of fuzzy probabilities provide extended insight in engineering problems and a workable basis to solve various problems in an elegant and efficient manner. This is illustrated in Fig. 2 by means of a repeated p-box analysis to calculate a fuzzy failure probability $\tilde{P}_f$. In this figure the fuzzy number $\tilde{x}$ represents a parameter of a probabilistic model for an engineering analysis, for example a variance for the distribution of the stiffness of the foundation soil. In the analysis the failure probability P_f of the engineering structure or system is calculated, and the imprecision of $\tilde{x}$ is mapped to this result. Using α-discretization this analysis can be performed with nested p-boxes. Each α-level set x_α of $\tilde{x}$ represents an interval parameter of a probability distribution and so defines a p-box. The engineering analysis with this p-box for the selected α yields an interval for the failure probability associated with the same α-level. Repeating the p-box analysis with several different α then leads to a nested set of α-level sets for the result P_f, which form the fuzzy result $\tilde{P}_f$.

The features of such fuzzy probabilistic analysis can be utilized to identify sensitivities of the failure probability with respect to the imprecision in the probabilistic model specification. Sensitivities of P_f are indicated when the interval size of P_f grows strongly with a moderate increase of the interval size of the input parameters. If this is the case, the membership function of $\tilde{P}_f$ shows outreaching or long and flat tails. An engineering consequence would be to pay particular attention to those model options in the input, which cause large intervals of P_f and to further investigate to verify the reasoning for these options and to possibly exclude these critical cases. A fuzzy probabilistic analysis also provides interesting features for design purposes. The analysis can be

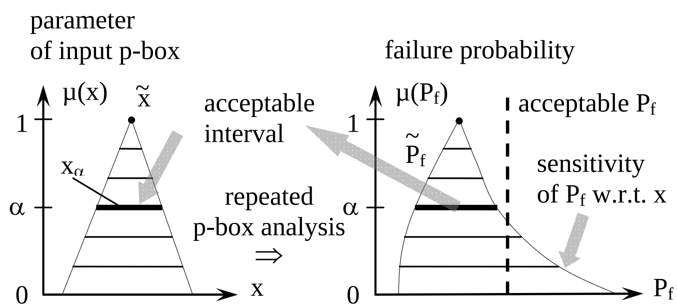

Figure 2: Repeated p-box analysis to calculate a fuzzy failure probability

performed with coarse specifications for design parameters and for probabilistic model parameters. From the results of this analysis, acceptable intervals for both design parameters and probabilistic model parameters can be determined directly without a repetition of the analysis. Indications are provided in a quantitative manner to collect additional specific information or to apply certain design measures to reduce the input imprecision to an acceptable magnitude. This implies a limitation of imprecision to only those acceptable magnitudes and so also caters for an optimum economic effort. For example, a minimum sample size or a minimum measurement quality associated with the acceptable magnitude of imprecision can be directly identified. Further, revealed sensitivities may be taken as a trigger to change the design of the system under consideration to make it more robust. A related method is described in [121] for designing robust structures in a pure fuzzy environment. These methods can also be used for the analysis of aged and damaged structures to generate a rough first picture of the structural integrity and to indicate further detailed investigations to an economically reasonable extent — expressed in form of an acceptable magnitude of input imprecision according to some α-level. A study in this direction is presented in [203] with focus on robustness assessment of offshore structures under imprecise marine corrosion. An engineering discussion of features, pros and cons of interval models and fuzzy probabilities versus rough probabilistic models in geotechnical applications, where information is usually quite vague and limited, is provided in [198].

