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An a Posteriori Error Estimator for the C0 Interior Penalty Approximations of Fourth Order Elliptic Boundary Value Problem on Quadrilateral Meshes

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to my

MOTHER and FATHER

with love

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by

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Abstract

Numerical solutions of fourth order elliptic problems with finite element methods has been the topic of research in computational mechanics for over 50 years. Traditional approaches to solve these problems include using C^1 conforming finite element methods which demand the C^1 continuity of the underlying shape functions, which is computationally very expensive. In this work, we will present the C^0 Interior Penalty Galerkin approximation of the fourth order elliptic problems which relies only on continuous i.e., C^0 shape functions, which is much cheaper to implement. The spatial discretization is based on quadrilateral meshes and the underlying C^0 shape functions are of degree at most 2. We are using the a residual-based a posteriori error estimator, for our a posteriori error analysis, and also we show and prove its reliability. We then used some benchmark problems in our numerical result section which illustrated the performance of the estimator.

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Chapter 1

Introduction

In the recent decade, discontinuous Galerkin (DG) methods have become popular and are widely used as a powerful simulation tool for solving partial differential equations, although the definition of discontinuous polynomial spaces developed in the 1970s [17]. For a detailed historical overview we refer to the survey paper [11]. The paper [2] is one of the earliest contributions to the theory of discontinuous Galerkin finite element methods for fourth order elliptic problems.

The fourth order elliptic boundary value problem is typically solved by nonconforming or mixed methods as opposed to the traditional H^2 conforming element such as the Bogner Fox Schmidt [5] for rectangular partitions. Even in two dimension, it is really complicated to solve a fourth order elliptic problem by conforming finite element method which involves C^1 finite elements [1, 12].

Among the nonconforming approaches, C^0 Interior Penalty methods have been around for about 15 years with the first work appearing in [13] and then Brenner and her coworkers refined and analyzed the C^0 interior penalty methods in [6] and [7]. C^0 interior penalty methods are simpler than C^1 conforming finite element methods. In fact, the lowest order C^0 interior penalty methods are as simple as the classical nonconforming finite element methods [10]. Even for complicated fourth order problems, by using only integration by parts, symmetrization, and penalization, we can directly design the quasi-optimal C^0 interior penalty method. We can find interior penalty method for fourth order problems using discontinuous piecewise polynomial functions in [15].

Regarding a posteriori error analysis, a residual-based a posteriori error estimator is presented here for a biharmonic problem on quadrilateral meshes. There are very few papers which discuss the error estimator for fourth order problems [19]. The error estimator of a biharmonic problem for C^0 interior penalty discontinuous Galerkin method is discussed in [6].

While in [6], the error analysis is on triangular meshes while our focus will be to present a posteriori error analysis on quadrilateral meshes. This is the first such a posteriori analysis presented for a rectangular mesh partition. Although C^0 methods have been introduced for quadrilaterals in [9], the a posteriori analysis for such a mesh subdivision has not been investigated so far. This is the novelty of our contribution.

Now the outline of the thesis is as follows:
In this Chapter, we will discuss a little about the finite element method and give some definitions which are important to know for building all our theory throughout the thesis paper.

In Chapter 2, we formulate the continuous weak form, which is also H^2 conforming and we also introduce the Bogner-Fox-Schmit Element.

Then in Chapter 3, we derive the C^0 interior penalty method for our problem. We prove the well-posedness of the problem by showing the continuity and V-ellipticity. Later, we construct the residual based error estimator.

In Chapter 4, we use the adaptive refinement algorithm for numerical solution of some problems on a square domain.

1.1 Definitions

Sobolev Spaces:

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\alpha = (\alpha_1, ..., \alpha_n)$ of order $|\alpha| = \alpha_1 + ... + \alpha_n = k$, then we define

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} .... \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

Sobolev semi-norms/norms when k is a positive integer, are defined as follows:

$$|v|_{H^k(\Omega)} = (\sum_{|\alpha|=k} ||D^\alpha v||^2_{L^2(\Omega)})^{1/2} \tag{1.1}$$

$$||v||_{H^k(\Omega)} = (\sum_{|\alpha|\leq k} ||D^\alpha v||^2_{L^2(\Omega)})^{1/2} \tag{1.2}$$

Define the Sobolev spaces $H^k(\Omega)$ as follows:

$$H^k(\Omega) = \{v \in L^2(\Omega) : ||v||_{H^k(\Omega)} < \infty\} \tag{1.3}$$

L^2 inner product: Let V be a vector space of real-valued continuous functions on an interval [a,b]. The L^2 inner product on V is defined by

$$(f, g) = \int_a^b f(x)g(x)dx$$

Simplicial mesh: A simplicial mesh $\mathscr{T}$ of the domain Ω is a finite collection of disjoint non-degenerate simplices $\mathscr{T} = \{T\}$ forming a partition of Ω.

$$\overline{\Omega} = \bigcup_{T\in\mathscr{T}} \overline{T} \tag{1.4}$$

Each $T \in \mathscr{T}$ is called a mesh element.

General mesh: A general mesh $\mathscr{T}$ of the domain Ω is a finite collection of disjoint polyhedra $\mathscr{T} = \{T\}$ forming a partition of Ω as in (1.4).

Element diameter, meshsize: Let $\mathscr{T}$ be a (general) mesh of the domain Ω. For all $T \in \mathscr{T}$, h_T denotes the diameter of T, and the meshsize is defined as the real number

$$h = \max_{T \in \mathscr{T}} h_T$$

We use the notation $\mathscr{T}_h$ for a mesh $\mathscr{T}$ with meshsize h.

The Broken Polynomial Space:

$$\mathbb{P}_k^d(\mathscr{T}_h) := \{v \in L^2(\Omega) | v|_T \in P_k^d(T), \forall T \in \mathscr{T}_h\}$$

where P_k^d is the space of polynomials of degree $\leq$ k of d variables.

Discrete trace inequality: There exists a constant $C_{tr} > 0$ which depends only on the shape regularity of $\mathscr{T}_h$. Then, for all $v_h \in \mathbb{P}_d^k(\mathscr{T}_h)$, all $T \in \mathscr{T}_h$, and all $e \in \mathscr{E}_h$

$$h_T^{\frac{1}{2}} ||v_h||_{L^2(e)} \leq C_{tr} ||v_h||_{L^2(T)} \,,$$

where $\mathscr{E}_h$ is the set of edges in $\mathscr{T}$. Also,

$$\| \frac{\partial^2 v_h}{\partial n_{\partial T}^2} \|_{0,\partial T} \leq C_{tr} h_T^{-1/2} \| v_h \|_{0,T} \ , \quad \forall T \in \mathscr{T}_h \ , \tag{1.5}$$

where $n_{\partial T}$ is the exterior unit normal to ∂T.

Finite Element We introduce the definition from [8, 14].

Let

- $K \subset \mathbb{R}^n$ be a bounded closed set with nonempty interior and piecewise smooth boundary (the element domain).
- P be a Finite-dimensional vector space, dim(P)=N, of functions $\phi : K \to \mathbb{R}$, which is the space of shape functions.

- Σ is a set $\{\psi\}_j$ of linear forms,

$$\psi_j : P \rightarrow \mathbb{R} \quad \forall j \in \{1, 2, ..., N\}$$
$$\phi \longmapsto \phi_j = \psi_j(\phi)$$

  the linear forms $\{\psi\}_j$ are called the local degrees of freedom.

Then (K, P, Σ) is called a finite element. Now the point evaluation at a_i is $\psi_i(\theta_j) = \theta_j(a_i)$ and first derivative at a_i, $\psi_{i,1} = \partial_1 \theta_j(a_i)$. Such ψ_i are dual basis. We assume for each ψ_i, there exists a unique $\theta_j \in P$ such that $p_i(\theta_j) = \delta_{ij} \quad \forall i, j \in \{1, 2, ..., N\}$. The basis $\{\theta_j\}_{1 \leq j \leq N}$ of P are called the local shape functions.

Chapter 2

H^2 conforming Finite Element Method on Quadrilateral Meshes

In this Chapter, we will formulate the continuous weak formulation for our fourth order problem. Then we will discuss its H^2 conformity. We will also introduce the Bogner-Fox-Schmit Element [5].

2.1 Formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. We consider the fourth order boundary value problem

$$\Delta^2 u = f \text{ in } \Omega \;, \tag{2.1}$$

$$u = 0 \text{ in } \partial\Omega \;, \tag{2.2}$$

$$\frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega \;, \tag{2.3}$$

where $f \in L^2(\Omega)$ and the boundary conditions are clamped. We say u is a solution of (2.1)-(2.3) if it satisfies equation (2.1)-(2.3).

We define V as

$$V = \{v \in H^2(\Omega) : v = 0 \text{ and } \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\} \;,$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative.

A weak form of the biharmonic problem is to find $u \in V$ such that

$$a(u,v) = (f,v) \qquad \forall v \in V \;, \tag{2.4}$$

where

$$a(w,v) = \int_T D^2 w : D^2 v dx \qquad \forall w, v \in H^2(\Omega) \ ,$$

$$D^2 w : D^2 v = \sum_{i,j=1}^{2} \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \ ,$$

and $(\cdot,\cdot)$ denotes the L^2 inner product on Ω.

Now we introduce all the notations we are going to use throughout the chapter. Let

- $\mathscr{T}_h$ be a quadrilateral subdivision of Ω
- $\mathscr{E}_h$ = the set of edges of the quadrilaterals in $\mathscr{T}_h$
- $\mathscr{E}_h^i$ = the subset of $\mathscr{E}_h$ consisting of the interior edges of Ω
- $\mathscr{E}_h^b$ = the subset of $\mathscr{E}_h$ consisting of the edges along $\partial\Omega$
- v_T= restriction of the function v to the quadrilateral T
- h_T = diameter of the quadrilateral T
- $h = \max\{h_T : T \in \mathscr{T}_h\}$
- $\mathscr{V}_h$ =set of all vertices in $\mathscr{T}_h$
- $|T|$ =area of the quadrilateral T
- the length of any edge $e \in \mathscr{E}_h$ will be denoted by h_e.

Next we define the jump and the average terms on the edge. Let

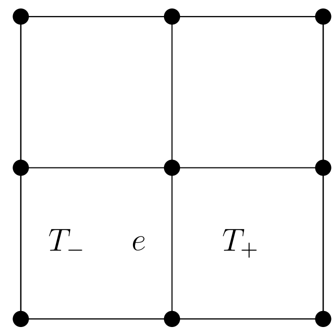
$$H^k(\Omega, \mathscr{T}_h) = \{v \in H^k(\Omega) : v_T \in H^k(T), \forall T \in \mathscr{T}_h\}.$$

Let $Q_k = \{\sum_j c_j p_j(x), q_j(y) : p_j, q_j$ polynomials of degree$\leq k\}$ and also

$$\dim Q_k = \dim\ (P_k^1)^2$$

where P_k^1 is the space of polynomials of degree $\leq k$ in 1 dimension, and C^0 finite element space is

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_T \in Q_2(T), T \in \mathscr{T}_h\}.$$



Let $e \in \mathscr{E}_h^i$ be shared by the two rectangles $T_{+,-} \in \mathscr{T}_h$. We take n to be the unit normal vector to e pointing from T_- to T_+ and define

$$\left[\left[\frac{\partial v_h}{\partial n}\right]\right] = \frac{\partial v_h|_{T_+}}{\partial n}\bigg|_e - \frac{\partial v_h|_{T_-}}{\partial n}\bigg|_e \qquad \forall v_h \in H^2(\Omega, \mathscr{T}_h)\ ,$$

$$\left[\left[\frac{\partial^2 v_h}{\partial n^2}\right]\right] = \frac{\partial^2 v_h|_{T_+}}{\partial n^2}\bigg|_e - \frac{\partial^2 v_h|_{T_-}}{\partial n^2}\bigg|_e \qquad \forall v_h \in H^3(\Omega, \mathscr{T}_h)\ ,$$

$$\left\{\left\{\frac{\partial v_h}{\partial n}\right\}\right\} = \frac{1}{2}\left(\frac{\partial v_h|_{T_+}}{\partial n}\bigg|_e + \frac{\partial v_h|_{T_-}}{\partial n}\bigg|_e\right) \qquad \forall v_h \in H^2(\Omega, \mathscr{T}_h)\ ,$$

$$\left\{\left\{\frac{\partial^2 v_h}{\partial n^2}\right\}\right\} = \frac{1}{2}\left(\frac{\partial^2 v_h|_{T_+}}{\partial n^2}\bigg|_e + \frac{\partial^2 v_h|_{T_-}}{\partial n^2}\bigg|_e\right) \ \forall v_h \in H^3(\Omega, \mathscr{T}_h).$$

Let $e \in \mathscr{E}_h^b$ be an edge of $T \in \mathscr{T}_h$. We take n to be the unit normal vector to e pointing outside Ω and define

$$\left[\left[\frac{\partial v_h}{\partial n}\right]\right] = -\frac{\partial v_h|_T}{\partial n}\bigg|_e \qquad \forall v_h \in H^2(\Omega, \mathscr{T}_h)\ ,$$

$$\left\{\left\{\frac{\partial^2 v_h}{\partial n^2}\right\}\right\} = \frac{\partial^2 v_h|_T}{\partial n^2}\bigg|_e \qquad \forall v_h \in H^3(\Omega, \mathscr{T}_h).$$

2.2 H^2- conformity of finite elements

In this section, we state and prove the criteria for a finite element space to be a H^2 conforming space. We begin by introducing the following finite element space V_h:

$$V_h = \{v_h : \overline{\Omega} \to \mathbb{R} |\ v_h|_T \in Q_k(T), \forall T \in \mathscr{T}_h\} \ ,$$

where $Q_k(T), T \in \mathscr{T}_h$, is the space of polynomials of degree $\leq k$ on T.
Now, we continue with our H^2 conformity proof.

