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In Applications, A Rigorous Proof Is Not Enough: It Is Also Important to Have an Intuitive Understanding

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Abstract

From a purely mathematical viewpoint, once a statement is rigorously proven, it should be accepted as true. Surprisingly, in applications, users are often reluctant to accept a rigorously proven statement until the proof is supplemented by its intuitive explanation. In this paper, we show that this seemingly unreasonable reluctance makes perfect sense: the proven statement is about the mathematical model which is an *approximation* to the actual system; an intuitive explanation provides some confidence that the statement holds not only for the model, but also for systems approximately equal to this model – in particular, for the actual system of interest.

Users are often reluctant to accept rigorously proven results: a problem. In *theoretical* mathematics, once a statement is (rigorously) proven, this statement is accepted as true. Of course, it is nice to also have an intuitive understanding of why this statement is true ("idea of the proof"), but even without such an understanding, the proven statement is still universally accepted as true.

In contrast, in *applications* of mathematics, often, users do not accept rigorously proven statements until they also get an intuitive understanding of why these statements are true. Why?

How this reluctance is explained now. To a mathematician, this reluctance to accept a formally proven statement sounds puzzling: the statement is proven, what else do we need? Mathematicians often ascribe this reluctance to the lack of a user's understanding of mathematics.

What we do in this paper. While we agree that sometimes reluctance comes from the lack of understanding of what is mathematical rigor, there are deeper – and meaningful – reasons for the users' reluctance.

Comment. Mathematicians are well aware of the users' reluctance. Because of this awareness, they (and, in particular, we) try to also add an intuitive explanation to the proof. For example, when we formally justified the existing semi-heuristic poverty measures [5], in addition to a formal justification, we also added intuitive explanations. Similarly, when we formally justified the use of Bernstein polynomials in function approximations [2], we also supplemented the proof with intuitive explanations.

Our explanation: main idea. Our explanation is straightforward:

- a rigorous proof proves that the mathematical model used to simulate the phenomena of interest has the desired property;
- if the mathematical model precisely described the real-life phenomena, then we would be able to conclude that the real phenomena also satisfies this property;
- in reality, however, mathematical models are usually only approximate; so, the fact that the model satisfies a certain property does not necessarily mean that this property is also satisfied for the real-life phenomena.

Towards a precise explanation: details. Different mathematical models can be characterized by different values of the corresponding parameters $x_1, \ldots, x_n$. Let $v_1, \ldots, v_n$ are the values of these parameters which are used in our model. The rigorous proof proves the following statement: the given property holds for the model with parameter values $v_1, \ldots, v_n$; let us denote this statement by $P(v_1, \ldots, v_n)$.

As we have mentioned, mathematical models are usually *approximate*. This means, in particular, that the actual (unknown) values x_i of the corresponding parameters are, in general, (somewhat) different from the selected value v_i. In general, the fact that a property P holds for the values $v_1, \ldots, v_n$, does not necessarily imply that it also holds for nearby values $x_1, \ldots, x_n$. To be able to conclude that the desired property holds for the real-life phenomenon, we therefore need to be able to show that we have $P(x_1, \ldots, x_n)$ not only for $x_i = v_i$ but also for $x_i \approx v_i$.

We will show that what we call an "intuitive explanation" actually provides such a justification for the correctness of $P(x_1, \ldots, x_n)$ for $x_i \approx v_i$.

What is "intuitive explanation"? In order to explain the above claim in precise terms, we need to understand what is usually meant by an intuitive explanation. Typically, an intuitive explanation means that instead of using the exact values v_i of the corresponding parameters, we use natural-language words such as "small", "negligible", etc.

For example, we can say that since in the expression $a_0 + a_1 \cdot t + a_2 \cdot t^2$, the coefficient a_2 is small, we can safely ignore it and make conclusions based on the linear approximation $a_0 + a_1 \cdot t$. Similarly, we can say that since the frequency

of the external signal is drastically different from the system's eigenfrequencies, resonance effects are small and can be safely ignored.