Whilst the access to fuzzy probabilities via p-box approach is intuitive and, thus, immediately attractive from a practical engineering point of view, the second access via fuzzy random variables is rather mathematical and provides ground for extensive theoretical considerations. Fuzzy random variables follow the idea that the observation of a random variable is imprecise. That is, the image of the random variable appears as fuzzy number rather than the random variable itself. In so far, traditional probability theory applies completely for the description of the underlying random variable in the probability space $[\Omega, \mathcal{F}, P]$, with the sample space Ω, the set of events (σ-algebra) $\mathcal{F}$, and the probability measure P. The key question is how to describe the fuzzy image of the random variable and its properties including the distribution function in $\mathbb{R}^n$. This includes the problem of measurability in $\mathbb{R}^n$. So far, there is no known σ-algebra constructed on $\mathbb{R}^n$ that can capture fuzzy realizations and hence fuzzy events. But this is natural because such a σ-algebra would permit assigning crisp measure values P to crisp events on $\mathbb{R}^n$ based on imprecise observations. This is a contradiction because imprecise observations cannot be assigned to crisp events in a binary manner without additional restrictions. One potential idea to escape from this situation could be to construct a σ-algebra with reference to a universe of fuzzy sets $\mathfrak{F}$ instead with reference to $\mathbb{R}^n$. But this would be abstract and not practicable for engineering purposes as we need to work on $\mathbb{R}^n$ and not on $\mathfrak{F}$, where the definition of our events of interest such as structural failure and the analysis would be problematic. A workable solution to the problem, in particular in view of engineering applications, can be found when the fuzzy image of the random variable is understood as a fuzzy set of real-valued images. In this manner, the analysis comes down to the consideration of a set of real-valued random variables, which are all plausible given the observation made, together with their membership values. And for each of these real-valued random variables the entire framework of traditional probability theory and mathematical statistics can be utilized. The membership values are processes in parallel using fuzzy set theory. The fuzziness of the observation is so carried forward to the measure values for events, which are then obtained as fuzzy sets of probability measures, or fuzzy probabilities for short. This complies with the intuitive understanding that the occurrence of a precisely defined event based on imprecise observations can only be determined in an imprecise manner and hence that the probability of occurrence can only be measured in an imprecise manner, as well. The construction of a fuzzy probabilistic framework in this manner (see [68]) leads exactly to the same model as that intuitively motivated by p-boxes; one deals with a fuzzy set of plausible probabilistic models. And it supports the idea that imprecision in data implies imprecision in the probability measure, which is also used in the p-box approach. In addition, the inclusion of the (purely random) generation scheme of realizations in the model provides a consistent access to the analysis of imprecise data. Some considerations and examples for the quantification of engineering data can be found in [201]. The consideration of fuzzy random variables as a fuzzy set of real-valued random variables also suggests an analysis, random variable by random variable, in engineering applications. The engineering analysis, such as a finite element analysis, can so be implemented as a black box analysis; see [82].