Theorem 1 *Let V_h be a finite element space and*

$$Q_k(T) \subset H^2(T), \quad T \in \mathscr{T}_h,$$

and,

$$V_h \subset C^1(\Omega).$$

Then,

$$V_h \subset H^2(\Omega).$$

.

Proof: Let $v_h \in V_h$, clearly $v_h \in L^2(\Omega)$.
We have to show that v_h admits weak derivatives $w_h^\alpha \in L^2(\Omega)$, $|\alpha| = 2$, i.e.,

$$\int_\Omega v_h D^\alpha z dx = (-1)^{|\alpha|} \int_\Omega w_h^\alpha z dx, \qquad z \in C_0^\infty$$

Now, by Green's theorem

$$\int_\Omega v_h D^2 z dx = \sum_{K \in \mathscr{T}_h} \int_K v_h D^2 z dx$$

$$= - \sum_{T \in \mathscr{T}_h} \int_T D v_h D z dx + \sum_{T \in \mathscr{T}_h} \int_{\partial T} v_h \hat{n}_2 D^2 z ds$$

$$= - \sum_{T \in \mathscr{T}_h} [- \int_T D^2 v_h z dx + \int_{\partial T} \hat{n}_1 D v_h z ds] + \sum_{T \in \mathscr{T}_h} \int_{\partial T} v_h \hat{n}_2 D^2 z ds$$

$$= \sum_{T\in\mathscr{T}_h} \int_T D^2 v_h z dx - \sum_{T\in\mathscr{T}_h} \int_{\partial T} \hat{n}_1 D v_h z ds + \sum_{T\in\mathscr{T}_h} \int_{\partial T} v_h \hat{n}_2 D^2 z ds$$

$$= \sum_{T\in\mathscr{T}_h} \int_T D^2 v_h z dx - \sum_{e\in\mathscr{E}_h(\Omega)} \int_e \hat{n}_1 [D v_h] z ds + \sum_{e\in\mathscr{E}_h(\Omega)} \int_e [v_h] \hat{n}_2 D^2 z ds,$$

where

$$[\![v_h]\!] = v_h|_{T_+} - v_h|_{T_-} \ ,$$

$$[\![Dv_h]\!] = Dv_h|_{T_+} - Dv_h|_{T_-} \ ,$$

across $e = T_1 \cap T_2, T_i \in \mathscr{T}_h, 1 \leq i \leq 2$. But $v_h \in C^1(\overline{\Omega})$, so $[\![v_h]\!] = 0$ and $[\![Dv_h]\!] = 0$.

□

The above Theorem 1 specifies the continuity requirements of a C^1 conforming finite element space i.e., we demand the continuity of the first and higher-order derivatives. To be specific we discuss the Bogner Fox Schmit element [5] below as an example and as an important conforming finite element space which we will need for our analysis in the next chapter.

2.2.1 Bogner-Fox-Schmit Element

In this section, we will introduce a C^1 type finite element called Bogner-Fox-Schmit Rectangle. Let T denote a rectangle with vertices a_i, $1 \leq i \leq 4$, then a polynomial in $\mathbb{Q}_4$ is uniquely determined by the following set of degrees of freedom:

$$\Sigma = \{\psi_i(\theta_j), \psi_{m_i}(\theta_j), \psi_{1234}(\theta_j), \psi_{i,1}(\theta_j), \psi_{i,2}(\theta_j), \psi_{i,12}(\theta_j), \psi_{\partial m_i}(\theta_j), \quad m_i \text{ is midpoint of edge } i; 1 \leq i \leq 4\} \quad (2.5)$$

where:

- Point evaluations at vertices, midpoints and center:
  $\psi_i(\theta_j) = \theta_j(a_i)$, $\psi_{m_i}(\theta_j) = \theta_j(m_i)$, & $\psi_{1234}(\theta_j) = \theta_j(a_{1234})$.
- First order derivatives, and normal derivatives:
  $\psi_{i,k} = \partial_k \theta_j(a_i)$, $\psi_{\partial m_i}(\theta_j) = \partial_n \theta_j(m_i)$, for $k = 1, 2$.

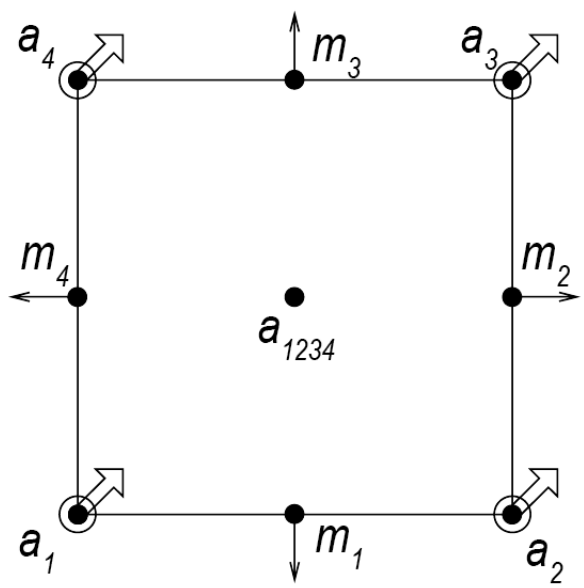


Figure 2.1: Bogner-Fox-Schmit Element [9].

- Second order mixed derivatives:

  $\psi_{i,12}(\theta_j) = \partial_{12}\theta_j(a_i)$.

and a_i, m_i and a_{1234} are as described in the Figure 2.1.

The resulting finite element is the Bogner-Fox-Schmit Rectangle. On each rectangle, the $\mathbb{Q}_4$ polynomials are determined by 25 nodal degrees of freedom.
Here the dots " ● " indicates point evaluation, the arrows "⇗" indicate the first and second order mixed derivatives evaluated at the vertices of the rectangle and " ↑ " indicate the outward normal derivatives. It is well known that this element is unisolvent.
This element is computationally expensive to implement due to the high degrees of freedom per cell. Instead, we would like to use a method which is easy to implement and doesn't involve a large degrees of freedom per cell. In [9], a new finite element method C^0 Interior Penalty Method was introduced that has the desirable properties of being computationally cheaper and easy to implement. Our next chapter will be devoted to studying this method.

Chapter 3

C^0 Interior Penalty Method: Fourth Order Problem

In this Chapter, we will formulate C^0 interior penalty methods for a fourth order problem. Then we will discuss its continuity and coercivity. We will also introduce a residual based error estimator and discuss its reliability.

3.1 Derivation of C^0 IP

Let C^0 finite element space is

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_T \in Q_k(T), T \in \mathscr{T}_h\} \ ,$$

let $v \in H^4(T)$ and multiply (2.1) by v, then by using the integration by parts on the left hand side

$$\begin{aligned}\int_T (\Delta^2 u) v dx &= \int_{\partial T} \frac{\partial \Delta u}{\partial n} v ds - \int_T \nabla(\Delta u) \nabla v dx \\ &= \int_{\partial T} \frac{\partial \Delta u}{\partial n} v ds - \int_{\partial T} \frac{\partial^2 u}{\partial n \partial t} \frac{\partial v}{\partial t} ds - \int_{\partial T} \frac{\partial^2 u}{\partial n^2} \frac{\partial v}{\partial n} ds + \int_T D^2 u : D^2 v dx \ ,\end{aligned} \tag{3.1}$$

where $D^2u : D^2v = \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}$.

Summing up over all $T \in \mathscr{T}_h$ and by using the boundary conditions, we have for $\forall v \in H^4(\Omega, \mathscr{T}_h)$,

$$\int_\Omega(\Delta^2 u)v dx = \sum_{e\in\mathscr{E}_h}\int_e \frac{\partial \Delta u}{\partial n} v ds - \sum_{e\in\mathscr{E}_h}\int_e \frac{\partial^2 u}{\partial n \partial t}\frac{\partial v}{\partial t} ds - \sum_{e\in\mathscr{E}_h}\int_e \frac{\partial^2 u}{\partial n^2}\left[\left[\frac{\partial v}{\partial n}\right]\right] ds + \sum_{T\in\mathscr{T}_h}\int_T D^2u : D^2 v dx$$
$$= \sum_{e\in\mathscr{E}_h}\int_e \frac{\partial^2 u}{\partial n^2}\left[\left[\frac{\partial v}{\partial n}\right]\right] ds + \sum_{T\in\mathscr{T}_h}\int_T D^2u : D^2 v dx. \tag{3.2}$$

So, with the right side we have

$$\sum_{e\in\mathscr{E}_h}\int_e \frac{\partial^2 u}{\partial n^2}\left[\left[\frac{\partial v}{\partial n}\right]\right] ds + \sum_{T\in\mathscr{T}_h}\int_T D^2u : D^2 v dx = \int_\Omega f v dx.$$

Now by using $ab - cd = \frac{(a+c)(b-d)}{2} + \frac{(b+d)(a-c)}{2}$ we have

$$\sum_{T\in\mathscr{T}_h}\int_T D^2u_h : D^2 v_h dx + \sum_{e\in\mathscr{E}_h}\int_e \left\{\left\{\frac{\partial^2 u_h}{\partial n^2}\right\}\right\}\left[\left[\frac{\partial v_h}{\partial n}\right]\right] ds = \int_\Omega f v_h dx,$$

where $u_h, v_h \in V_h$.

Then we add the term $\sum_{e\in\mathscr{E}_h}\int_e \{\{\frac{\partial^2 v_h}{\partial n^2}\}\}[[\frac{\partial u_h}{\partial n}]] ds$ in both sides, since $[[\frac{\partial u_h}{\partial n}]] = 0$. Finally we add the penalty term.

So, the C^0 interior penalty method for (2.4) is to find $u_h \in V_h$ such that

$$A_h(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h \ , \tag{3.3}$$

where

$$A_h(u_h, v_h) = \sum_{T\in\mathscr{T}_h}\int_T D^2u_h : D^2 v_h dx + \sum_{e\in\mathscr{E}_h}\int_e \left\{\left\{\frac{\partial^2 u_h}{\partial n^2}\right\}\right\}\left[\left[\frac{\partial v_h}{\partial n}\right]\right] ds + \sum_{e\in\mathscr{E}_h}\int_e \left\{\left\{\frac{\partial^2 v_h}{\partial n^2}\right\}\right\}\left[\left[\frac{\partial u_h}{\partial n}\right]\right] ds$$
$$+ \sum_{e\in\mathscr{E}_h}\frac{\sigma}{h_e}\int_e \left[\left[\frac{\partial u_h}{\partial n}\right]\right]\left[\left[\frac{\partial v_h}{\partial n}\right]\right] ds \ , \tag{3.4}$$

where $\sigma \geq 1$ is a constant called penalty parameter.

Now we define $\|.\|_h$

$$\|v\|_h^2 = a_h(v,v) + \sum_{e\in\mathscr{E}_h} \frac{\sigma}{h_e} \int_e \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]^2 ds \qquad \forall v \in H^2(\Omega, \mathscr{T}_h)\ , \tag{3.5}$$

where the bilinear form $a_h(\cdot,\cdot)$ is defined by

$$a_h(w,v) = \sum_{T\in\mathscr{T}_h} \int_T D^2w : D^2v dx \qquad \forall v, w \in H^2(\Omega, \mathscr{T}_h). \tag{3.6}$$

By using the Poincaré inequality for $H_0^1(\Omega)$ and $H_0^2(\Omega)$ as described in [16, 18], for instance

$$\|u\|_{L^2(\Omega)}^2 \le C\|\nabla u\|_{L^2(\Omega)}^2 \quad \forall u \in H_0^1(\Omega)\ , \tag{3.7}$$

$$\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \le C\|\Delta u\|_{L^2(\Omega)}^2 \quad \forall u \in H_0^2(\Omega)\ , \tag{3.8}$$

wee see that $\|.\|_h$ is a norm.