In order to describe natural-language words, it is reasonable to use *fuzzy logic*, technique specifically designed to describe such imprecise ("fuzzy") knowledge in computer-understandable terms; see, e.g., [1, 4]. When we claim that some quantity is small or large, we do not describe the exact value of this quantity, we only describe the *range* of possible values. In addition to this range – which is guaranteed to contain all "small" values – experts can also provide narrower intervals that contain all small values with a given degree of certainty.

In other words, an expert is 100% sure that the value v belongs to a wide interval, and with some degree of confidence, the expert believes that v belongs to a narrower interval. For different degrees of confidence, we have different interval ranges; in order to increase the expert's degree of confidence, we need to add extra points to the range, i.e., make the interval wider. Thus, instead of a *single* interval range $\mathbf{v} = [\underline{v}, \overline{v}]$, we have a *family* of ranges $\mathbf{v}(d) = [\underline{v}(d), \overline{v}(d)]$ corresponding to different degrees of confidence d. This family is *nested* in the sense that when $d < d'$, we have $\mathbf{v}(d) \subseteq \mathbf{v}(d')$. Nested intervals $\mathbf{v}(d)$ form what is usually called a *fuzzy number* $V = \{\mathbf{v}(d)\}_d$; see, e.g., [1, 3, 4].

Comment. Instead of nested intervals, we can also describe, for each value x, the largest confidence level $d(x)$ for which $x \in \mathbf{v}(d)$. The value $\mu(x) \stackrel{\text{def}}{=} 1 - d(x)$ is called a *membership function* of the fuzzy number. Vice versa, once we know the membership function, we can reconstruct each nested interval $\mathbf{v}(d)$ as the *α-cut* $\{x : \mu(x) \geq 1 - d\}$.

Properties of fuzzy numbers are defined in a straightforward way. If the only information that we have about each value x_i is that it is contained in the corresponding range $\mathbf{v}_i$, then the only possibility to guarantee that the property P holds for the *actual* (unknown) values x_i is to prove that the property P holds for *all* possible tuples $(x_1, \ldots, x_n)$ for which $x_i \in \mathbf{v}_i$. In line with this idea, it is reasonable to say that the property P holds for the intervals $\mathbf{v}_1, \ldots, \mathbf{v}_n$ if this property holds for all the tuples $(x_1, \ldots, x_n)$ for which $x_i \in \mathbf{v}_i$.

For n fuzzy numbers $V_1 = \{\mathbf{v}_1(d)\}_d, \ldots, V_n = \{\mathbf{v}_n(d)\}_d$, with degree of confidence d, we have $x_i \in \mathbf{v}_i(d)$. So, if the property $P(x_1, \ldots, x_n)$ holds for all the tuples for which $x_i \in \mathbf{v}_i(d)$, then our degree of confidence that P holds for V_i is at least d. Thus, for n fuzzy numbers $V_i = \{\mathbf{v}_i(d)\}_d$, the degree $P(V_1, \ldots, V_n)$ with which the property P holds for $V_1, \ldots, V_n$ can be naturally defined as the largest of the degrees d for which the property P holds for the intervals $\mathbf{v}_1(d), \ldots, \mathbf{v}_n(d)$.

Why intuitive explanation is necessary in applications. Now, we can explain, in precise terms, wy intuitive explanation is necessary in applications. Indeed, an intuitive explanation means that instead of the *approximate* values v_i of the model's parameters, we consider fuzzy numbers V_i, i.e., nested families of intervals $\mathbf{v}_i(d)$ that contain the *actual* (unknown) values x_i with different degrees

of certainty d. The fact that we are able to conclude that P is true – with some high confidence d_0 – means that the property $P(V_1, \ldots, V_n)$ holds with degree d_0. This, in its turn, means that the property $P(x_1, \ldots, x_n)$ holds for all values $x_i \in \mathbf{v}_i(d_0)$. With degree of confidence d_0, the actual values x_i are contained in the interval ranges $\mathbf{v}_i(d_0)$ and, therefore, the property $P(x_1, \ldots, x_n)$ holds for the actual values x_i.

This is exactly what we were trying to prove. Thus, the intuitive explanation provides us with confidence that the property P holds not only for the approximate values v_i, but also for the *actual* values x_i – and this is exactly what the users want.

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