Fuzzy random variables were first introduced in [42, 204] and further developed with significant steps in[205], [206], [207], and [70]. These developments show differences in terminology, concepts, and in the associated consideration of measurability; and the investigations are ongoing [208, 209, 210, 211, 212, 213, 214, 215, 216]. In [70] it was shown that the different concepts can be unified to a certain extent. An overview with specific comments on the different developments is provided in [70] and [217]. Generally, it is noted that α-discretization is utilized as a helpful instrument. Investigations were pursued on independent and dependent fuzzy random variables, for which parameters were defined with particular focus on variance and covariance [218, 219, 220, 221, 222, 223]. Fuzzy random processes were examined to reveal properties of limit theorems and martingales associated with fuzzy randomness [224, 225]; see also [226, 227] and for a survey [228]. Particular interest was devoted to the strong law of large numbers [229, 230, 231]. Further, the differentiation and the integration of fuzzy random variables was investigated in [124, 232]. Considerable effort was made in the statistical evaluation of imprecise data. Fundamental achievements were reported in [206], [233, 234], and [235]. Classical statistical methods were extended in order to take account of statistical fluctuations/variability and imprecision simultaneously, and the specific features associated with the imprecision of the data were investigated. Research in this direction is reported, for example, in [236, 237] in view of evaluating measurements, in [124, 125] for decision making, and in [238, 239, 240] for regression analysis. Methods for evaluating imprecise data with the aid of generalized histograms are discussed in [241, 242]. Also, the application of resampling methods is pursued; bootstrap concepts are utilized for statistical estimations [243] and hypothesis testing [48] based on imprecise data. Another method for hypothesis testing is proposed in [244], which employs fuzzy parameters in order to describe a fuzzy transition between rejection and acceptance. BAYESian methods have also been extended by the inclusion of fuzzy variables to take account of imprecise data; see [235, 49] for a comprehensive overview. A contribution to BAYESian statistics with imprecise prior distributions is presented in [245]. This leads to imprecise posterior distributions, imprecise predictive distributions, and may be used to deduce imprecise confidence intervals. The effects of imprecise prior distributions and imprecise data in an engineering context are investigated in [246]. A combination of the BAYESian theorem with kriging based on imprecise data is described in [247]. A BAYESian test of fuzzy hypotheses is discussed in [248], while in [126] the application of a fuzzy BAYESian method for decision making is presented. The variety of theoretical developments provides reasonable margins for the formulation of fuzzy probability theory, whereby conceptual choices have to be made depending on the underlying problem and the envisaged application; see [68]. In view of engineering applications the following choices seem most reasonable, lead to a complete agreement with p-box approach and correspond to the initial explanations above. The question of measurability is solved employing the concept of measurable bounding functions [42, 204]. The integration of a fuzzy-valued function is realized according to [249] so that any fuzziness, whether in the integral bounds or in the integrand, is translated into fuzziness of the result and not averaged. The distance between fuzzy numbers, which is needed in statistics with fuzzy realizations, is calculated according to the extension principle, [250, 66, 33], leading to fuzzy distances between fuzzy numbers. Moments, and other parameters, of a fuzzy random variable are so obtained as fuzzy numbers. These selections comply with the definitions in [206] and [42]; see also [217]. They have been used in the engineering developments and applications in [82].

A fuzzy random variable is defined as the mapping

$$\tilde{\mathrm{X}} : \Omega \rightarrow \mathfrak{F}(\mathbf{X}) \qquad (22)$$

where Ω be the sample space with the random elementary events $\omega \in \Omega$, and $\mathfrak{F}(\mathbf{X})$ is the collection of all fuzzy numbers $\tilde{x}$ on $\mathbf{X} = \mathbb{R}^n$. The fuzzy numbers $\tilde{x}$ are described with membership functions $\mu(x)$ using a membership scale μ perpendicular to the hyperplane $\Omega \times \mathbf{X}$. Each random elementary event ω from Ω is so connected to a fuzzy realization $\tilde{x}$ without interaction between Ω and μ. That is, randomness induced by Ω and fuzziness described by μ—only for the images in x-direction—are not mixed with one another; see Fig. 3.

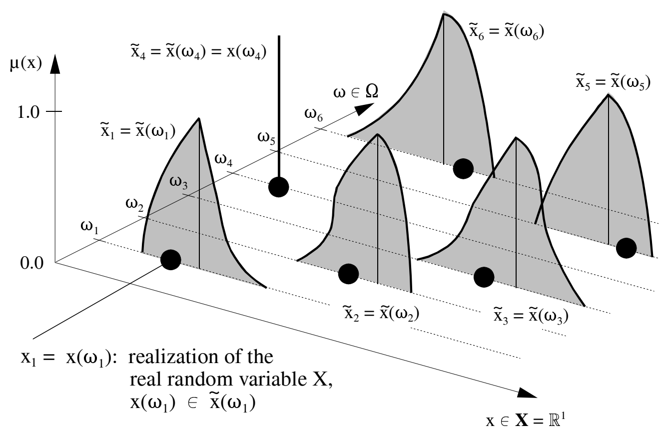

Figure 3: Fuzzy random variable

Generally, a fuzzy random variable can be discrete or continuous with respect to both fuzziness and randomness. The further consideration refers to the continuous case, from which the discrete case may be derived.