Now $|.|_{a_h}$ is the seminorm corresponding to $a_h(\cdot,\cdot)$, i.e.

$$|v|_{a_h}^2 = a_h(v,v) = \sum_{T\in\mathscr{T}_h} \int_T D^2v : D^2v dx = \sum_{T\in\mathscr{T}_h} |v|_{H^2(T)}^2\ ,$$

$$|v|_{a_h} = |v|_{H^2(\Omega)} \qquad \forall v \in H^2(\Omega).$$

3.1.1 Continuity of $A_h(u_h, v_h)$

Theorem 2 *There exists a constant C_{bnd} which depends only on the shape regularity of $\mathscr{T}_h$ such that*

$$\forall u_h, v_h \in V_h \qquad |A_h(u_h,v_h)| \le C_{bnd}||u_h||_h||v_h||_h. \tag{3.9}$$

Proof: From (3.4) we know

$$\begin{aligned} A_h(u_h,v_h) &= \sum_{T\in\mathscr{T}_h} \int_T D^2u_h : D^2v_h dx + \sum_{e\in\mathscr{E}_h} \int_e \left\{\!\!\left\{\frac{\partial^2 u_h}{\partial n^2}\right\}\!\!\right\} \left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right] ds + \sum_{e\in\mathscr{E}_h} \int_e \left\{\!\!\left\{\frac{\partial^2 v_h}{\partial n^2}\right\}\!\!\right\} \left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right] ds \\ &\quad + \sum_{e\in\mathscr{E}_h} \frac{\sigma}{h_e} \int_e \left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right] \left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right] ds \\ &= \mathfrak{E}_1 + \mathfrak{E}_2 + \mathfrak{E}_3 + \mathfrak{E}_4\ , \end{aligned} \tag{3.10}$$

where

$$\mathfrak{E}_1 = \sum_{T\in\mathscr{T}_h} \int_T D^2 u_h : D^2 v_h dx \ ,$$

$$\mathfrak{E}_2 = \sum_{e\in\mathscr{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 u_h}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] ds \ ,$$

$$\mathfrak{E}_3 = \sum_{e\in\mathscr{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial u_h}{\partial n} \right]\!\!\right] ds \ ,$$

$$\mathfrak{E}_4 = \sum_{e\in\mathscr{E}_h} \frac{\sigma}{h_e} \int_e \left[\!\!\left[\frac{\partial u_h}{\partial n} \right]\!\!\right] \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] ds.$$

Now by using the Cauchy-Schwarz inequality we get

$$\begin{aligned} |\mathfrak{E}_1| = |\sum_{T\in\mathscr{T}_h} \int_T D^2 u_h : D^2 v_h dx| &\leq \Big(\sum_{T\in\mathscr{T}_h} \int_T |D^2 u_h|^2 \Big)^{1/2} \Big(\sum_{T\in\mathscr{T}_h} \int_T |D^2 v_h|^2 \Big)^{1/2} \\ &\leq \|u_h\|_h \|v_h\|_h. \end{aligned} \tag{3.11}$$

Then by using the trace inequality and standard inverse estimate

$$\begin{aligned} |\mathfrak{E}_2| = |\sum_{e\in\mathscr{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 u_h}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] ds| &\leq \left(\sum_{e\in\mathscr{E}_h} h_e || \left\{\!\!\left\{ \frac{\partial^2 u_h}{\partial n^2} \right\}\!\!\right\} ||^2_{L^2(e)} \right)^{1/2} \left(\sum_{e\in\mathscr{E}_h} h_e^{-1} || \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] ||^2_{L^2(e)} \right)^{1/2} \\ &\leq \left(C_{tr} \sum_{T\in\mathscr{T}_h} |u_h|^2_{H^2(T)} \right)^{1/2} \left(\sum_{e\in\mathscr{E}_h} h_e^{-1} || \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] ||^2_{L^2(e)} \right)^{1/2} \\ &\leq (C_{tr})^{1/2} \|u_h\|_h \|v_h\|_h. \end{aligned} \tag{3.12}$$

Now, similarly we get

$$\begin{aligned} |\mathfrak{E}_3| = |\sum_{e\in\mathscr{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial u_h}{\partial n} \right]\!\!\right] ds| \\ \leq (C_{tr})^{1/2} \|u_h\|_h \|v_h\|_h. \end{aligned} \tag{3.13}$$

By Cauchy-Schwarz inequality from our last term

$$\begin{aligned}
|\mathfrak{E}_4| &= |\sum_{e\in\mathscr{E}_h} \frac{\sigma}{h_e}\int_e \left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right]\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right] ds| \\
&\le \left(\sum_{e\in\mathscr{E}_h} \frac{\sigma}{h_e}\|\left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right]\|^2_{L^2(e)}\right)^{1/2} \left(\sum_{e\in\mathscr{E}_h} \frac{\sigma}{h_e}\|\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right]\|^2_{L^2(e)}\right)^{1/2} \\
&\le \|u_h\|_h \|v_h\|_h.
\end{aligned} \tag{3.14}$$

Finally by collecting all from (3.11), (3.12), (3.13) and (3.14) we get

$$|A_h(u_h, v_h)| \le |\mathfrak{E}_1| + |\mathfrak{E}_2| + |\mathfrak{E}_3| + |\mathfrak{E}_4| \le C_{bnd}||u_h||_h||v_h||_h. \tag{3.15}$$

□

3.1.2 Coercivity of $A_h(u_h, v_h)$

Theorem 3 *There exists a constant $C_\sigma > 0$ and $\sigma^* > 0$ which depends only on the shape regularity of $\mathscr{T}_h$ such that if $\sigma > \sigma^*$, then*

$$\forall u_h, v_h \in V_h \qquad A_h(u_h, u_h) \ge C_\sigma ||u_h||_h^2. \tag{3.16}$$

Proof: $\forall u_h \in V_h$ from (3.4) we know

$$\begin{aligned}
A_h(u_h, u_h) &= \sum_{T\in\mathscr{T}_h} \int_T D^2 u_h : D^2 u_h dx + 2\sum_{e\in\mathscr{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 u_h}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right] ds + \sum_{e\in\mathscr{E}_h} \frac{\sigma}{h_e}\int_e \left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right]^2 ds \\
&= \sum_{T\in\mathscr{T}_h} |u_h|^2_{H^2(T)} + 2\sum_{e\in\mathscr{E}_h}\int_e \left\{\!\!\left\{\frac{\partial^2 u_h}{\partial n^2}\right\}\!\!\right\}\left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right] ds + \sum_{e\in\mathscr{E}_h} \frac{\sigma}{h_e}\|\left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right]\|^2_{L^2(e)}.
\end{aligned} \tag{3.17}$$

Now following (3.12) we get

$$
\begin{aligned}
|\sum_{e\in\mathscr{E}_h}\int_e \left\{\left\{\frac{\partial^2 u_h}{\partial n^2}\right\}\right\}\left[\left[\frac{\partial u_h}{\partial n}\right]\right] ds| &\leq \left(\sum_{e\in\mathscr{E}_h} h_e ||\left\{\left\{\frac{\partial^2 u_h}{\partial n^2}\right\}\right\}||^2_{L^2(e)}\right)^{1/2}\left(\sum_{e\in\mathscr{E}_h} h_e^{-1}||\left[\left[\frac{\partial u_h}{\partial n}\right]\right]||^2_{L^2(e)}\right)^{1/2} \\
&\leq (C_{tr})^{1/2}\sum_{T\in\mathscr{T}_h}|u_h|_{H^2(T)}\sum_{e\in\mathscr{E}_h} h_e^{-1/2}||\left[\left[\frac{\partial u_h}{\partial n}\right]\right]||_{L^2(e)}.
\end{aligned}
\tag{3.18}
$$

Now,

$$
\begin{aligned}
A_h(u_h,u_h) \geq \sum_{T\in\mathscr{T}_h}|u_h|^2_{H^2(T)} - 2(C_{tr})^{1/2}\sum_{T\in\mathscr{T}_h}|u_h|_{H^2(T)}\sum_{e\in\mathscr{E}_h} h_e^{-1/2}||\left[\left[\frac{\partial u_h}{\partial n}\right]\right]||_{L^2(e)} \\
+ \sum_{e\in\mathscr{E}_h}\frac{\sigma}{h_e}\|\left[\left[\frac{\partial u_h}{\partial n}\right]\right]\|^2_{L^2(e)}
\end{aligned}
$$

Then we use the following inequality, let β be a positive real number and $\sigma > \beta^2$ then for all $x, y \in \mathbb{R}$

$$
x^2 - 2\beta xy + \sigma y^2 \geq \frac{\sigma-\beta^2}{1+\sigma}(x^2+y^2).
$$

By applying this inequality with $\beta = 2(C_{tr})^{1/2}, x = |u_h|_{H^2(T)}, y = h_e^{-1/2}\|\left[\left[\frac{\partial u_h}{\partial n}\right]\right]\|_{L^2(e)}$

$$
\begin{aligned}
A_h(u_h,u_h) &\geq \frac{\sum_{e\in\mathscr{E}_h}\frac{\sigma}{h_e} - \frac{4C_{tr}}{h_e}}{1+\sum_{e\in\mathscr{E}_h}\frac{\sigma}{h_e}}\left(\sum_{T\in\mathscr{T}_h}|u_h|^2_{H^2(T)} + \sum_{e\in\mathscr{E}_h} h_e^{-1}\|\left[\left[\frac{\partial u_h}{\partial n}\right]\right]\|^2_{L^2(e)}\right) \\
&= C_\sigma\|u_h\|^2_h.
\end{aligned}
$$

□

3.2 Enriching operator

In this section, we will introduce a linear operator E_h.

$$
E_h : V_h \to \tilde{V}_h
$$

where $\tilde{V}_h \subset H_0^2(\Omega)$ is the Bogner-Fox-Schmit element space associated with $\mathscr{T}_h$. The linear operator E_h connects our C^0 finite element space V_h to C^1 finite element space $\tilde{V}_h$.

The enriching operator E_h is constructed by averaging,

$$(E_h v_h)p = v_h(p) \qquad \forall p \in \mathscr{V}_h, v_h \in V_h. \tag{3.19}$$

We specify the degrees of freedom N in Bogner-Fox-Schmit element

$$N(E_h v_h) = \frac{1}{|\mathbb{T}_N|} \sum_{T \in \mathscr{T}_N} N(v_T) \ , \tag{3.20}$$

where $\mathbb{T}_N$ is the set of rectangles that share the degree of freedom N and $|\mathbb{T}_N|$ is the number of elements of $\mathbb{T}_N$.

The linear operator E_h satisfies:

$$|v_h - E_h v_h|^2_{a_h} \leq C \sum_{e \in \mathscr{E}_h} \frac{1}{h_e} \| \left[\left[\frac{\partial v_h}{\partial n} \right]\right] \|^2_{L_2(e)} \qquad \forall v_h \in V_h \ , \tag{3.21}$$

which is proved in [9].

3.3 Interpolation operator

The standard Lagrange nodal interpolation operator $\pi_h : H^2_0(\Omega) \to V_h$ has the following properties

$$|\phi - \pi_h \phi|_{H^l(T)} \leq C h_T^{2-l} |\phi|_{H^2(T)} \qquad \text{for } 0 \leq l \leq 2, \tag{3.22}$$

$$\int_{\partial T} \left| \frac{\partial}{\partial n}(\phi - \pi_h \phi) \right|^2 ds \leq C h_T |\phi|^2_{H^2(T)}. \tag{3.23}$$

3.4 A reliable *a posteriori* error estimator

We will develop a reliable residual-based a posteriori error estimator η_h for (3.3). This was introduced for triangular meshes in [6].

