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Equation of State of a Dense and Magnetized Neutron System

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EQUATION OF STATE OF A DENSE AND MAGNETIZED NEUTRON SYSTEM

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Master's Program in Physics

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Chapter 1

Introduction

White dwarf, neutron stars and black holes are termed together as astrophysical compact stars. The matter inside these compact stars is glued together so compactly that they are also known as the smallest and densest astrophysical objects to exist[9]. Among them, white dwarf and neutron stars are more accessible than black holes where gravity is so strong that even light cannot get out of it; though the recent detection of gravitational waves will definitely enable us to access these compact objects in a better way[19]. White dwarf is a result of gravitational collapse of low-mass stars (less than $8M_{\odot}$) when the thermal pressures produced by nuclear fusion are no longer able to resist its inward pull of gravity, and supported from further collapses by the pressure of some kinetic effects other than of thermal pressure. This pressure mainly comes from electron-degeneracy pressure contribution. In addition to that, for high-mass stars, a remnant left after supernova explosion, is the ultimate end-point for these heavy stars where the remnants exceed maximum mass limit- the so-called Chandrasekhar limit $\sim 1.4M_{\odot}$[10, 20]. Exceeding this limit will result in the formation of either neutron stars or black holes depending on the initial mass of the stars. Thus, neutron stars are stellar remnant of rather massive stars; which supported from further collapses by neutron degeneracy pressure. Ideally, the name 'neutron star' misleads us that, it exclusively consists of neutron particles; but, in practical, a neutron star contains few percent of protons and electrons. So it is called neutron stars just in order to emphasize that it is comprised of dominantly neutron particles. Finally, for stars with masses more than $20M_{\odot}$, black holes are formed after the collapse, we will be kept then out of our discussion.

In order to reaching the stability in compact stars, the degeneracy pressure plays a very

important rule. The degeneracy pressure originates from Pauli Exclusion Principle. The principle, proposed by well-known physicist Wolfgang Pauli in 1925, states that all fermions, due to Fermi-Dirac statistics, satisfy that two identical fermions are forbidden to occupy the same quantum state at a same time[15, 11]. Typically, the compact stars, all possible lowest energy levels are filled by the particles. As a consequence of this principle, the pressure is emergent against the compression of matter into smaller volumes of space. According to Heisenberg principle the position is more localized for less the volumes of particle, and hence more the uncertainties in momentum for particles. This momentum uncertainty gives rise to the degeneracy pressure. Thus, matter in compact stars, since it is immensely squeezed and dense, becomes degenerate producing a high Pauli pressure. Before the formation of compact stars, the normal stars begin to run out of fuel through burning gases.

In the recent advanced-telescopes like Chandra, ESA's XMM-Newton, and NASA's Rossi X-ray and RXTE has enabled us for the improved measurements leading to better understanding of the properties of the densest objects (white dwarf, neutron star, etc.) in our universe. The improved observational accessibility on distant compact stars is also making it active research area as physics in extreme conditions-such as high density, strong magnetic field, etc. It will not only help us for deeper understanding of those exotic objects but also it has paved a way to check the validity of established theories at extreme conditions which is not possible in laboratories on earth. Since we mostly interested in neutron stars, that means we have to deal with objects which have density close to nuclear density $\sim 10^{15} gm/cm^3$, and surface magnetic field more than $10^{12} G$, which is at least around 10^{14} times more than earth's field [6].

For high-density matter in neutron stars, the structure and composition of the neutron stars are not yet well- understood due to lack of proper construction of the corresponding equation of state (EoS). The EoS plays a very important rule to understand better the neutron stars' structure and composition. The EoS involves the relation between state variables

such as temperature (T), pressure (P), chemical potential (μ), energy density ($\mathcal{E}$). With the knowledge of the EoS we can find the stars' mass-radii (M-R) relation. In spite of varieties of EoS found to make predictions, the description of dense matter in neutron stars, unfortunately still shows uncertainties. Those high densities, the EoS is not only a useful tool to describe more precisely the structure and composition, but also some exotic phenomena such as deconfined of quark matter, super-fluidity and superconductivity.