Let $\tilde{\mathrm{X}}$ be a fuzzy random variable with realizations $\tilde{x}_i$. Each fuzzy realization $\tilde{x}_i$ represents a fuzzy set of real-valued realizations, which are all plausible. Let x_{ji} be a plausible realization out of $\tilde{x}_i$, $x_{ji} \in \tilde{x}_i$. Then, each series of x_{ji}, $i = 1, 2, ...$, with $x_{ji} \in \tilde{x}_i$ for each i can be considered as a series of real-valued realizations x_{ji} of a real-valued random variable X_j; and the real-valued random variable X_j can be considered as contained in $\tilde{\mathrm{X}}$, $\mathrm{X}_j \in \tilde{\mathrm{X}}$. X_j is also called original of $\tilde{\mathrm{X}}$ in the sense that X_j is one plausible real-valued random variable behind the blurred image $\tilde{\mathrm{X}}$. The X_j carry membership values, which may be utilized to express their degree of plausibility or only for instrumental purpose as described at the beginning of this section. Consequently, the fuzzy random variable $\tilde{\mathrm{X}}$ can be described as the fuzzy set of all originals X_j contained in $\tilde{\mathrm{X}}$,

$$\tilde{\mathrm{X}} = \{(\mathrm{X}_j, \mu(\mathrm{X}_j)) \,|\, x_{ji} \in \tilde{x}_i \forall i\} \ . \tag{23}$$

The special case of a real-valued random variable X is so obtained when a fuzzy random variable $\tilde{\mathrm{X}}$ includes only one original $\mathrm{X} = \mathrm{X}_j$. And a pure fuzzy variable is obtained in the case of no randomness. Interestingly, if a repeated observation yields exactly the same fuzzy realization multiple times, this does not necessarily imply that no randomness exists in this problem. The model then covers the case that the imprecision of the observation is too strong to detect the random properties; it considers all possible variants of random variation that could be in the problem but are not explicitly detected. It is noted that this feature would not occur for choices of distance measures between fuzzy sets other than made herein. Coverage of these special cases enables a simultaneous treatment of real-valued random variables, fuzzy random variables and fuzzy variables within the same environment and sharing the numerical algorithms. Vice versa, it enables the utilization of theoretical results from traditional probability theory and fuzzy set theory and of established numerical techniques from stochastic mechanics and from interval and fuzzy analysis.

It is assumed that the α-level sets of the realizations $\tilde{x}$ are connected, compact and convex — which is generally the case in engineering applications and only very special exceptions might exist. This enables an efficient and convenient numerical treatment. Again, the min-max operator and the extension principle [250, 66, 33] are used to process fuzzy information. Then, α-discretization can be applied to the fuzzy random variable $\tilde{\mathrm{X}}$ to obtain random α-level sets

$$\mathrm{X}_\alpha = \{\mathrm{X} = \mathrm{X}_j \mid \mu(\mathrm{X}_j) \geq \alpha\} \ , \tag{24}$$

which represent closed random intervals $[\mathrm{X}_{\alpha\,\mathrm{l}}, \mathrm{X}_{\alpha\,\mathrm{r}}]$ in the one-dimensional case. This representation of fuzzy random variables with α-discretization establishes full compliance with p-box approach for each α-level. Consideration of the realizations of the random α-level sets X_α as focal sets in the framework of evidence theory leads to distributions for plausibility and belief, which are bounding probability distributions to all real-valued random variables $\mathrm{X} \in \mathrm{X}_\alpha$. These bounding distributions represent p-boxes. And, in the same manner as in p-box approach, these bounding functions can also be formulated directly at the model level with the imprecision arising from vagueness in the probabilistic model specification, in addition to imprecision carried forward from observations.