The error estimator η_h is defined by

$$\eta_h = (\sum_{T \in \mathscr{T}_h} \eta_T^2 + \sum_{e \in \mathscr{E}_h} \eta_{e,1}^2 + \sum_{e \in \mathscr{E}_h^i} \eta_{e,2}^2)^{1/2}.$$

Here, η_h consists of three parts η_T,$\eta_{e,1}$ and $\eta_{e,2}$. The first part is the residual η_T over the rectangles $T \in \mathscr{T}_h$:

$$\begin{aligned}\eta_T &= h_T^2 \|f - \Delta^2 u_h\|_{L^2(T)} \\ &= h_T^2 \|f\|_{L^2(T)}, \qquad \text{since } u_h \in Q_2(T).\end{aligned} \tag{3.24}$$

The second part is the residual $\eta_{e,1}$ associated with the jump of the first-order normal derivative of u_h across the edges $e \in \mathscr{E}_h$:

$$\eta_{e,1} = \frac{\sigma}{\sqrt{h_e}} \|[\partial u_h / \partial n]\|_{L^2(e)}. \tag{3.25}$$

The third part is the residual $\eta_{e,2}$ associated with the jump of the second-order normal derivatives of u_h across the edges $e \in \mathscr{E}_h^i$:

$$\eta_{e,2} = \sqrt{h_e} \|[\partial^2 u_h / \partial n^2]\|_{L^2(e)}. \tag{3.26}$$

So, our error estimator η_h is then defined by

$$\begin{aligned}\eta_h &= (\sum_{T \in \mathscr{T}_h} \eta_T^2 + \sum_{e \in \mathscr{E}_h} \eta_{e,1}^2 + \sum_{e \in \mathscr{E}_h^i} \eta_{e,2}^2)^{1/2} \\ &= (\sum_{T \in \mathscr{T}_h} h_T^4 \|f\|_{L^2(T)}^2 + \sum_{e \in \mathscr{E}_h} \sigma^2 h_e^{-1} \|[\partial u_h / \partial n]\|_{L^2(e)}^2 + \sum_{e \in \mathscr{E}_h^i} h_e \|[\partial^2 u_h / \partial n^2]\|_{L^2(e)}^2)^{1/2}.\end{aligned} \tag{3.27}$$

The following theorem shows that the error estimator η_h is reliable, i.e. provides an upper bound.

3.4.1 Reliability of error estimator

Theorem 4 *Let $u \in H_0^2(\Omega)$ be the solution of* (2.4) *and $u_h \in V_h$ be the solution of* (3.3). *There exists a positive constant C depending only on the shape regularity of $\mathscr{T}_h$ such that*

$$\|u - u_h\|_h \leq C\eta_h. \tag{3.28}$$

Proof: By using (3.5) we get,

$$\|u - u_h\|_h^2 = a_h(u - u_h, u - u_h) + \sum_{e \in \mathscr{E}_h} \frac{\sigma}{h_e} \int_e \left[\!\left[\frac{\partial u - u_h}{\partial n} \right]\!\right]^2 ds. \tag{3.29}$$

Since,

$$[\![\partial(u-u_h)/\partial n]\!] = \partial u/\partial n - \partial u_h^+/\partial n - \partial u/\partial n + \partial u_h^-/\partial n = -[\![\partial u_h/\partial n]\!].$$

We have then

$$\sum_{e\in\mathscr{E}_h} \frac{\sigma}{h_e}\|[\partial(u-u_h)/\partial n]\|^2_{L_2(e)} = \sum_{e\in\mathscr{E}_h} \frac{\sigma}{h_e}\|[\partial u_h/\partial n]\|^2_{L_2(e)} \le \sum_{e\in\mathscr{E}_h} \eta^2_{e,1}. \tag{3.30}$$

Now we know that

$$\begin{aligned} a_h(u-u_h, u-u_h) &= \sum_{T\in\mathscr{T}_h} \int_T D^2(u-u_h) : D^2(u-u_h)dx \\ &= \sum_{T\in\mathscr{T}_h} \|D^2(u-u_h)\|^2_{L^2(T)} \\ &= \sum_{T\in\mathscr{T}_h} \|D^2(u-E_hu_h+E_hu_h-u_h)\|^2_{L^2(T)}\ , \end{aligned}$$

where E_h is our enriching operator (3.2). Then by using the triangle inequality we have

$$\sum_{T\in\mathscr{T}_h} \|D^2(u-E_hu_h+E_hu_h-u_h)\|_{L^2(T)} \le \|D^2(u-E_hu_h)\|_{L^2(T)} + \|D^2(E_hu_h-u_h)\|_{L^2(T)}. \tag{3.31}$$

From the second term of the right side of (3.31), and by using the (3.21) we get

$$\|D^2(E_hu_h-u_h)\|^2_{L^2(T)} \le C\sum_{e\in\mathscr{E}_h} \eta^2_{e,1}. \tag{3.32}$$

Now we need to estimate the first term on the right hand side of (3.31).

Let $v \in H_0^2(\Omega)$,

$$\begin{aligned} (D^2(u-E_hu_h), D^2v)_\Omega &= (D^2u, D^2v)_\Omega - (D^2E_hu_h, D^2v)_\Omega \\ &= (f,v)_\Omega - \sum_{T\in\mathscr{T}_h} (D^2(E_hu_h-u_h+u_h), D^2v)_T \\ &= (f,v)_\Omega - \sum_{T\in\mathscr{T}_h} (D^2u_h, D^2v)_T + \sum_{T\in\mathscr{T}_h} (D^2(u_h-E_hu_h), D^2v)_T. \end{aligned} \tag{3.33}$$

Then

$$\sum_{T\in\mathscr{T}_h}(D^2u_h,D^2v)_T=\sum_{T\in\mathscr{T}_h}(D^2u_h,D^2v-D^2\Pi_hv+D^2\Pi_hv)_T$$
$$=\sum_{T\in\mathscr{T}_h}(D^2u_h,D^2v-D^2\Pi_hv)_T+\sum_{T\in\mathscr{T}_h}(D^2u_h,D^2\Pi_hv)_T\ , \tag{3.34}$$

where Π_h is the standard Lagrange interpolation operator and say, $v_h=\Pi_hv$. Now from (3.3) and (3.4) we have

$$\sum_{T\in\mathscr{T}_h}(D^2u_h,D^2v_h)_T=\sum_{T\in\mathscr{T}_h}(f,v_h)_T-\sum_{e\in\mathscr{E}_h}(\left\{\!\!\left\{\frac{\partial^2u_h}{\partial n^2}\right\}\!\!\right\},\left[\!\!\left[\frac{\partial v_h}{\partial n}\right]\!\!\right])_e-\sum_{e\in\mathscr{E}_h}(\left\{\!\!\left\{\frac{\partial^2v_h}{\partial n^2}\right\}\!\!\right\},\left[\!\!\left[\frac{\partial u_h}{\partial n}\right]\!\!\right])_e$$
$$-\sum_{e\in\mathscr{E}_h}\frac{\sigma}{h_e}(\left[\!\!\left[\frac{\partial u_h}{\partial n}\right]\!\!\right],\left[\!\!\left[\frac{\partial v_h}{\partial n}\right]\!\!\right])_e. \tag{3.35}$$

Now by using (3.35) in (3.33) we get

$$(D^2(u-E_hu_h),D^2v)_\Omega=(f,v)_\Omega-\sum_{T\in\mathscr{T}_h}(f,v_h)_T+\sum_{e\in\mathscr{E}_h}(\left\{\!\!\left\{\frac{\partial^2u_h}{\partial n^2}\right\}\!\!\right\},\left[\!\!\left[\frac{\partial v_h}{\partial n}\right]\!\!\right])_e$$
$$+\sum_{e\in\mathscr{E}_h}(\left\{\!\!\left\{\frac{\partial^2v_h}{\partial n^2}\right\}\!\!\right\},\left[\!\!\left[\frac{\partial u_h}{\partial n}\right]\!\!\right])_e$$
$$+\sum_{e\in\mathscr{E}_h}\frac{\sigma}{h_e}(\left[\!\!\left[\frac{\partial u_h}{\partial n}\right]\!\!\right],\left[\!\!\left[\frac{\partial v_h}{\partial n}\right]\!\!\right])_e-\sum_{T\in\mathscr{T}_h}(D^2u_h,D^2(v-v_h))_T$$
$$+\sum_{T\in\mathscr{T}_h}(D^2(u_h-E_hu_h),D^2v)_T$$
$$=\sum_{T\in\mathscr{T}_h}(f,v-v_h)_T+\sum_{e\in\mathscr{E}_h}(\left\{\!\!\left\{\frac{\partial^2u_h}{\partial n^2}\right\}\!\!\right\},\left[\!\!\left[\frac{\partial v_h}{\partial n}\right]\!\!\right])_e \tag{3.36}$$
$$+\sum_{e\in\mathscr{E}_h}(\left\{\!\!\left\{\frac{\partial^2v_h}{\partial n^2}\right\}\!\!\right\},\left[\!\!\left[\frac{\partial u_h}{\partial n}\right]\!\!\right])_e$$
$$+\sum_{e\in\mathscr{E}_h}\frac{\sigma}{h_e}(\left[\!\!\left[\frac{\partial u_h}{\partial n}\right]\!\!\right],\left[\!\!\left[\frac{\partial v_h}{\partial n}\right]\!\!\right])_e-\sum_{T\in\mathscr{T}_h}(D^2u_h,D^2(v-v_h))_T$$
$$+\sum_{T\in\mathscr{T}_h}(D^2(u_h-E_hu_h),D^2v)_T.$$

Now by using integration by parts

$$\sum_{T\in\mathscr{T}_h}(D^2u_h, D^2(v-v_h))_T = \sum_{T\in\mathscr{T}_h}(\Delta^2 u_h, v-v_h)_T - \sum_{e\in\mathscr{E}_h^b}(\frac{\partial^2 u_h}{\partial n^2}, \frac{\partial(v-v_h)}{\partial n})_e + \sum_{e\in\mathscr{E}_h^b}(\frac{\partial u_h}{\partial n}, v-v_h)_e$$
$$= -\sum_{e\in\mathscr{E}_h^b}(\frac{\partial^2 u_h}{\partial n^2}, \frac{\partial(v-v_h)}{\partial n})_e. \tag{3.37}$$

So,

$$\begin{aligned}\sum_{T\in\mathscr{T}_h}(D^2u_h, D^2(v-v_h))_T &= -\sum_{e\in\mathscr{E}_h^b}\int_e \left\{\left\{\frac{\partial^2 u_h}{\partial n^2}\right\}\right\}\left[\left[\frac{\partial(v-v_h)}{\partial n}\right]\right]ds \\ &= \sum_{e\in\mathscr{E}_h^b}\int_e \left\{\left\{\frac{\partial^2 u_h}{\partial n^2}\right\}\right\}\left[\left[\frac{\partial v_h}{\partial n}\right]\right]ds.\end{aligned} \tag{3.38}$$

Then we use (3.38) in (3.36).

Then by Cauchy Schwarz inequality and (3.22) we get

$$\begin{aligned}(f, v-v_h) &\leq \sum_{T\in\mathscr{T}_h}\|f\|_{L^2(T)}\|v-v_h\|_{L^2(T)} \\ &\leq C\sum_{T\in\mathscr{T}_h} h_T^2\|f\|_{L^2(T)}|v|_{H^2(T)} \\ &\leq C\left(\sum_{T\in\mathscr{T}_h}\eta_r^2\right)^{1/2}|v|_{H^2(\Omega)}.\end{aligned} \tag{3.39}$$

First by using Cauchy Schwarz inequality, trace inequality and (3.22) we get

$$\begin{aligned}\sum_{e\in\mathscr{E}_h}\int_e \left\{\left\{\frac{\partial^2 v_h}{\partial n^2}\right\}\right\}\left[\left[\frac{\partial u_h}{\partial n}\right]\right]ds &\leq \left(\sum_{e\in\mathscr{E}_h} h_e\int_e \left\{\left\{\frac{\partial^2 v_h}{\partial n^2}\right\}\right\}^2 ds\right)^{1/2}\left(\sum_{e\in\mathscr{E}_h} h_e^{-1}\int_e \left[\left[\frac{\partial u_h}{\partial n}\right]\right]^2 ds\right)^{1/2} \\ &\leq C\left(\sum_{T\in\mathscr{T}_h}|v_h|^2_{H^2(T)}\right)^{1/2}\left(\sum_{e\in\mathscr{E}_h}\eta_{e,1}^2\right)^{1/2} \\ &\leq C|v|_{H^2(\Omega)}\left(\sum_{e\in\mathscr{E}_h}\eta_{e,1}^2\right)^{1/2}.\end{aligned} \tag{3.40}$$