Applying an uniform magnetic field breaks SO(3) symmetry of the given system giving rise to the anisotropy in the energy-momentum tensor, and hence to pressure anisotropy resulting in the splitting of the pressure into two components: One along the field and another one transverse to the field direction [4, 7]. In spite of significant effects on the EoS, there exist some authors who still disagree with the anisotropic nature of the pressure although they do not have proved result [17]. Another way to say that, this anisotropy results from the rotational symmetry breaking in the presence of a magnetic field [4]. Because of this, the anisotropic effects due to strong magnetic field on the system even could force the system into instabilities. In this work, we will mainly investigate some aspects of EoS for highly dense magnetized neutron system, which are the main components of neutron stars. Supposing that the system is completely made of this charged neutral particles, it is found that the magnetic field can only affect the system through its interaction with the neutron anomalous magnetic moment. The next subsection will be devoted to the discussion of gross characteristics of neutron stars in details.

1.1 Overview on neutron stars

Firstly, it is common to come up with the natural question, why compact stars are so important in our consideration? Straightforwardly the answer is because it is comprised of some

exotic properties, i.e. high density, strong magnetic fields, temperature, etc.That open the possibility to study the physics of systems under extreme conditions. In this subsection we will mainly discuss briefly about these exotic properties of compact stars. However, compact stars are smaller, colder and comparatively denser version of normal stars, and these are formed through gigantic gravitational collapses when kinetic effects or fundamental interactions, i.e. electromagnetic and nuclear forces, are failed to oppose inward pull of gravity. The reason behind the birth of compact stars like neutron stars is when thermal pressure of normal high-mass stars is no longer able to win over gravity. During this period of time, particles become so compressed that they start to combine with protons, forming neutrons with giving off neutrino as byproduct. The trapped neutrinos basically increase the interior temperature tremendously resulting in a gigantic explosion called Supernova type II explosion [8]. After the supernovae II explosion, the remained astronomical remnant is called neutron star (NSs), which is mostly made up of neutrons at relative low temperature due to the huge energy release delivered in the supernovae explosion. As a result, it is found that masses of neutron stars range between $1.4M_{\odot}$ (corresponds to the solar mass) and $3M_{\odot}$, which is known as Tolman- Oppenheimer-Volkof limit [2]. Beyond of this limit, it is assumed that NSs turn into black holes.

It is still a matter of debate on the initial mass of high-mass stars corresponding to neutron stars, but typically the masses of stars range between 8 to 20-30 times $M_{\odot}$ [13, 16]. The supernova-collapse theory tells us that the possible masses $(1.4_{\odot})$ of the remnants (neutron stars) after the explosion takes place becomes they are highly dense, with 2-12 km radius of matter. Since the matter inside stars is so compressed; the mean density closes to nuclear density level $\sim 10^{15} gm/cm^3$. Prior to the supernova or gravitational collapses, matter in the cores of the stars have densities $10^{gm/cm^3}$ with very low entropies [12]. However, the densities are not uniformly distributed, rather it varies from surface to the center of the neutron star. As we look from surface to center, the density increases towards the center of the star. The EoS of matter there is not easy to verify at this level of matter density,

which requires precise observational data and well foundation of quantum gravity. The table below describes the densities and the constituents at the different layers of neutron stars [18].

Layers	Core or crust length (km)	Density (gm/cm^3)	Composition
Outer Crust	0.3 – 0.5	$10^6 - 4\times10^{11}$	Heavy nuclei and free electrons
Inner Crust	1 – 2	$4\times10^{11} - 2\times10^{14}$	Heavier nuclei and free e^- and protons
Outer Core	~ 9	$2\times10^{14} - 10^{15}$	Neutrons with few e^- and p.
Inner Core	0 – 3	$> 10^{15}$	Exotic particles-hyperons, quarks

