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Towards Unique Physically Meaningful Definitions of Random and Typical Objects

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TOWARDS UNIQUE PHYSICALLY MEANINGFUL DEFINITIONS OF RANDOM AND TYPICAL OBJECTS

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To distinguish between random and non-random sequence, Kolmogorov and Martin-Löf proposed a new definition of randomness, according to which an object (e.g., a sequence of 0s and 1s) if random if it satisfies all probability laws, i.e., in more precise terms, if it does not belong to any definable set of probability measure 0. This definition reflect the usual physicists' idea that events with probability 0 cannot happen. Physicists – especially in statistical physics – often claim a stronger statement: that events with a very small probability cannot happen either. A modification of Kolmogorov-Martin-Löf's (KLM) definition has been proposed to capture this physicists' claim. The problem is that, in contrast to the original KLM definition, the resulting definition of randomness is not uniquely determined by the probability measure: for the same probability measure, we can have several different definitions of randomness. In this paper, we show that while it is not possible to define, e.g., a unique *set* $\mathcal{R}$ of random objects, we can define a unique *sequence* $\mathcal{R}_n$ of such sets (unique in some reasonable sense).

1. Definitions of Random and Typical Objects: Reminder

Some sequences are random, some are not. Traditional probability theory does not allow us to distinguish between random and non-random numbers, sequences, etc., it only allows us to determine the probability that, e.g., a random sequence of 0s and 1s – corresponding to flipping a fair coin – belongs to a given set of sequences. However, from the physical viewpoint, some sequences of 0s and 1s are random – in the sense that they can appear as a result of the actual coin tosses – while other cannot. For example, it is intuitively clear that a periodic sequence 0101. . . cannot result from the actual coin tosses.

Kolmogorov-Martin-Löf randomness. To formalize the above intuitive notion, A. N. Kolmogorov and P. Martin-Löf proposed the following idea. How do we know that a sequence is not random? For example, a sequence 00. . . (consisting of all 0s) is not random because we know that:

- for almost all sequence, the proportion of 0s among the first n symbols tends to $1/2$ as $n \to \infty$, while

- for the sequence $00\ldots$, the proportion of 0s is equal to 1.

Similarly, a periodic sequence $0101\ldots$ is not random because:

- for almost all sequence, the frequency of 11 tends to $1/4$, while

- for the periodic sequence $0101\ldots$, this frequency is 0.

Frequency limit is a simple case, we may have more complex reasons why a given sequence is not random.

In general, it is reasonable to say that a sequence is random if it satisfies all the *probability laws*, i.e., all the statements (defined in a certain language L) which are true for almost all sequences. To be more precise, a probability law on the set X of all sequences is an L-definable subset $S \subseteq X$ for which $P(S) = 1$ – or, equivalently, for whose (similarly definable) complement $-S$, we have $P(-S) = 0$. So, we arrive at the following definition.

Definition 1. We can say that an object $x \in X$ is *random* in the sense of Kolmogorov-Martin-Löf if it does not belong to any L-definable set of probability 0.

Remark 1. Definable simply means that there is a formula in the corresponding language that uniquely determines the object. For example, in the usual theory of real numbers, 1, 2, π, $\sqrt{2}$, e, any number that can we thing of is definable. This does not means, of course, that every real number if definable: in each language L, there are no more countably many words, so there are no more than countably many L-definable real numbers. Thus, there are continuum many real numbers which are not definable.

The notion of an L-definable object cannot be formally described within the language L itself. The proof of this impossibility is a simple modification of the known Berry paradox [1]: if we could describe definability within L, we could, in particular, define the smallest integer that cannot be described by fewer than 100 words. By definition, this integer cannot be described by fewer than 100 words, but the above description "the smallest integer that cannot be described by fewer than 100 words" is a description of this integer in fewer than 100 words – a contradiction.

Thus, to formally talk about L-definability, we need to use a stronger language L_1 in which L-definability can be formally described. For each language L, the existence of such a stronger language L_1 can be easily proven; see, e.g., [2].