Then by using trace inequality and (3.23) we get

$$\begin{aligned}
\sum_{e\in\mathscr{E}_h}\frac{\sigma}{h_e}\int_e\left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right]\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right]ds &\le \left(\sum_{e\in\mathscr{E}_h}\int_e\frac{1}{h_e}\int_e\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right]^2 ds\right)^{1/2}\left(\sum_{e\in\mathscr{E}_h}\int_e\frac{\sigma^2}{h_e}\left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right]^2 ds\right)^{1/2}\\
&= \left(\sum_{e\in\mathscr{E}_h}\int_e\frac{1}{h_e}\int_e\left[\!\left[\frac{\partial (v-v_h)}{\partial n}\right]\!\right]^2 ds\right)^{1/2}\left(\sum_{e\in\mathscr{E}_h}\int_e\frac{\sigma^2}{h_e}\left[\!\left[\frac{\partial u_h}{\partial n}\right]\!\right]^2 ds\right)^{1/2}\\
&\le C|v|_{H^2(\Omega)}\left(\sum_{e\in\mathscr{E}_h}\eta_{e,1}^2\right)^{1/2}.
\end{aligned} \tag{3.41}$$

Now by using (3.34) - (3.41), from (3.33) we get

$$\begin{aligned}
(D^2(u-E_hu_h),D^2v)_\Omega &= a_h((u-E_hu_h),v)\\
&\le C|v|_{H^2(\Omega)}\eta_h.
\end{aligned} \tag{3.42}$$

By the definition of norm operator and duality, we can write

$$\|D^2(u-E_hu_h)\|_{L^2(T)} = |u-E_hu_h|_{a_h} = |u-E_hu_h|_{H^2(\Omega)} = \sup_{v\in H_0^2(\Omega)/\{0\}}\frac{a(u-E_hu_h,v)}{|v|_{H^2(\Omega)}}. \tag{3.43}$$

So by combining (3.42) and (3.43) we get

$$\|D^2(u-E_hu_h)\|_{L^2(T)} \le C\eta_h. \tag{3.44}$$

Finally by using (3.29), (3.30), (3.31), (3.32) and (3.44) we get

$$\|u-u_h\|_h \le C\eta_h.$$

□

Chapter 4

Numerical Results

In this Chapter, we will implement the theory for numerical simulations of some biharmonic problems. In all our tests, we have $[-1, 1]^2$ as the domain. The codes are implemented using the C++ software library `deal.II` [4, 3].

4.1 Adaptive refinement strategy

We used the adaptive refinement strategy in our test problems to get a better result with less computations. In this section, we will discuss about the adaptive algorithm and the technique we are applying in our codes. This algorithm is discussed in detail in [20].
The implemented adaptive algorithm is,

$$\textbf{SOLVE} \Rightarrow \textbf{ESTIMATE} \Rightarrow \textbf{MARK} \Rightarrow \textbf{REFINE}$$

Let $\mathscr{T}_0$ be the initial mesh and $\{\mathscr{T}_i\}_i$ be the sequence of meshes after using the adaptive refinement.

Solve:

In first step, we compute a finite element approximation u_h of u. For this we use the `deal.II` and UMFPACK package.

Estimate:

In this step, we consider the residual type a posteriori error estimator that we introduced in Chapter 3 as follows:

$$\eta_h = (\sum_{T\in\mathscr{T}_h} \eta_T^2 + \sum_{e\in\mathscr{E}_h} \eta_{e,1}^2 + \sum_{e\in\mathscr{E}_h^i} \eta_{e,2}^2)^{1/2}.$$

Mark:

In this step, we mark the element which needs more refinement based on the value of our local estimator. We are using the Dörfler marking strategy for refinement. Let $\theta \in (0,1)$; and $\mathbb{S}_1$ be the set of elements $T \in \mathscr{T}_h$, $\mathbb{S}_2$ be the set of edges of $e \in \mathscr{E}_h$ satisfying:

$$\theta\eta_h \leq \tilde{\eta_h}\ ,$$

where

$$\tilde{\eta_h} = (\sum_{T\in\mathbb{S}_1} \eta_T^2 + \sum_{e\in\mathbb{S}_2}(\eta_{e,1}^2 + \eta_{e,2}^2))^{1/2}.$$

Let $\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2$, where $\mathbb{S}$ is our set of all the marked elements and edges.

Refine: We already have our marked element, so we will refine those marked elements in this step. Since in our case, we have quadrilateral meshes, for refining first we choose the midpoint of the edges and then by connecting the midpoints, we get the desired refinements. We assume the shape regularity in our refinement. Since, it is impossible to avoid the occurrence of hanging nodes in our refinement, we impose a restriction to the hanging nodes that we will have one irregular mesh in our refinement. We are defining one irregular mesh as follows: If $L(T)$ is the level of the mesh refinement, we need to achieve $T \in \mathscr{T}_k$ from $\mathscr{T}_0$, then

$$|L(T_1) - L(T_2)| \leq 1$$

where T_1 and T_2 are any two neighboring cells.

The order of convergence we used in all our problems were computed by $\frac{log(|e_{n-1}/e_n|)}{log(|DOF_n/DOF_{n-1}|)}$ through the inbuilt `deal.II` function reduction_rate and here DOF_n is the total dofs at n refinement cycle. Also, in all our problems we used $\sigma = 5$.

4.2 First Example

The exact solution in this test, $u = x^2y^2(1-x)^2(1-y)^2$.

$$\begin{aligned} f &= 24x^4 - 48x^3 + 288x^2y^2 - 288x^2y + 72x^2 - 288xy^2 + 288xy - 48x + 24y^4 \\ &\quad - 48y^3 + 72y^2 - 48y + 8 \qquad \text{in } \Omega \\ u &= 0 \text{ and } \frac{\partial u}{\partial n} = 0 \qquad \text{in } \partial\Omega. \end{aligned}$$

Here in the beginning, we solve on a sequence of uniform mesh. We also compute the error and the error estimator for each solution. Then we use the adaptive refinement strategy over the mesh and compute the solution.

Table 4.1: Error and error estimator with uniform refinement($\theta = 1$).

cycle	DOF	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{a_h}$	$\frac{\sigma}{h_e}\|[\frac{\partial(u-u_h)}{\partial n}]\|_{L^2}^2$	$\|u-u_h\|_h$	η_h	order of convergence
0	2.500e+01	7.55769e-4	5.3891e-2	1.80116e-2	7.19026e-2	1.19855	
1	7.100e+01	4.26237e-4	3.66107e-2	1.10516e-2	4.76623e-2	6.64661e-1	3.704e-01
2	2.380e+02	1.46996e-4	2.07531e-2	6.58638e-3	2.73395e-2	1.8458e-1	4.693e-01
3	8.600e+02	4.3798e-05	1.04896e-2	2.46298e-3	1.29526e-2	4.95272e-2	5.311e-01
4	3.256e+03	1.23105e-05	5.2146e-3	9.05627e-4	6.12022e-3	1.33755e-2	5.250e-01
5	1.266e+04	3.28333e-06	2.5937e-3	3.27156e-4	2.92086e-3	3.73712e-3	5.144e-01

Table 4.2: Error and error estimator when $\theta = 0.5$.

cycle	DOF	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{a_h}$	$\frac{\sigma}{h_e}\|[\frac{\partial(u-u_h)}{\partial n}]\|^2_{L^2}$	$\|u-u_h\|_h$	η_h	order of convergence
0	2.5e+01	7.55769e-4	5.3891e-2	1.80116e-2	7.19026e-2	1.19855	
1	5.7e+01	5.50414e-4	4.32259e-2	1.39326e-2	5.71586e-2	8.80823e-1	2.676e-01
2	8.100e+01	2.67979e-4	2.8262e-2	6.91224e-3	3.51742e-2	3.22948e-1	1.209e+00
3	1.810e+02	1.74019e-4	2.86074e-2	1.38693e-2	4.24767e-2	1.85635e-1	-1.511e-02
4	2.820e+02	9.78429e-05	0.0167289	0.00573779	0.0224667	0.0971769	1.210e+00
5	5.490e+02	6.83163e-05	1.40996e-2	5.04965e-3	1.91493e-2	5.4851e-2	2.567e-01
6	9.230e+02	3.18476e-05	1.06616e-2	4.38191e-3	1.50435e-2	3.03312e-2	5.380e-01
7	1.697e+03	2.12895e-05	6.9985e-3	1.45563e-3	8.45413e-3	1.74069e-2	6.912e-01
8	3.013e+03	1.26316e-05	0.00588924	2.60342e-3	8.49266e-3	1.01303e-2	3.006e-01

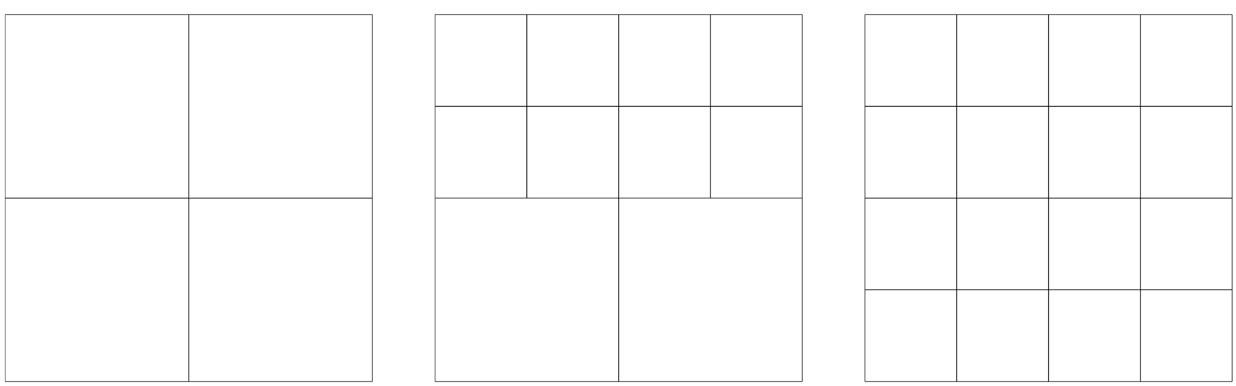


Figure 4.1: Adaptive Meshes: cycle 0,1 and 2 when $\theta = 0.5$.

Remark: From Table 4.3 we see that after cycle 0 of refinement, there is a symmetry with respect to the calculations of the cell and edge errors. However, this is not reflected in the numerical realization of the marking step within the software deal.II. For example, if we change θ to 0.25, we notice that only top right element gets mark in spite of the error indicators being same as for $\theta = 0.5$. The tables below presents the symmetric values of the indicators when $\theta = 0.5$ (for Figure 4.1) and $\theta = 0.25$.

Table 4.3: Error indicators when $\theta = 0.5$.

cycle	cell number	cell error indicator	edge error indicator
0	0	0.59493	0.0043457
	1	0.59493	0.0043457
	2	0.59493	0.0043457
	3	0.59493	0.0043457

Table 4.4: Error indicators when $\theta = 0.25$.

cycle	cell number	cell error indicator	edge error indicator
0	0	0.59493	0.0043457
	1	0.59493	0.0043457
	2	0.59493	0.0043457
	3	0.59493	0.0043457

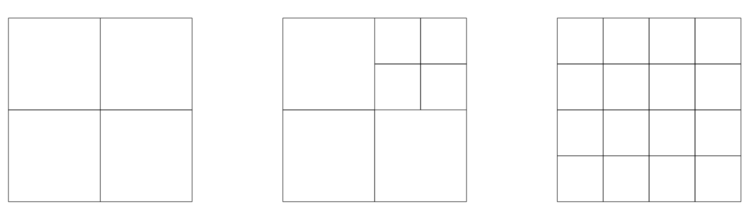


Figure 4.2: Adaptive Meshes: cycle 0,1 and 2 when $\theta = 0.25$.

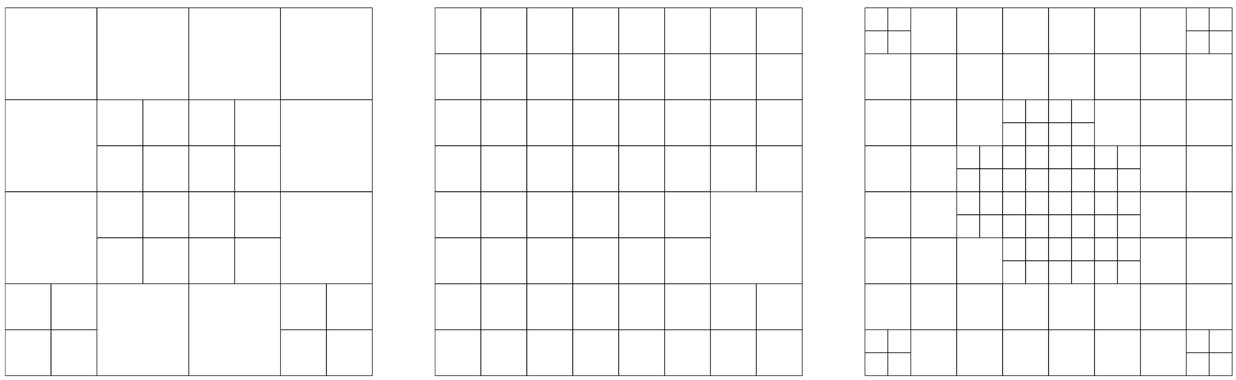


Figure 4.3: Adaptive Meshes: cycle 3,4 and 5 when $\theta = 0.5$.