Neutron stars contain the strongest magnetic field in the universe ever. Magnetic field of a typical pulsar $\sim 10^{12} - 10^{13}G$, which is around 105 – 106 times larger than highest man made fields. It has been found that some of the neutron stars(10% of all neutron stars) are possessing strongest magnetic field and they are called magnetars.The common nature of the magnetars is to have ultra-strong surface magnetic fields around 1014- 1015 G, and temperature less than 107 K. The inner magnetic field has been estimated around $10^{18} - 10^{20}G$, which is estimated from equipartition of energy principle. The idea is based on the conversation of field flux that amount for a flux density increase with the decrease of surface area, it happens when stars shrinks their size during the collapse. The magnetic field of a pulsar is greater than the field estimated using flux conservation, which has been compensated later with the theory of Pauli paramagnetism of Fermi system. In addition, the fact is also added that induced magnetic field increases with decreasing temperature at their late evolutionary stage. But for the magnetars, strength of the fields can not be explained by the flux conservation principle, moreover the other proposed theory, like Magnetodynamics (MDM) could not give the correct explanation of this strongest field. So, it is assumed that to come up with correct explanation of such high field we need to have better understanding of the composition of magnetars. For example, it was proposed that if magnetars are formed by quarks in a color superconducting state the original field can be boosted by the existence of g'von vortices [5].

Todays' advanced and improved observatories over the world has delved into the deeper space for finding thousands of neutron stars; whereas it has been a great challenge after the initial proposal for its detection done by astronomers -Walter Baade and Fritz Zwiky in 1934 after studying supernova explosion [1]. In 1967 Anthony Hewish and Jocelyn Bell at Cambridge discovered radio pulses with regular interval from PSR $B1919+1$ was named by Pulsar, and later it is interpreted as isolated and rotating neutron star.

1.2 Objectives of the thesis

To understand the properties of NSs under the prevailing extreme physical conditions is a hot and challenging topic of investigation in nuclear astrophysics. Our main goal in this thesis is to investigate the EoS of a neutron system under the extreme conditions existing in NSs. In particular,we will consider a neutron system at very high density and in the presence of a strong magnetic fields, we attempt to analyze numerically the critical value of the magnetic field from where the pressure anisotropy begins to be significant. Then, we will compare one respect to that obtained for charged particles as electron [7]. Besides, we also investigate numerically how the pure magnetic field-Maxwell term effects the energy and pressures, as well as to find the behavior of system with the applied magnetic field. Obviously this analysis will help us to understand how much a strong magnetic field can affect EoS.

1.3 Outline

First chapter is fully devoted to the introduction of our work; mainly it discusses briefly about characteristics of neutron stars, and purposes of our work. In addition to the theoretical background is introduced on the second chapter, the third chapter introduces the

mathematical tools we need to study the system thermodynamics and in particular its EoS. Then, we have calculated statistical average of the energy -momentum tensor components (i.e. the energy density, parallel and transverse pressures). Thus, using the results of the quantum-statistical average of the energy-momentum components, we calculate the system energy density and anisotropic pressure components. Chapter 4 will cover numerical results of energy density and pressure components at various range of magnetic field and density. We then compare with other similar results obtained for charged particles as electrons.

Chapter 2

Theoretical Overview

At the beginning of this chapter, we will discuss the necessary mathematical tools, and thus to frame the theoretical background needed to treat the problems in our study. The problem we are investigating demands applications of different branches of physics. Throughout this chapter, one of the powerful physical law that will be applied repeatedly is the Least Action Principle. The term the Lagrangian (L) is found by using the least action principle, which encodes all the physical information from which the equations of motion of the given system can be obtained by Euler-Lagrange equation. Another important theoretical object to be used is the energy-momentum tensor that is also expressed in terms of system Lagrangian (L).

In the first section of this chapter, co- and contra-variant tensor notation are introduced, as well as the metric tensor form. The later sections will introduce some important concepts of the theory of relativity, quantum field theories, and the statistical approaches towards the problem etc.