In view of practical applications, probability distribution functions are defined for fuzzy random variables [82, 235, 242, 251, 68]. These represent a fuzzy set of the probability distribution functions $F_j\,(\underline{x})$ of all originals X_j of $\tilde{\mathrm{X}}$ with their membership values $\mu\,(F_j\,(x))$,

$$\tilde{F}\,(x) = \left\{ \left(F_j\,(x)\,, \mu\,(F_j\,(x))\right) \mid \mathrm{X}_j \in \tilde{\mathrm{X}},\, \mu\,(F_j\,(x)) = \mu\,(\mathrm{X}_j)\ \forall j \right\} \ , \tag{25}$$

and p-boxes $[F_{\alpha\,l}\,(x)\,, F_{\alpha\,r}\,(x)]$ for each α-level with the distribution bounds

$$F_{\alpha\,l}\,(x = (x_1, ..., x_n)) = 1 - \max_{\mathrm{X}_j \in \mathrm{X}_\alpha} P\,(\mathrm{X}_j = t = (t_1,\, \ldots,\, t_n) \mid\ x, t \in \mathbf{X} = \mathbb{R}^n,$$
$$\exists t_k \geq x_k,\, 1 \leq k \leq n)\,, \tag{26}$$

$$F_{\alpha\,r}\,(x = (x_1, ..., x_n)) = \max_{\mathrm{X}_j \in \mathrm{X}_\alpha} P\,(\mathrm{X}_j = t = (t_1, \ldots, t_n) \mid\ x, t \in \mathbf{X} = \mathbb{R}^n,$$
$$t_k < x_k,\, k = 1,\, ...,\, n)\ . \tag{27}$$

so that

$$\tilde{F}(x) = \{(F_\alpha(x), \mu(F_\alpha(x))) \quad | \quad F_\alpha(x) = [F_{\alpha\,l}(x), F_{\alpha\,r}(x)], \\ \mu(F_\alpha(x)) = \alpha\ \forall\alpha \in (0,1]\}. \qquad (28)$$

Each original X_j determines precisely one trajectory $F_j(x)$ within the bunch of distributions covered by $\tilde{F}(x)$. And, for each x, the $[F_{\alpha\,l}(x), F_{\alpha\,r}(x)]$ appear as closed connected intervals based on the choices and restrictions made above.

These distribution functions can easily be formulated in a parametric manner by substituting fuzzy numbers for the parameters in probabilistic model descriptions; see [206, 82, 68]. The fuzzy model parameters can be estimated using mathematical statistics, and also non-parametric formulations can be established; see [206, 235, 201, 49]. Fuzzy distribution functions can easily be used for further calculations, but they do not uniquely describe a fuzzy random variable; see [252]. This theoretical lack (which also applies to evidence theory and p-boxes) is, however, generally without an effect in practical applications so that stochastic simulations may be performed according to the distribution functions [104, 105]. Alternative simulation methods were proposed based on parametric [253] and non-parametric [252, 254] descriptions of fuzziness. The approach according to [254] enables a direct generation of fuzzy realizations based on a concept for an incremental representation of fuzzy random variables. This method is designed to simulate and predict fuzzy time series; it circumvents the problems of artificial uncertainty growth or bias of non-probabilistic uncertainty, which is frequently concerned with numerical simulations. In overall, an engineering analysis with fuzzy probabilities can be realized by combining stochastic techniques applied to the included individual real-valued random variables with fuzzy analysis techniques in order to process the fuzziness in the probabilistic model description. A generally applicable fuzzy analysis technique based on a global optimization approach using α-discretization is described in [255] as a basis for various analyses including reliability assessment and robust design; see [95, 82, 256, 107, 120, 106]; whereby the overall analysis is performed in a nested scheme. If the analysis provides some special features, such as monotonicities or linearities, numerically efficient methods from interval mathematics [32] may be employed for the α-level mappings instead of a global optimization approach; see [173, 104]. The examples show that the analysis is feasible even for solving large problems if numerically efficient methods are chosen for the components of the analysis. From the present point of view, a combination of a Finite Element based structural analysis with a spectral approach of Monte Carlo simulation including a response surface method for dealing with random fields and processes and α-level optimization for processing fuzziness represents the most general and powerful symbiosis. The fuzzy probabilistic analysis, eventually, enables best-case and worst-case studies in terms of probabilities, within a range of plausible probabilistic models and nuanced with various magnitudes of imprecision. This can be utilized for various kinds of engineering analysis including sensitivity analyses and robust design. Fuzzy probabilistic models combine, without mixing, randomness and fuzziness. These are considered simultaneously but viewed separately at any time during the analysis and in the results. Fuzzy probabilities may be understood as an imprecise probabilistic model which allows a simultaneous consideration of all plausible probabilistic models that are relevant to describing the problem in a nuanced manner.