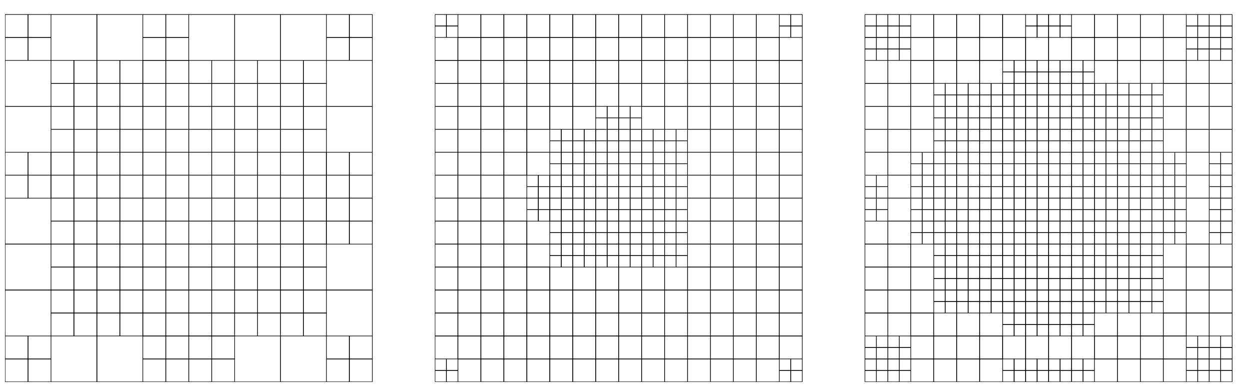


Figure 4.4: Adaptive Meshes: cycle 6,7 and 8 when $\theta = 0.5$.

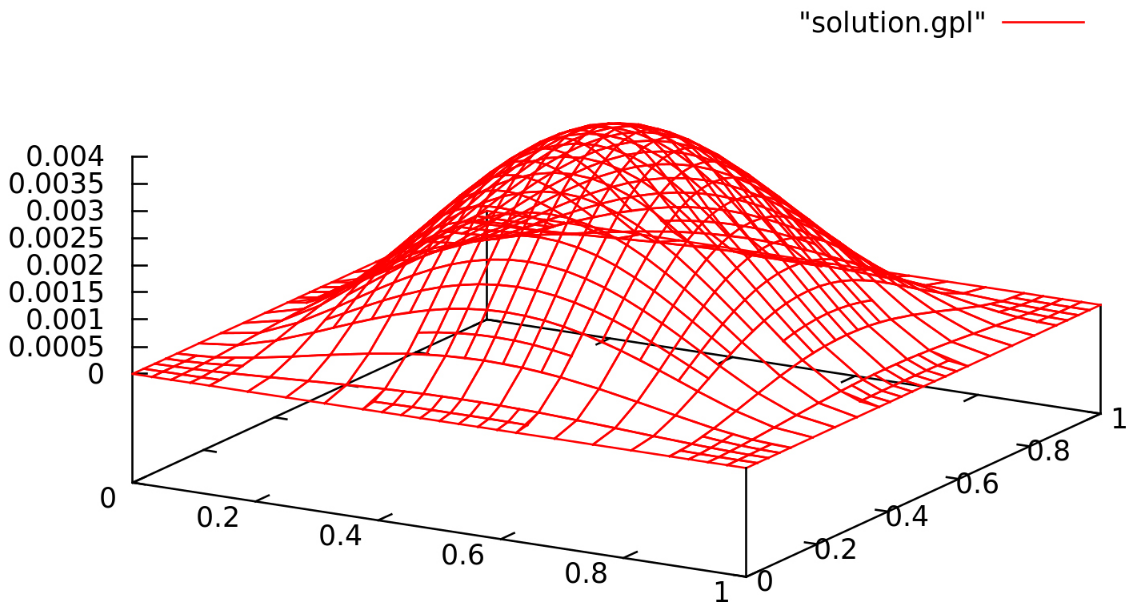


Figure 4.5: Computed solution.

4.3 Second Example

The exact solution in this test, $u = (\sin(\pi x))^2(\sin(\pi y))^2$.

$$f = 8\pi^4(8\sin(\pi x)^2\sin(\pi y)^2 - 3\sin(\pi x)^2 - 3\sin(\pi y)^2 + 1) \qquad \text{in } \Omega$$

$$u = 0 \text{ and } \frac{\partial u}{\partial n} = 0 \qquad \text{in } \partial\Omega.$$

Here in the beginning, we solve on a sequence of uniform mesh. We also compute the error and the error estimator for each solution. Then we use the adaptive refinement strategy over the mesh and compute the solution.

Table 4.5: Error and error estimator with uniform refinement($\theta = 1$).

cycle	DOF	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{a_h}$	$\frac{\sigma}{h_e}\|[\frac{\partial(u-u_h)}{\partial n}]\|_{L^2}^2$	$\|u-u_h\|_h$	η_h	order of convergence
0	2.5e+1	2.15994e-1	1.31342e+1	5.79096	1.89252e+1	4.299e+2	
1	7.1e+1	1.08184e-1	8.15743	3.18353	1.1341e+1	2.380e+2	4.563e-01
2	2.35e+2	2.49479e-2	3.66784	9.17189e-1	4.58503	6.488e+1	6.678e-01
3	8.57e+2	6.54796e-3	1.8497	4.03704e-1	2.2534	1.675e+1	5.291e-01
4	3.253e+3	1.67022e-3	9.23527e-1	1.36535e-1	1.06006	4.37136	5.207e-01
5	1.2650e+4	4.20289e-4	4.60958e-1	4.62886e-2	5.07246e-1	1.16156	5.117e-01
6	4.9876e+4	1.05546e-4	2.30242e-1	1.56236e-2	2.45866e-1	3.159e-1	5.060e-01

Table 4.6: Error and error estimator when $\theta = 0.9$.

cycle	DOF	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{a_h}$	$\frac{\sigma}{h_e}\|[\frac{\partial(u-u_h)}{\partial n}]\|_{L^2}^2$	$\|u-u_h\|_h$	η_h	order of convergence
0	2.5e+1	2.159e-1	1.313e+1	5.7909	1.8925e+1	4.299e+2	
1	7.1e+1	1.081e-1	8.15743	3.18353	1.1341e+1	2.380e+2	4.563e-01
2	2.15e+2	2.4746e-2	3.78145	9.6247e-1	4.74392	6.6528e+1	6.939e-01
3	6.61e+2	6.4083e-3	2.50836	1.64082	4.14917	1.8552e+1	3.655e-01
4	1.941e+3	2.5264e-3	1.26763	5.2268e-1	1.79032	5.39464	6.336e-01
5	6.061e+3	8.136e-4	6.5816e-1	1.6848e-1	8.2665e-1	1.63371	5.756e-01
6	1.8749e+4	2.168e-4	3.4467e-1	8.0122e-2	4.24797e-1	5.0834e-1	5.728e-01
7	5.9001e+4	6.028e-05	1.7621e-1	2.6047e-2	2.0226e-1	1.6418e-1	5.852e-01
8	1.99261e+5	1.762e-05	9.883e-2	2.802e-2	1.2686e-1	5.684e-2	4.751e-01

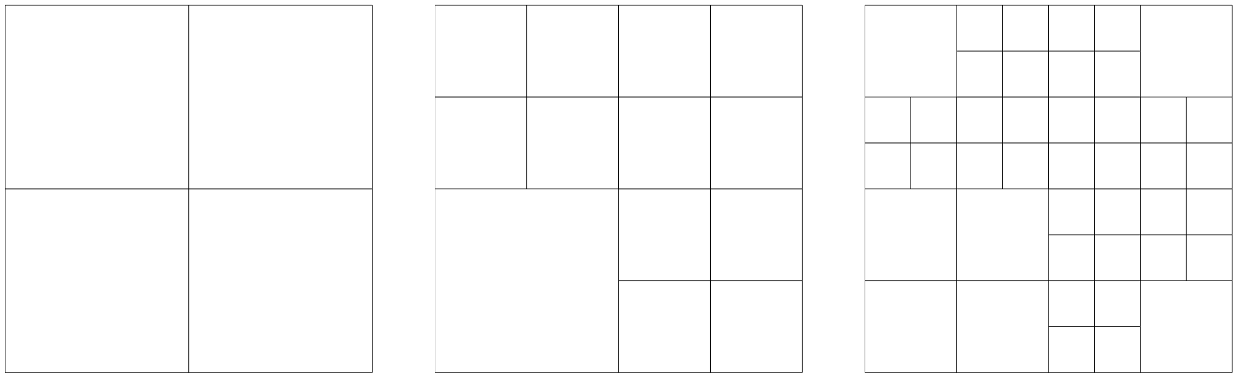


Figure 4.6: Adaptive Meshes: cycle 0,1 and 2 when $\theta = 0.9$.

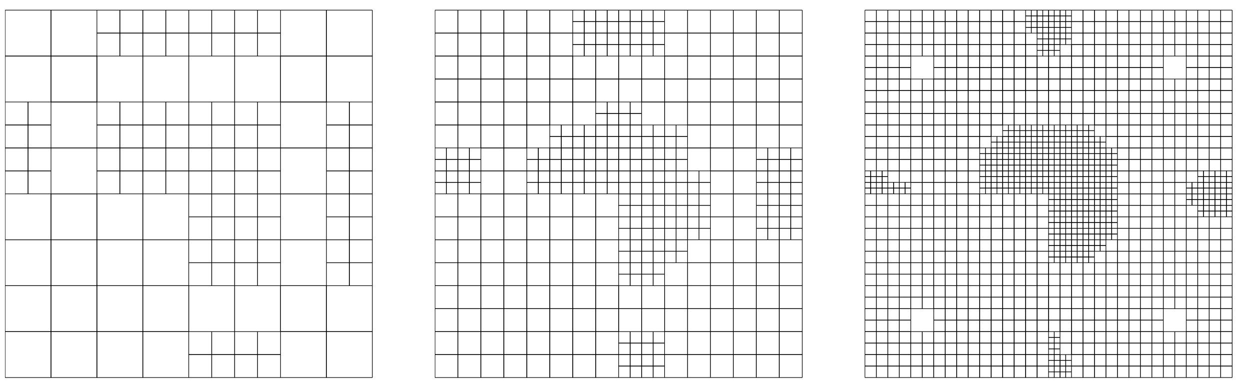


Figure 4.7: Adaptive Meshes: cycle 3,4 and 5 when $\theta = 0.9$.

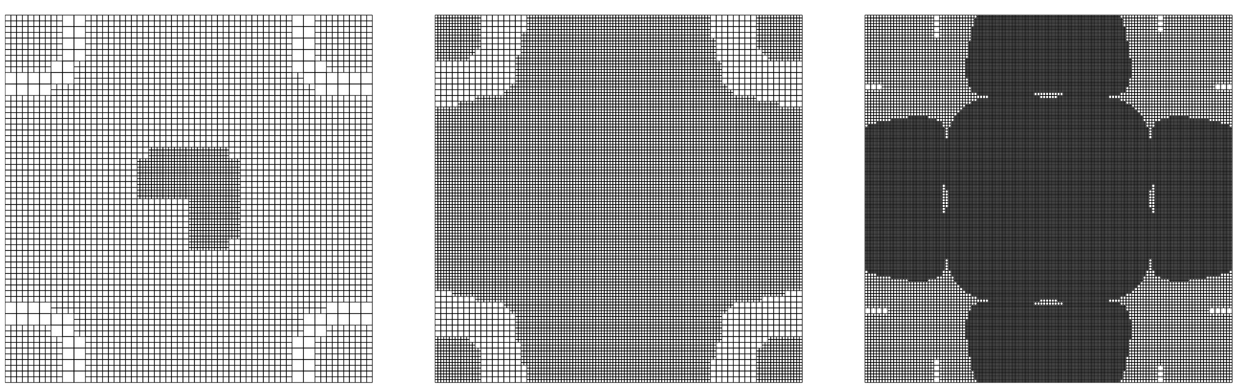


Figure 4.8: Adaptive Meshes: cycle 6,7 and 8 when $\theta = 0.9$.