2.1 Index Notation : Co-and Contra -variant Tensor

The theory of relativity is framed by the usage of tensor notation, which deals with 4-dimensional space and time coordinate system. So, to grab this theory well, it requires a least knowledge of tensor notational rules, which is the concerns of this section. However, a vector's components in the 3-dimensional coordinate system can be represented by the index

notation usually in the following way

$$x = \{x_1, x_2, x_3\} = \{x, y, z\}$$

In the 4-dimension, the vector is represented in contra and co- variant form respectively as with μ = 0,1,2,3. So,

$$x^\mu = \left\{x^0, x^1, x^2, x^3\right\} = \{ct, x, y, z\} \tag{2.1}$$

$$x_\mu = \{x_0, x_1, x_2, x_3\} = \{ct, -x, -y, -x\} \tag{2.2}$$

Here, we are using the metric $\eta_{\mu\nu} = (1, -\vec{1})$. The geometry of spacetime is represented by metric tensor $g_{\mu\nu}$. In the case of flat space the metric $g_{\mu\nu}$ becomes $g_{\mu\nu} = \eta_{\mu\nu}$. The $\eta_{\mu\nu}$ is metric in flat space called Minkowskian metric is defined by

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \eta^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.3}$$

The gradient of the four-vectors also can be written using the same notation as, for the contra-variant form

$$\partial^\mu = \left\{\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\} \tag{2.4}$$

and in the covariant form as

$$\partial_\mu = \left\{\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\} \tag{2.5}$$

To change the coordinate system under a coordinate transformation, we have to use partial differential rule as to get new coordinate system. For the contra-variant coordinates

we have

$$x'^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\nu}} x^{\nu}, \tag{2.6}$$

while for the covariant components it is

$$x'_{\mu} = \frac{\partial x^{\nu}}{\partial x^{\mu}} x_{\nu} \tag{2.7}$$

Under the above transformation, a scaler field is invariant,i.e.$S'(x'^{\mu}) = S(x^{\mu})$. So,

$$\frac{\partial S'(x'^{\mu})}{\partial x'^{\nu}} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} \frac{\partial S(x)}{\partial x^{\nu}} \tag{2.8}$$

In the contra-variant form, the components of a physical quantity, for example, a four-vector, can be represented compactly as

$$A^{\mu} = \left\{A^0, A^1, A^2, A^3\right\} \tag{2.9}$$

We can rewrite the same quantity, but in covariant(lowered index) form as

$$A_{\mu} = \{x_0, x_1, x_2, x_3\} \tag{2.10}$$

So, we can have contravariant and covariant vector components just making up and down the indices. One of them can be transformed into the other by using the metric tensor, where we used the Einstein summation equation,

$$A_{\mu} = \eta_{\mu\nu} A^{\nu} \tag{2.11}$$

Similarly, we can go from a contra-variant vector to a covariant one

$$A^{\nu} = \eta^{\mu\nu} A_{\mu} \tag{2.12}$$

The four vectors in equations (2.9) and (2.10) both follow the same coordinate transformation as Eqn.(2.6) & (2.7). The inner product between contravariant and covariant components gives a scalar quantity, which is also invariant under the coordinate transfromation

$$A'^{\mu}B'_{\mu} = \left(\eta^{\mu}_{\alpha}A^{\alpha}\right)\left(\eta^{\beta}_{\mu}B_{\beta}\right) = \eta^{\mu}_{\alpha}\eta^{\beta}_{\mu}A^{\alpha}B_{\beta} = \delta^{\beta}_{\alpha}A^{\alpha}B_{\beta} = A^{\alpha}B_{\beta}, \tag{2.13}$$

where δ^{β}_{α} is called Kronecker-delta , which is defined by

$$\delta^{\alpha\beta} = \delta_{\alpha\beta} = \delta^{\alpha}_{\beta} \tag{2.14}$$

The two rank contravariant tensors, $T^{\mu\nu}$, can be written as the product of two four-vectors as

$$T^{\mu\nu} = A^{\mu}B^{\nu} \tag{2.15}$$

can be transformed in the following way

$$T'^{\mu\nu} = \eta^{\mu}_{\alpha}\eta^{\nu}_{\beta}A^{\alpha}B^{\beta} = \eta^{\mu}_{\alpha}\eta^{\nu}_{\beta}T^{\alpha\beta} \tag{2.16}$$