Fuzzy probabilities have been employed in various engineering applications; see, for example, [257, 85, 82, 37]. In [107] the time-dependent reliability of reinforced concrete structures is analyzed using efficient simulation techniques. This includes a consideration of imprecise dependencies in form of a fuzzy correlation length. Time-dependent reliability under corrosion is investigated in [258]. A method for the prediction of fuzzy structural responses, which operates on the basis of a fuzzy ARMA process simulation starting from imprecise measured data, is presented in [259]. Applications to the numerical simulation of the controlled demolition of structures by blasting are reported in [260, 261]. The reliability of offshore structures with a fuzzy probabilistic model for marine corrosion is investigated in [106] using importance sampling. Developments and applications in structural design and in robustness assessment with fuzzy probabilities can be found in [262]. In [263] a robust optimization of tuned mass dampers is solved in an environment with fuzzy mean and fuzzy variance in the description of the structural performance. An application to the analysis of the fatigue problems is reported in [264]. The prediction of surface subsidence due to mining activities is investigated in[265] with fuzzy parameters in the probabilistic model description. In [266] damage state and performance of structures are analyzed and indicators are formulated with fuzzy parameters in a probabilistic model. A neural network based approach to simulate fuzzy time series in fuzzy stochastic process is proposed in [267] and applied to forecast settlements. A related work on forecasting fuzzy-time series with neural networks is presented in [268] in the context of simulating material behavior. These examples indicate the broad spectrum of possible engineering applications for fuzzy probabilities and the associated benefits and further potential.

5. Conclusions

In solving engineering problems, it is extremely important to properly take uncertainty and imprecision into consideration. In engineering applications, there are two main sources of uncertainty and imprecision. First, the values of many important parameters change: weather parameters change, water levels change, and even for mass manufactured objects, the values of the corresponding parameters are allowed to change within the required tolerance bounds. Such an uncertainty is called *aleatory uncertainty* or *variability.*

Second, even for an individual object, an object with fixed values of the corresponding physical characteristics, we usually only know the values of these characteristics with some uncertainty. Indeed, our knowledge of these values comes either from measurements or from expert estimates; measurements are never absolutely accurate, and expert estimates are not absolutely accurate either. Such an uncertainty is called *epistemic.*

For both types of uncertainty, we do not know the exact value of the corresponding quantity. It is therefore desirable to find out what are the possible values and what are the frequencies with which different possible values can occur. For example, when designing a bridge in a windy area, we want to know the possible values of the wind speed, and we want to know the frequencies with which winds of different strengths can occur. Similarly, when we measure the wind and get an approximate value of the wind speed, it is desirable to know what are the possible values of the measurement error and how frequent are measurement errors of different sizes.

In other words, for both types of uncertainty, ideally, we should know the range of possible values, and we should know the probability distribution on this range. The traditional engineering approach to uncertainty (the one which is usually taught to engineering students) assumes that we indeed know this probability distribution.