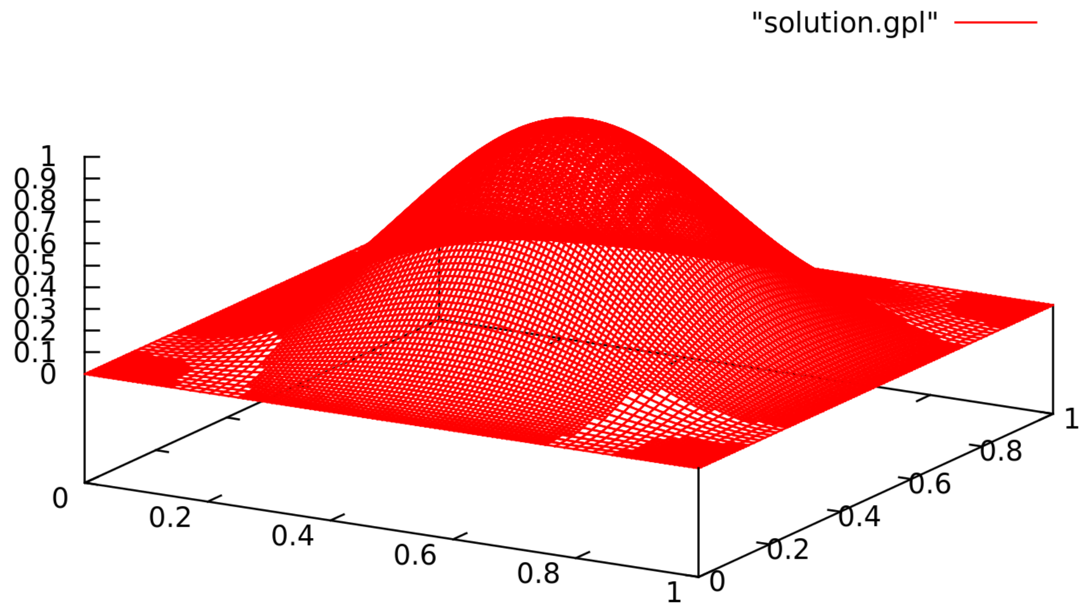


Figure 4.9: Computed solution.

4.4 Third Example

The exact solution in this test, $u = (1 - \cos(2\pi x))(1 - \cos(2\pi y))$.

$$f = -16\pi^4(\cos(2\pi x) + \cos(2\pi y) - 4\cos(2\pi x)\cos(2\pi y)) \qquad \text{in } \Omega$$
$$u = 0 \text{ and } \frac{\partial u}{\partial n} = 0 \qquad \text{in } \partial\Omega.$$

Here in the beginning, we solve on a sequence of uniform mesh. We also compute the error and the error estimator for each solution. Then we use the adaptive refinement strategy over the mesh and compute the solution.

Table 4.7: Error and error estimator with uniform refinement($\theta = 1$).

cycle	DOF	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{a_h}$	$\frac{\sigma}{h_e}\|[\frac{\partial(u-u_h)}{\partial n}]\|_{L^2}^2$	$\|u-u_h\|_h$	η_h	order of convergence
0	2.5e+1	8.639e-1	5.253e+1	2.316e+1	7.570e+1	1.719e+3	
1	7.1e+1	4.327e-1	3.262e+1	1.273e+1	4.53638e+1	9.523e+2	4.563e-01
2	2.35e+2	9.979e-2	1.467e+1	3.66875	1.834e+1	2.595e+2	6.678e-01
3	8.57e+2	2.619e-2	7.3988	1.61481	9.01361	6.702e+1	5.291e-01
4	3.253e+3	6.6809e-3	3.69411	5.461e-1	4.24025	1.748e+1	5.207e-01
5	1.2650e+4	1.68116e-3	1.84383	1.851e-1	2.02899	4.64625	5.117e-01
6	4.9876e+4	4.2218e-4	9.2097e-1	6.2494e-2	9.83464e-1	1.26381	5.060e-01

Table 4.8: Error and error estimator when $\theta = 0.9$.

cycle	DOF	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{a_h}$	$\frac{\sigma}{h_e}\|[\frac{\partial(u-u_h)}{\partial n}]\|_{L^2}^2$	$\|u-u_h\|_h$	η_h	order of convergence
0	2.5e+1	8.639e-1	5.253e+1	2.316e+1	7.5700e+1	1.719e+3	
1	7.1e+1	4.327e-1	3.262e+1	1.273e+1	4.536e+1	9.523e+2	4.563e-01
2	2.15e+2	9.898e-2	1.512e+1	3.84988	1.897e+1	2.661e+2	6.939e-01
3	6.61e+2	2.56335e-2	1.0033e+1	6.56327	1.6596e+1	7.421e+1	3.655e-01
4	1.941e+3	1.01058e-2	5.07053	2.09075	7.16129	2.157e+1	6.336e-01
5	6.061e+3	3.25476e-3	2.63267	6.739e-1	3.30662	6.53482	5.756e-01
6	1.8749e+4	8.675e-4	1.3787	3.204e-1	1.69919	2.03338	5.728e-01
7	5.9001e+4	2.4115e-4	7.0484e-1	1.04191e-1	8.0904e-1	6.5672e-1	5.852e-01
8	1.99261e+5	7.04851e-05	3.95348e-1	1.121e-1	5.074e-1	2.273e-1	4.751e-01

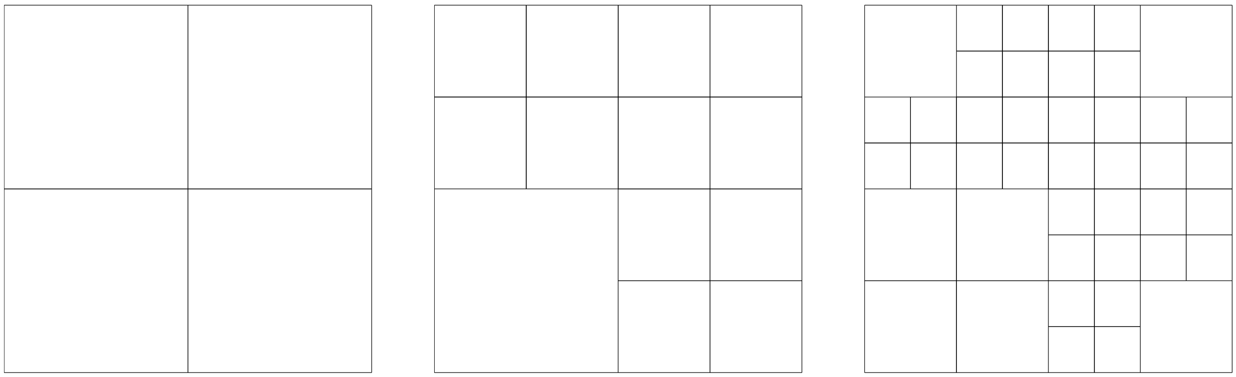


Figure 4.10: Adaptive Meshes: cycle 0,1 and 2 when $\theta = 0.9$.

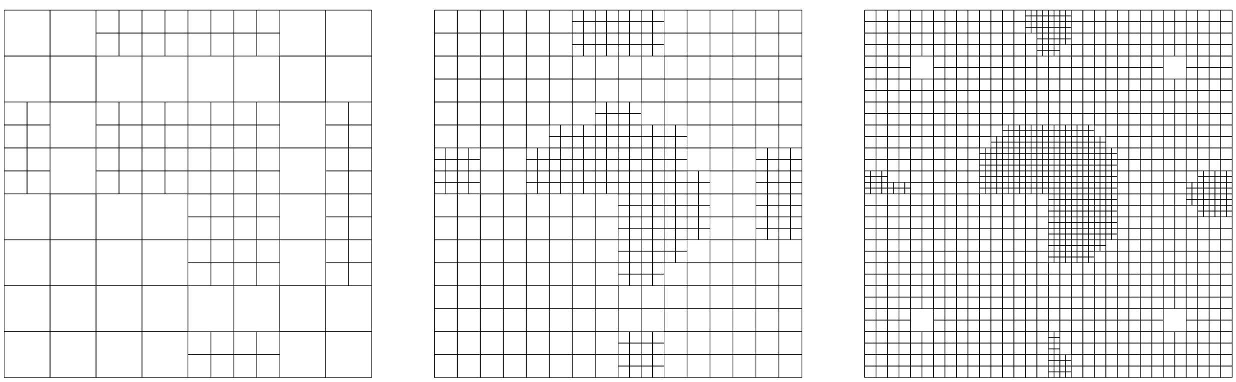


Figure 4.11: Adaptive Meshes: cycle 3,4 and 5 when $\theta = 0.9$.

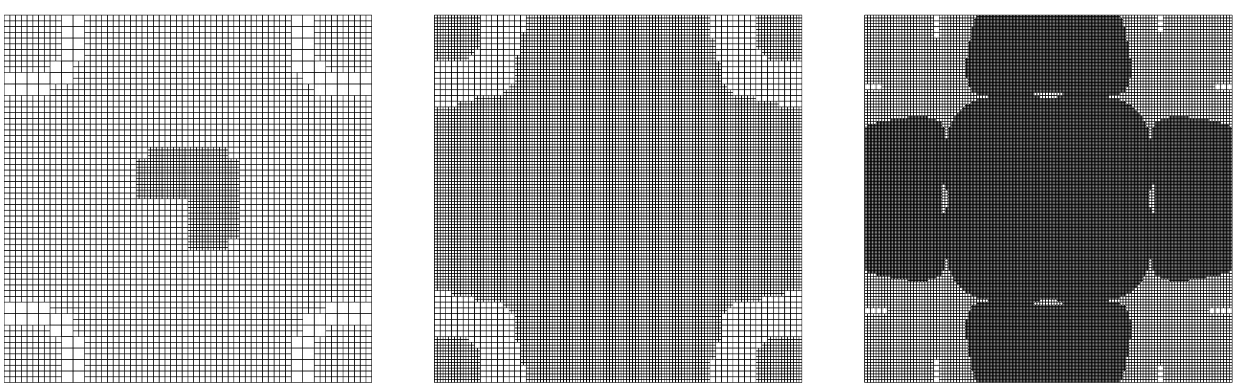


Figure 4.12: Adaptive Meshes: cycle 6,7 and 8 when $\theta = 0.9$.

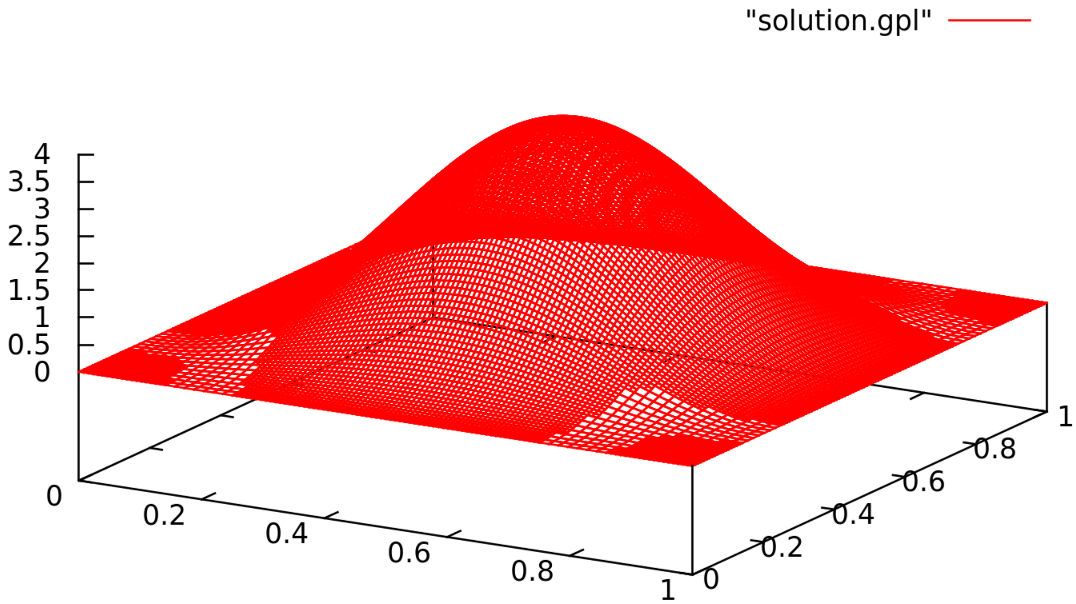


Figure 4.13: Computed solution.