The Lorentz transformation matrix, in a general direction, is defined by

$$\Lambda^{\mu}_{\nu} = \left\{\begin{matrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\gamma_y\gamma_x}{\gamma^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\gamma_y\gamma_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\gamma_z\gamma_x}{\beta^2} & (\gamma-1)\frac{\gamma_z\gamma_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{matrix}\right\}, \tag{2.17}$$

and for the case of two inertial reference frames with relative velocity v in the x_3 direction,

the Lorentz transformation (Λ^{μ}_{ν}) is defined as

$$\Lambda^{\mu}_{\nu} = \left\{\begin{matrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{matrix}\right\}. \tag{2.18}$$

Where $\gamma = (1 - v_i^2)^{-1/2}$ and $\beta = \frac{v}{c}$. So far the metric we have used is applicable for flat space (so called Minkowski space). In the curved space, the metric is replaced by general covariant metric $g_{\mu\nu}$ defined by

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}, \tag{2.19}$$

where the matrix coefficients are functions of space and time. The metric tensor in curve space ,$g_{\mu\nu}$ obeys all the transformation rules as in the flat space. The metric $g_{\mu\nu}$ is a symmetric tensor means, $g_{\mu\nu} = g_{\mu\nu}$, and the determinant is denoted by $g = det\ g_{\mu\nu}$.

The anti-symmetric electromagnetic two ranked field tensor is defined by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{2.20}$$

In our study we will also use Dirac matrices defined as

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{2.21}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.22}$$

2.2 Relativity

Before Einstein's proposal of theory of relativity, space and time were considered as separate entities. In the Newtonian mechanics, the time is the same everywhere, and independent of the inertial reference frame. In 1905, Albert Einstein made a breakthrough towards the Newtonian ideas on space and time. According to his theory of special relativity (SR) space and time is the same physical entity is termed as space-time together. His theory of special relativity is based on two postulates.

First postulate : Physics is the same for every inertial reference frame.

Second postulate : The speed of light is the same to every observer, no matter what is the relative speed of them.

One of the most famous finding of special relativity (SR) is the mass-energy equivalence, $E = mc^2$.

The special theory of relativity is a particular case of his another most profound work: The general theory of relativity. In this second theory, he put gravity to the inertial reference frame or what is equivalent to consider non-inertial reference frames, that can move one

with respect to others with same acceleration. After twelve years of special theory, he could formulate, in 1916, that in the presence of a gravitation field (mass) space-time becomes curved, that is, more the mass more the curvature of space-time. The space-time curvature is represented by the metric $g_{\mu\nu}$ of Riemannian geometry. one of the prediction of general relativity is the existence of gravitational waves. It has been observed last year making a strong confirmation of this theory. In our study, we deal with compact stars, which are very massive astrophysical objects. So, the general theory of relativity comes into effective to account for their macroscopic properties. Actually, these stars are such place combines different branches of area, i.e. nuclear,particle physics and general relativity etc. In the framework of micro-physics determining the EoS's, the effect of the gravitational field is not noticeable. Therefore, in the microscopic region physics behaves like that of SR.

Thus, for EoS of compact objects, we can consider a Minkowskian metric. The change of metric by a factor 2.5 over the dimension of stars. As a result, in the dimension of nucleons its effect is extremely small around 10^{-19} of that factor [3]. Nevertheless, in finding the star mass-radius relation the metric mofification became significant.

2.2.1 Formalism of Special Relativity

As we have mentioned that special relativity is based on two postulates, one of them stating that physics is invariant for all inertial reference frame, it means that the laws of physics have to be Lorentz invariant. In this way, a system of reference x^ν will be transformed by the Lorentz transformation as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.23}$$

The distance or proper time interval, being a scalar is also invariant under the Lorentz transformation, can be written as below

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2.24}$$

or

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 d\tau^2, \tag{2.25}$$

where τ is called proper time, which also shows invariance p under Lorentz transformation. That is to say, the proper time is the same to every inertial reference frame.