Almost all sequences are Kolmogorov-Martin-Löf randomness. As we have just mentioned, in each language L, there are no more countably many words, so there are no more than countably many L-definable sets of probability 0. Hence, to describe all sequence which are Kolmogorov-Martin-Löf random, we delete, from the original set X of probability measure 1, a union of countably many L-definable sets of measure 0. A union of countably many set of measure 0 is also of measure 0, so almost all sequences are random in this sense.

Towards a more physically adequate notion of randomness. Kolmogorov-Martin-Löf definition prevents sequences 00... from being called random. This prevention makes perfect physical sense: if we flip a coin and every time get tail, this clearly is not a fair coin.

However, one can easily check that for each random sequence ω, a sequence $0...0$ (1,000,000 times) followed by ω is also random in the sense of Kolmogorov-Martin-Löf. This is not physically meaningful: it is clearly not realistically possible to toss a fair coin million times and get tail every time. Physicists usually argue that this situation is not physical because its probability is too low – in the above example, this probability is equal to $2^{-1,000,000}$. This argument is behind the usual applications of statistical physics: e.g., from the purely mathematical viewpoint, it is possible that all the randomly moving molecules of a human body will start going into the same vertical direction, and the person will float on air – but the probability of this event is so small, then it is not physically possible.

This idea cannot be described by simply setting a small threshold $p_0 \ll 1$ and claiming that no event with a probability $\leq p_0$ is possible: indeed, for a sequence of n coin tosses, every sequence of 0s and 1s has the same probability 2^{-n}; so, for a sufficiently large n, for which $2^{-n} \leq p_0$, we would arrive at a conclusion that no such sequence is possible at all. In other words, the threshold p_0 is not a universal constant, it depends on the property.

In particular, if we are looking for the impossibility to have too many 0s at the beginning, then we have a sequence of sets

$$A_n = \{\omega \text{ starts with } n \text{ zeroes}\} = \{\omega : \omega_1 = \ldots = \omega_n = 0\}$$

for which $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ and $P(A_n) \to 0$ as $n \to \infty$. In general, we have a definable sequence of sets A_n for which $A_n \supseteq A_{n+1}$ and $P(A_n) \to 0$. Our claim is that for each such sequence, there is an N for which $P(A_N)$ is so small that a truly random sequence cannot belong to the set A_N. In other words, we arrive at the following definition of the class $\mathcal{R}$ of random sequences:

Definition 2. We say that a non-empty set $\mathcal{R} \subseteq X$ is *a set of random objects* if for every definable sequence A_n for which $A_n \supseteq A_{n+1}$ and $P(A_n) \to 0$, there exists an N for which $A_N \cap \mathcal{R} = \emptyset$.

Discussion. The existence of such a set follows if we consider a language in which L-definability can be explicitly described; see, e.g., [2]. It turns out that, in contrast to the Kolmogorov-Martin-Löf definition, we cannot have a set of random elements $\mathcal{R}$ for which $P(\mathcal{R}) = 1$. However, we can have a set of random elements which is "almost" of measure 1 [2]. To formulate this result, let us recall that in the continuous case, not every set is measurable, but for every set S, we can define:

- an *inner measure* $\underline{P}(S)$, defined as the supremum of all the values $P(S')$ for all measurable sets $S' \subseteq S$, and

- an *outer measure* $\overline{P}(S)$, defined as the infimum of all the values $P(S')$ for all measurable sets $S' \supseteq S$.

The existence result is that for every $\varepsilon > 0$, we can have a set of random elements $\mathcal{R}$ for which $\underline{P}(\mathcal{R}) \geq 1 - \varepsilon$.

From random to typical. In the above examples, we assume that we know the probability distribution on the set of all possible objects. In some cases, physicists talk about "typical" objects even when no such probability distribution is known. For example, they argue that since "almost all" cosmological solutions of General Relativity theory have a certain asymptotic behavior, the actual University must exhibit the same behavior; see, e.g., [3]. The possibility of such conclusions comes from the fact that if $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$, then $P(A_n) \to 0$ no matter what the probability distribution P is. Thus, we arrive at the following definition:

Definition 3. We say that a non-empty set $\mathcal{T} \subseteq X$ is *a set of typical objects* if for every definable sequence A_n for which $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$, there exists an N for which $A_N \cap \mathcal{T} = \emptyset$.