4.5 Fourth Example

The exact solution in this test, $u = \sin(\pi x)\sin(\pi y)$.

$$f = 4\pi^4 \sin(\pi x)\sin(\pi y) \qquad \text{in } \Omega$$
$$u = 0 \text{ and } \frac{\partial u}{\partial n} = 0 \qquad \text{in } \partial\Omega.$$

Here in the beginning, we solve on a sequence of uniform mesh. We also compute the error and the error estimator for each solution. Then we use the adaptive refinement strategy over the mesh and compute the solution.

Table 4.9: Error and error estimator with uniform refinement($\theta = 1$).

cycle	DOF	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{a_h}$	$\frac{\sigma}{h_e}\|[\frac{\partial(u-u_h)}{\partial n}]\|_{L^2}^2$	$\|u-u_h\|_h$	η_h	order of convergence
0	2.5e+1	4.796e-2	3.24431	1.25228	4.4966	9.801e+1	
1	7.1e+1	2.599e-2	2.1523	7.288e-1	2.88115	5.417e+1	3.931e-01
2	2.35e+2	8.09572e-3	1.11155	3.225e-1	1.43411	1.458e+1	5.521e-01
3	8.51e+2	2.274e-3	5.511e-1	1.319e-1	6.8311e-1	3.87275	5.451e-01
4	3.235e+3	6.0866e-4	2.729e-1	4.532e-2	3.182e-1	1.0353	5.264e-01
5	1.2610e+4	1.57845e-4	1.358e-1	1.514e-2	1.5098e-1	2.817e-1	5.128e-01
6	4.9792e+4	4.02311e-05	6.777e-2	5.0697e-3	7.284e-2	7.868e-2	5.063e-01

Table 4.10: Error and error estimator when $\theta = 0.7$.

cycle	dof	$\|u-u_h\|_{L^2}$	$\|u-u_h\|_{a_h}$	$\frac{\sigma}{h_e}\|[\frac{\partial(u-u_h)}{\partial n}]\|_{L^2}^2$	$\|u-u_h\|_h$	η_h	order of convergence
0	2.5e+1	4.796e-2	3.24431	1.25228	4.4966	9.801e+1	
1	7.1e+1	2.599e-2	2.1523	7.288e-1	2.88115	5.417e+1	3.931e-01
2	1.48e+2	9.755e-2	1.66157	1.03733	2.6989	1.867e+1	3.523e-01
3	2.79e+2	5.101e-3	8.63957e-1	2.6126e-1	1.12522	7.27367	1.032e+00
4	6.61e+2	2.0775e-3	7.9018e-1	5.467e-1	1.33692	3.3365	1.035e-01
5	1.633e+3	1.2063e-3	4.0901e-1	1.5066e-1	5.59686e-1	1.43803	7.281e-01
6	3.301e+3	4.01704e-4	3.1335e-1	2.0929e-1	5.22649e-1	5.8565e-1	3.786e-01
7	8.849e+3	2.62227e-4	1.8651e-1	7.58423e-2	2.6236e-1	2.8701e-1	5.261e-01
8	2.1457e+4	8.28842e-05	1.0705e-1	4.007e-2	1.4713e-1	1.2285e-1	6.268e-01

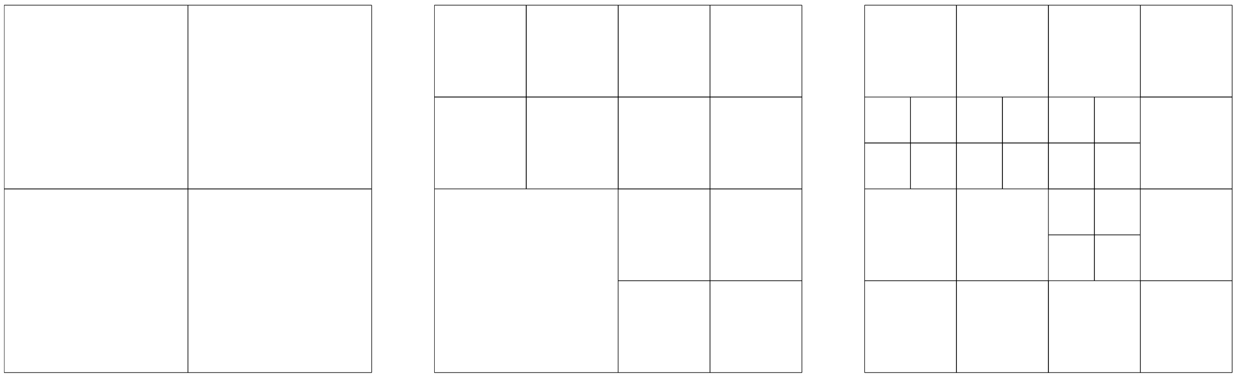


Figure 4.14: Adaptive Meshes: cycle 0,1 and 2 when $\theta = 0.7$.

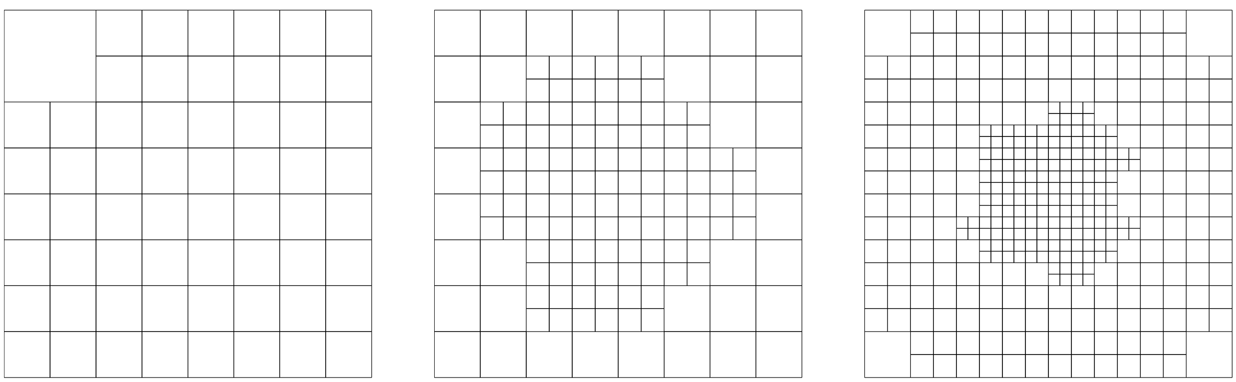


Figure 4.15: Adaptive Meshes: cycle 3,4 and 5 when $\theta = 0.7$.

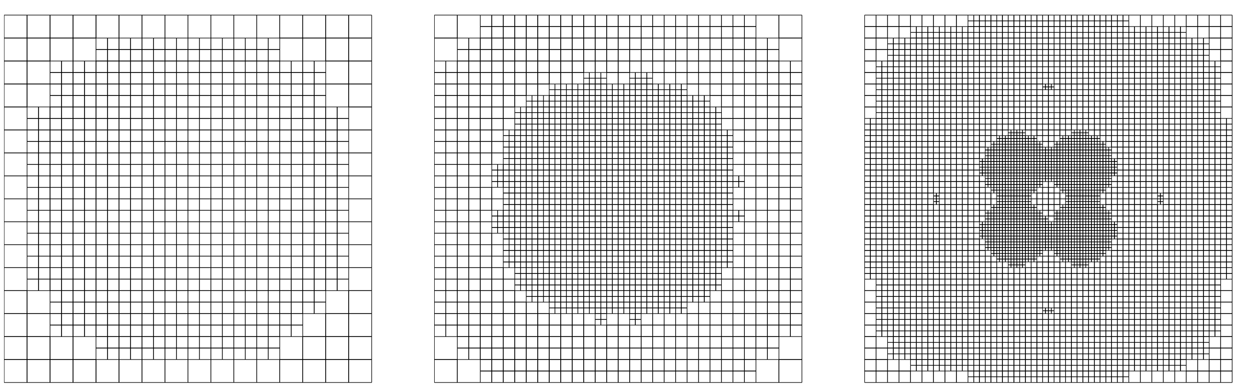


Figure 4.16: Adaptive Meshes: cycle 6,7 and 8 when $\theta = 0.7$.

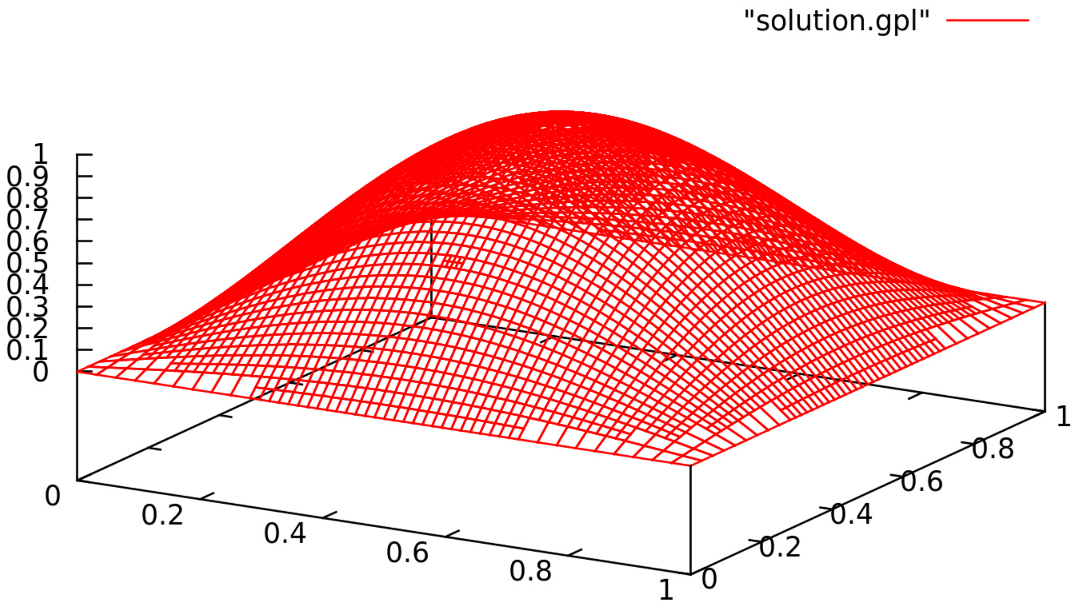


Figure 4.17: Computed solution.

In the above test examples, we observed the reliability of the estimator upto a certain threshold of DOFS $\approx 10^5$, except for the last example where the threshold is 10^4. Beyond this threshold, the discretization error dominates the estimator. This is possibly because h_T becomes very small.

Chapter 5

Conclusion and Future Work

5.1 Conclusion

In this thesis, we introduced and discussed a linear biharmonic boundary value problem. Then, we developed Interior Penalty Discontinuous Galerkin (IPDG) or C^0 interior penalty method for our fourth order problem. Rigorous analysis on H^2 conformity, continuity and V-ellipticity is obtained. A reliable residual based a posteriori error estimator is obtained and for our computations, we considered an adaptive mesh refinement. Finally, we demonstrated our analysis by numerical calculations for some selected functions.

5.2 Future Work

As we have already mentioned, in this thesis, we did our analysis of the linear fourth order problem. We will continue to develop C^0 interior penalty method and error analysis of a nonlinear fourth order problem.

In my thesis work, we spent some time on numerical calculations with the following problem. Let the Ω be the L-shaped domain $(-1,1)^2$ $([0,1)\times(-1,0])$ with the exact singular solution

$$u(r,\theta) = (r^2\cos^2\theta - 1)^2(r^2\sin^2\theta - 1)^2 r^{(1+z)} g(\theta) \tag{5.1}$$

where $z = 0.5444837367$ is a non characteristic root of $\sin^2(z\omega) = z^2 \sin^2(z\omega), \omega = \frac{3\pi}{2}$ and

$$
\begin{aligned}
g(\theta) = & \left[\frac{1}{z-1} \sin((z-1)\omega) - \frac{1}{z+1} \sin((z+1)\omega) \right] \\
& \times [\cos((z-1)\theta - \cos((z+1)\theta] \\
& - \left[\frac{1}{z-1} \sin((z-1)\theta) - \frac{1}{z+1} \sin((z+1)\theta) \right] \\
& \times [\cos((z-1)\omega - \cos((z+1)\omega]
\end{aligned}
$$

This is a benchmark problem that has a singularity and we would like to apply the adaptive algorithm to this problem.

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Curriculum Vitae

Mohammad Arifur Rahman was born on July 12, 1989. The youngest of all the three children of Rafiq Ullah and Aleya Begum, he completed his Bachelor of science in Mathematics from University of Dhaka, one of the best educational institute for higher education in Bangladesh in 2011.

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