The action of a system of relativistic particles can be written as

$$S = \int_{t_1}^{t_2} dt \mathcal{L}(x^\mu, \dot{x^\mu}) = \int_{\tau_1}^{\tau_2} \gamma d\tau \mathcal{L}(x^\mu, \dot{x^\mu}) \tag{2.26}$$

According to the Hamilton's least action principle as followed by

$$\delta S = \int_{t_1}^{t_2} dt \delta \mathcal{L}(x^\mu, \dot{x^\mu}) = \int_{\tau_1}^{\tau_2} d\tau \delta \big(\gamma \mathcal{L}(x^\mu, \dot{x^\mu})\big) = 0 \tag{2.27}$$

The Lagrangian for free classical particles is given by

$$\mathcal{L} = -mc^2 \sqrt{1 - \beta^2} \tag{2.28}$$

In the form of tensor notation it is given by

$$\mathcal{L} = -mc^2 \sqrt{(1 - \dot{x^\mu} \eta_{\mu\nu} \dot{x^\nu})} \tag{2.29}$$

Most importantly, this given Lagrangian is Lorentz invariant under the Lorentz transforma-

tion.

2.2.2 General Relativity: Einstein's Field Equation

The Poisson equation of Newtonian gravitational physics is expressed by

$$\nabla^2\phi = k_N\rho \tag{2.30}$$

where ϕ is gravitational potential, ρ is the mass density, and k_N is a constant number. Einstein gives, later, the corrected version through his theory of general relativity when he considers gravity into SR framework. The great triumph of Einstein theory is that he could express space-time curvature in the presence of gravity in the form of one single equation, which is known as Einstein's field equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = k_E T_{\mu\nu} \tag{2.31}$$

As we already mentioned, on the right side of Poisson equation (2.30) ρ is mass density, which is considered as an intrinsic property, while not in the Einstein field equation, (where the right hand side term in (2.51)) provides the energy-momentum tensor. The left side is formed by the Ricci curvature tensor $R_{\mu\nu}$, the Ricci scalar R and the metric tensor $g_{\mu\nu}$; which denote the properties of space-time that can be modified by the momentum and energy density of matter.

The action of for general relativistic system with matter field ϕ and metric field $g_{\mu\nu}$ is

$$S = \int_\Omega \mathcal{L}(\phi, \partial_\mu\phi, g_{\mu\nu}, \partial_\sigma g_{\mu\nu}, \partial_\rho\partial_\sigma g_{\mu\nu})\, d\Omega \tag{2.32}$$

The Lagrangian for the gravitational field is

$$\mathcal{L}_G = \sqrt{-g}R \tag{2.33}$$

If the only matter field present in the system is a free scalar field,ϕ we have that the Lagrangian density is given by

$$\mathcal{L}_\phi = \frac{\sqrt{-g}}{2} g_{\mu\nu}\partial_\mu\phi\partial_\nu \tag{2.34}$$

In the case of a free electromagnetic field $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$; the Lagrangian density is

$$\mathcal{L}_{EM} = \frac{\sqrt{-g}}{8\pi} g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta} \tag{2.35}$$

2.3 Lagrangian Formalism

Lagrangian formalism is a convenient way to make possible the application of the "Principle of Least Action". In our study, we will use Lagrangian formalism to derive the quantum statistical EoS of the system. From there we will define the thermodynamic quantities as the system effective potential magnetization, etc. In this section, to describe the quantum-statistical properties of the system will be covered shortly.

2.3.1 Classical Mechanics and The Field Concept

The action of a given system is defined as

$$S = \int_{t_0}^{t_1} dt L(q_r, \dot{q}_r), \tag{2.36}$$

where L is the Lagrangian, and $q, \dot{q}$ are the generalize coordinates and velocities, respectively. For a system in N dimension, r = 1,2,3....N. According to the Hamilton's principle, the action

of a system of particles follows a path during time interval dt for which the variation of the action is stationary.

$$\delta S = \int_{t_0}^{t_1} dt \delta L(q_r, \dot{q}_r) = 0 \tag{2.37}$$

Using the variational calculus, we can derive the solution of (2.37), which is called Euler-Lagrange equations

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_r}) - \frac{\partial L}{\partial q_r} = 0 \tag{2.38}$$

The Lagrangian L is defined by

$$L = T - V \tag{2.39}$$

Where T and V are the kinetic and potential energy respectively. The Lagrangian L codify all the system physical information. Once we know the Lagrangian L, we can derive equations of motion for the given system of particles using Euler- Lagrangian equation (2.38).