2. Formulation of the Problem

In contrast to the Kolmogorov-Martin-Löf definition, the new notions are not uniquely determined. The problem dealt with in this paper comes from the fact that the above physically adequate notions of random and typical objects are not uniquely determined.

Indeed, in the Kolmogorov-Martin-Löf's definition, once we know the probability measure P and fix the language L, we can tell which objects are random and which are not. In contrast, we may have several different sets $\mathcal{R}$ and $\mathcal{T}$ that satisfy Definitions 2 and 3.

Let us show that this non-uniqueness is inevitable. Indeed, let us consider a simple situation when we have a uniform probability distribution P on a unit disc

$$D = \{(x, y) : x^2 + y^2 \leq 1\}.$$

This probability distribution is invariant w.r.t. arbitrary rotations around the disk's center $(0, 0)$. So, any definable rotation preserves the situation. Thus, if we had a meaningful way to select a unique set $\mathcal{R}$, this selection would not change under such rotations. It turns out that such an invariance is not possible:

Предложение 1. *Let X be a unit disc D with a uniform distribution P, and let $\mathcal{P} \subseteq X$ be a set of random objects. Then, there exists a definable rotation under which the set $\mathcal{P}$ is not invariant.*

Remark 2. For readers' convenience, all the proofs are placed in the special (last) Proofs section.

Possible reason for non-uniqueness. A possible reason for non-uniqueness is that, as we have mentioned earlier,

- while we cannot have a set of random elements $\mathcal{R}$ with $P(\mathcal{R}) = 1$,

- for every $\varepsilon > 0$, we can have a set of random elements $\mathcal{R}$ for which $\underline{P}(\mathcal{R}) \geq 1 - \varepsilon$.

Thus, to capture the general intuitive meaning of randomness, it is not enough to have *one* set of random elements, it is more appropriate to have a *sequence* of such sets – corresponding to decreasing values of ε.

This is what we will do in this paper.

3.   Definitions and the Main Result

Comment about languages.   As we have mentioned, we start with a language L in which we describe definable objects. Then, we need a (stronger) language L_1 in which we can formally describe L-definability. It is possible to define L_1-definable sets of random elements and sets of typical elements [2]. We want to formally discuss L_1-definable sets, so we need to introduce yet stronger language L_2 in which such discussions are formally possible. This language can be built from L_1 in the same manner as the language L_1 is built from L.

Definition 4. By a *description of randomness*, we mean a sequence of L_1-definable sets of random elements $\mathcal{R}_n$ for which $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$.

Definition 5. By a *description of typicality*, we mean a sequence of L_1-definable sets of typical elements $\mathcal{T}_n$ for which $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$.

When is a description universal? When are two descriptions equivalent? To answer these questions, we can take into account that every non-empty subset of a set of random elements is also a set of random elements, and that every non-empty subset of the set of all typical elements is also a set of typical elements:

Лемма 1. *If $\mathcal{R}$ is a set of random elements, and $\mathcal{S}$ is a non-empty subset of $\mathcal{R}$, then $\mathcal{S}$ is also a set of random elements.*

Лемма 2. *If $\mathcal{T}$ is a set of typical elements, and $\mathcal{S}$ is a non-empty subset of $\mathcal{T}$, then $\mathcal{S}$ is also a set of typical elements.*

Remark 3. The proofs follow directly from Definitions 2 and 3. Now, we can formally describe universality and equivalence.

Definition 6. We say that a description of randomness $\mathcal{R}_n$ is *universal* if for every L_1-definable set of random elements $\mathcal{R}$, there exists n for which $\mathcal{R} \subseteq \mathcal{R}_n$.

Definition 7. We say that a description of typicality $\mathcal{T}_n$ is *universal* if for every L_1-definable set of typical elements $\mathcal{T}$, there exists n for which $\mathcal{T} \subseteq \mathcal{T}_n$.