In many practical situations, we indeed have this information. For example, when we have a large number of observations, we can determine the probability distribution corresponding to wind variability. For some measuring instruments, we have a large sample of comparative measurement results performed by this instrument and by a much more accurate ("standard") measuring instrument. Based on this sample, we can find the probability distribution for this instruments' measurement uncertainty.

However, in many important engineering problems, we only have *partial* information about these probability. This may be because the sample is too small. This may also be because the actual probability distribution within tolerance intervals may be different depending on the manufacturer: the only thing which all manufacturers guarantee is that these values are within the tolerance limits. An expert may not be comfortable describing his or her uncertainty by exact probability values. In all these practically important cases, we have *imprecise* probability.

Sometimes, the range of possible values is the only information we have; this corresponds to interval and set-valued uncertainty. Sometimes, we do not know the exact values of the probabilities but we know *bounds* on these probabilities. Sometimes, instead of guaranteed bounds, we only know bounds which are valid with some certainty, a situation which is often efficiently described by fuzzy-valued probabilities and fuzzy random variables.

From the theoretical viewpoint, imprecise probabilities are a thriving area of research. There has been a large amount of interesting research in imprecise probability, both in general theory of imprecise probability and in specific imprecise probability areas such as interval uncertainty, interval-valued probabilities, fuzzy-valued probabilities, etc. However, in engineering practice, people still mostly use traditional probabilistic methods, even when it is clear that the corresponding probabilities are only known imprecisely.

There are several reasons for this scarcity of engineering applications. First, in order to use imprecise probability techniques, we need to develop efficient algorithms and methodologies for using them. In contrast to classical statistical methods–which have been developed and perfected for decades– many imprecise probability techniques are not yet very computationally efficient. Second, applications are rarely a straightforward application of algorithms: usually, engineering knowledge and engineering intuition helps in solving the corresponding problems. During the centuries of applying traditional statistical methods, engineers and applied mathematicians have gained a lot of intuition about their use in engineering applications. For many promising imprecise probability techniques, such an intuition still needs to be acquired.

The current special issue is one of the steps towards a wider use of imprecise probability techniques. With this objective in mind, we solicited papers that resolve both issues described above. We have papers that provide new algorithms and methodologies for using imprecise probabilities in engineering, and we have papers that describe and analyze practical engineering applications of imprecise probability techniques. We hope that both types of papers will help practitioners apply these techniques – and the remaining open problems highlighted in many of the papers will inspire theoreticians in making these techniques more practically useful.

As the reader can see from the previous sections of this overview (and from the actual papers) different applications use different imprecise probability techniques. At first glance, these methods may sound different, but, as we have emphasized several times, most of these methods are closely related and reflect different aspects of the same concept of imprecise probability. We hope that the reader gets the impression that we have been trying to convey: that the interconnections and mutual complementarities between these methods are much stronger than the differences between them. There is a unity in these methods, both on the theoretical and on the algorithmic level.

For example, whether we have a measurement-induced interval uncertainty about the values coming from a known probability distribution, or we have the results of very accurate measurements of the quantity whose probability distribution is only known with interval uncertainty, we end up with the same technique: the techniques of p-boxes. And this techniques enables us to introduce *both* types of interval uncertainty – at no additional computational cost. Similarly, whether we have expert-induced fuzzy uncertainty about the values coming from a known probability distribution, or we have the results of very accurate measurements of the quantity whose probability distribution is only known with fuzzy uncertainty, we end up with the the techniques of fuzzy probabilities – which, from computational viewpoint, reduced to processing p-boxes corresponding to different thresholds α.

In short, our take-home message to the readers of this special issue is that whether we are using probabilistic methods, interval methods, p-boxes, fuzzy techniques, we are drinking the same water of truth from different sides of the same well. The results of using different imprecise probability techniques are rewarding, and the more we take into account the unity of these methods, the more we complement different techniques, the better our solutions to engineering problems. Let the hundreds applications of imprecise probability bloom!

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