In classical field theory, the particle's coordinates q_r are replaced by continuous fields in space and time, $\phi(q_r, t)$. In the case of a relativistic field, the action of the system can be written as

$$S[\phi(x^\mu)] = \int d\Omega L\big(\phi_r(x^\mu), \partial_\mu \phi(x^\mu)\big) \tag{2.40}$$

where $d\Omega = dt\ d^3x$. The infinitesimal variation in $S[\phi(x^\mu)]$ is

$$\delta S[\phi(x^\mu)] = \int d\Omega \delta L\big(\phi_r(x^\mu), \partial_\mu \phi(x^\mu)\big) \tag{2.41}$$

According to the Hamilton's principle the action must be stationary with respect to infinitesimal variations of the fields that leave the fields values invariant at the initial and final times, i.e $\phi(x, t_1) = \phi_1(x)$ and $\phi(x, t_2) = \phi_2(x)$. Hence, we have $\delta\phi(t_1, x) = \delta\phi(t_2, x) = 0$. Using these conditions we derive the Eular-Lagrange equation as:

$$\frac{d}{dx^\mu} \frac{\partial L}{\partial\big(\partial_\mu \phi_r\big)} - \frac{\partial L}{\partial \phi_r} = 0 \tag{2.42}$$

2.4 Electromagnetic Field

The most familiar examples of classical field theory is the Maxwell theory with electric and magnetic fields $\vec{E}(\vec{x}, t)$, $\vec{B}(\vec{x}, t)$ respectively. Using the Hamilton Principle together with some topological properties we find that there are four Maxwell equations to describe all electromagnetic phenomena. The electric and magnetic fields are given in terms of a single four-vector component field $A^\mu(\vec{x}, t) = (\phi, \vec{A})$ as

$$\tilde{E} = -\nabla\phi - \frac{\partial \tilde{A}}{\partial t} \tag{2.43}$$

and

$$\tilde{B} = \nabla \times \vec{A} \tag{2.44}$$

Using curl and divergence operation on previous two equation, two of four Maxwell equation can derived

$$\nabla.\vec{B} = 0 \tag{2.45}$$

$$\nabla \times \vec{E} = -\frac{d\tilde{B}}{dt} \tag{2.46}$$

The other two Maxwell equations are inhomogeneous derived from the least action principle, and they are

$$\nabla.\vec{E} = \rho \tag{2.47}$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{1}{c}\vec{J} \tag{2.48}$$

where ρ and $\vec{J}$ are the charge and current densities respectively. From the above two equations, we can write the continuity equation, which preserve charge density locally.

$$\frac{\partial \rho}{t} + \nabla.\vec{J} = 0 \tag{2.49}$$

The four-vector current can be introduced as $J_\mu = (c\rho, \tilde{J})$. So, the charge conservation can be written as $\partial_\mu J^\mu = 0$.

To construct the Lagrangian for electromagnetic field, the potential fields $\phi(\tilde{x}, t)$ and $\vec{A}(\vec{x}, t)$ are used as generalized coordinates of the electromagnetic field. We can start from the Lagrangian,

$$\mathcal{L} = \frac{1}{2}(E^2 - B^2) - \rho\phi + \frac{1}{c}\vec{J}.\vec{A} \tag{2.50}$$

Potentials ϕ and $\tilde{A}$ constitute a four-vector $A_\mu = (-\phi, \tilde{A})$, hence the equation(39) can be rewritten as

$$\mathcal{L} = \frac{1}{2}(E^2 - B^2) + \frac{1}{c}J^\mu A_\mu \tag{2.51}$$

By introducing the electromagnetic strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.52}$$

the Lagrangian can be expressed as:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{c}J^\mu A_\mu \tag{2.53}$$

The second term in the right hand side of (2.53) arises from the interaction of the electromagnetic field with the charged particles. This term will be in our study, since we will only consider charge neutral fermions.