Definition 8. We way that two descriptions of randomness $\mathcal{R}_n$ and $\mathcal{R}'_n$ are *equivalent* if the following two conditions hold:

- for every natural number n, there exists n' for which $\mathcal{R}_n \subseteq \mathcal{R}'_{n'}$; and

- for every natural number n', there exists n for which $\mathcal{R}'_{n'} \subseteq \mathcal{R}_n$.

Definition 9. We way that two descriptions of typicality $\mathcal{T}_n$ and $\mathcal{T}_n'$ are *equivalent* if the following two conditions hold:

- for every natural number n, there exists n' for which $\mathcal{T}_n \subseteq \mathcal{T}_{n'}'$; and

- for every natural number n', there exists n for which $\mathcal{T}_{n'}' \subseteq \mathcal{T}_n$.

Предложение 2. *There exists a universal description of randomness, and this description is unique modulo equivalence.*

Предложение 3. *There exists a universal description of typicality, and this description is unique modulo equivalence.*

Remark 4. In other words:

- there exists a universal description of randomness, and all universal descriptions of randomness are equivalent to each other, and

- there exists a universal description of typicality, and all universal descriptions of typicality are equivalent to each other.

The proof of Propositions 2 and 3 is based on the following lemmas:

Лемма 3. *For every set X, if $\mathcal{R}'$ and $\mathcal{R}''$ are sets of random elements, then their union $\mathcal{R} \stackrel{\text{def}}{=} \mathcal{R}' \cup \mathcal{R}''$ is also a set of random elements.*

Лемма 4. *For every set X, if $\mathcal{T}'$ and $\mathcal{T}''$ are sets of typical elements, then their union $\mathcal{T} \stackrel{\text{def}}{=} \mathcal{T}' \cup \mathcal{T}''$ is also a set of typical elements.*

4. Proofs

Proof of Proposition 1.   Let $\mathcal{P} \subseteq D$ be a set of random objects. Every point $d \in D$ can be described by its polar coordinates (r, θ), where $0 \leq r \leq 1$ is the distance from this point to $(0,0)$, and $\theta \in [0, 2\pi)$ is the angle between the direction $0d$ from 0 to this point d and the x-axis. Let us consider the following sequence of sets A_n:
$$A_n = \{(r, \theta) : |\theta| \leq 2^{-n}\}.$$
Then, clearly, $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$, so $P(A_n) \to 0$. Thus, by definition of the set of random objects, there exists an integer N for which $A_N \cap \mathcal{P} = \emptyset$. In other words, no point $(r, \theta) \in \mathcal{P}$ can have a value $|\theta| \leq 2^{-N}$.

Let us now prove, by contradiction, that the set $\mathcal{P}$ cannot be invariant relative to all possible definable rotations. Indeed, if the set $\mathcal{P}$ was invariant with respect to arbitrary definable rotations, then it would be invariant with respect to rotation by 2^{-N} radians, by $2 \cdot 2^{-N}$ radians, etc. Since no points with $|\theta| \leq 2^{-N}$ are in the set $\mathcal{P}$, by rotating by 2^{-N}, we conclude that no points with $\theta \in [0, 2 \cdot 2^{-N}]$ can be in this set, then that no points with $\theta \in [2^{-N}, 3 \cdot 2^{-N}]$ can be in this set – etc., until we conclude that no points with any value θ are possible and thus, that $\mathcal{P}$ is an empty set. The proposition is proven.

Proof of Lemma 3. To prove that the union $\mathcal{R} = \mathcal{R}' \cup \mathcal{R}''$ is a set of random elements, let us assume that A_n be a definable sequence of sets for which $A_n \supseteq A_{n+1}$ and $P(A_n) \to 0$. We need to prove that there exists an integer N for which $A_N \cap \mathcal{R} = \emptyset$.

Indeed, since $\mathcal{R}'$ is a set of random elements, there exists an integer N' for which $A_{N'} \cap \mathcal{R}' = \emptyset$. Since $m < n$ implies $A_m \supseteq A_n$, we conclude that $A_N \cap \mathcal{R}' = \emptyset$ for all $N \geq N'$.

Similarly, since $\mathcal{R}''$ is a set of random elements, there exists an integer N'' for which $A_{N''} \cap \mathcal{R}'' = \emptyset$. Since $m < n$ implies $A_m \supseteq A_n$, we conclude that $A_N \cap \mathcal{R}'' = \emptyset$ for all $N \geq N''$.

For $N \stackrel{\text{def}}{=} \max(N', N'')$, we have $N \neq N'$ and $N \geq N''$ and hence, $A_N \cap \mathcal{R}' = \emptyset$ and $A_N \cap \mathcal{R}'' = \emptyset$. So, no element of A_N belongs to $\mathcal{R}'$ or to $\mathcal{R}''$. In other words, no element of A_N belongs to the union $\mathcal{R} = \mathcal{R}' \cup \mathcal{R}''$. So, $A_N \cap \mathcal{R} = \emptyset$. Lemma is proven.

Proof of Lemma 4. To prove that the union $\mathcal{T} = \mathcal{T}' \cup \mathcal{T}''$ is a set of typical elements, let us assume that A_n be a definable sequence of sets for which $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$. We need to prove that there exists an integer N for which $A_N \cap \mathcal{T} = \emptyset$.

Indeed, since $\mathcal{T}'$ is a set of typical elements, there exists an integer N' for which $A_{N'} \cap \mathcal{T}' = \emptyset$. Since $m < n$ implies $A_m \supseteq A_n$, we conclude that $A_N \cap \mathcal{T}' = \emptyset$ for all $N \geq N'$.

Similarly, since $\mathcal{T}''$ is a set of typical elements, there exists an integer N'' for which $A_{N''} \cap \mathcal{T}'' = \emptyset$. Since $m < n$ implies $A_m \supseteq A_n$, we conclude that $A_N \cap \mathcal{T}'' = \emptyset$ for all $N \geq N''$.

For $N \stackrel{\text{def}}{=} \max(N', N'')$, we have $N \neq N'$ and $N \geq N''$ and hence, $A_N \cap \mathcal{T}' = \emptyset$ and $A_N \cap \mathcal{T}'' = \emptyset$. So, no element of A_N belongs to $\mathcal{T}'$ or to $\mathcal{T}''$. In other words, no element of A_N belongs to the union $\mathcal{T} = \mathcal{T}' \cup \mathcal{T}''$. So, $A_N \cap \mathcal{T} = \emptyset$. Lemma is proven.

Proof of Proposition 2. Let us first prove that every two universal descriptions of randomness $\mathcal{R}_n$ and $\mathcal{R}'_n$ are equivalent to each other. Indeed, for every n, by definition of a description of randomness, the set $\mathcal{R}_n$ is a L_1-definable set of random elements. Due to the fact that the description $\mathcal{R}'_n$ is universal, there exists a natural number n' for which $\mathcal{R}_n \subseteq \mathcal{R}'_{n'}$. Similarly, we can prove that for every n', there exists a natural number n for which $\mathcal{R}_n \subseteq \mathcal{R}'_{n'}$. Equivalence is proven.

Let us now prove existence of a universal description of randomness. Indeed, in the language L_2, we can formally describe L_1-definability, so we can have a sequence $\mathcal{R}_{(1)}, \mathcal{R}_{(2)}, \ldots,$ of all L_1-definable sets of random elements. Now, we can take

$$\mathcal{R}_n \stackrel{\text{def}}{=} \mathcal{R}_{(1)} \cup \ldots \cup \mathcal{R}_{(n)}.$$

Each of these sets is also L_1-definable. Due to our construction, we have $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$, so this is indeed a description of randomness.

Let us prove that this description is universal. Indeed, each L_1-definable description of randomness $\mathcal{R}$ is in the sequence $\mathcal{R}_{(1)}, \mathcal{R}_{(2)}, \ldots,$ so it has the form $\mathcal{R} = \mathcal{R}_{(n)}$

for some n. Then, by definition of $\mathcal{R}_n$, we have $\mathcal{R} = \mathcal{R}_{(n)} \subseteq \mathcal{R}_n$. Universality is proven, and so it the proposition.

Proof of Proposition 3 is similar.

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