2.5 Quantum Mechanics

In the microscopic world of matter, the physical quantities are measured in terms of probabilities rather than certainty. In his uncertainty principle, Heisenberg proposed that nature forbids us to precisely measure momentum and position of the particles simultaneously at a microscopic level, Heisenberg's principle:

$$\Delta p \Delta x \geq \frac{\hbar}{2} \tag{2.54}$$

where $\hbar = \frac{h}{2\pi}$, with $h \approx 6.626 \times 10^{-34} J/s$, being the Planck constant.

On the hand, Schrödinger describes the dynamics of probability in quantum physics through an equation, which is known as Schroedinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t) \tag{2.55}$$

where $\Psi(x,t)$ is the wave function of particles and $\hat{H}$ is the Hamiltonian operator of the system of particles given by $\hat{H} = \frac{\hat{p}^2}{2m} + V$. Here, the momentum operator is expressed as $\hat{p} = -i\hbar\nabla$. So, Schridinger equation can be rewritten as

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\Psi(x,t) \tag{2.56}$$

2.5.1 Relativistic Quantum Mechanics

The Schrödinger equation is applicable in the non-relativistic world where the velocities of particles are $v \ll c$; or when the kinetic energy of the quantum particles do not exceed their rest energies.Beyond that, relativity starts to affects quantum particles considerably.

2.5.2 Klein -Gordon Equation

First, relativity and quantum mechanics are reconciled through Klein-Gordon equation, which is known as the relativistic version of Schroedinger equation. For the particles with spin zero, applying quantization procedure to the relativistic equation for the free spinless

particle we have

$$P^\mu = (E/c, \vec{P}) \tag{2.57}$$

$$P^\mu P_\mu = (E/c)^2 - P^2 = m^2c^2 \tag{2.58}$$

$$E^2 = P^2c^2 + m^2c^4 = -\hbar^2\nabla^2 + m^2c^4 \tag{2.59}$$

Choosing the natural units c, $\hbar = 1$, quantum mechanical equation for spinless particle becomes

$$\begin{aligned} \partial_t{}^2\phi(x,t) &= \big(-\nabla^2 + m^2\big)\phi(x,t) \\ \big(\partial_\mu^2 + m^2\big)\Psi(x,t) &= 0 \end{aligned} \tag{2.60}$$

This equation is called Klein-Gordon equation.

2.5.3 Dirac Equation

Later on, Pauli Dirac derived the corresponding equation for spin-1/2 particles. He sued the following physical arguments: The relativistic equation must be first order in time derivative, and in second one order it should reduce to the Klein-Gordon equation. Dirac approach was aimed to factorize the Klein-Gordon equation

$$\big(-i\gamma^\mu\hat{p_\mu} - m\big)\big(i\gamma^\mu\hat{p_\mu} - m\big)\mathit{\Psi} = 0 \tag{2.61}$$

as

$$\big(i\gamma^\mu\partial_\mu - m\big)\mathit{\Psi} = 0 \tag{2.62}$$

where γ^μ are the Dirac-gamma matrices ,satisfying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \equiv 2g^{\mu\nu}$ and $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

The wave function of particles Ψ is a considered as field variable in the quantum field approach. The variation of the Lagrangain depending on the field Ψ gives rise to the different types of field equation for the different particles. In other way. particles are represented by fields Ψ in the quantum field theory (QFT) approach.

Taking into account that the Euler-Lagrange equation is given by

$$\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\Big) - \frac{\partial\mathcal{L}}{\partial\Psi} = 0 \tag{2.63}$$

The Lagrangian density $\mathcal{L}$ for free particles that gives rise to the Dirac equation is given by

$$\mathcal{L} = \bar{\Psi}\Big(i\gamma^\mu\partial_\mu - m\Big)\Psi \tag{2.64}$$

In our study we will use particles as neutrons, which are neutral spin-1/2 particles with magnetic moment $\vec{\mu}$. This magnetic moment interacts with an applied magnetic field , $\hat{H} = -\vec{\mu}.\vec{B}$. The magnetic moment is given by $\vec{\mu} = K_N\vec{\sigma}$, where K_N and σ are the neutron anomalous magnetic moment and Pauli spin operator, respectively
We can have the generalized Lagrangian density adding this interaction term in (2.62)

$$\mathcal{L} = \bar{\Psi}\Big(i\gamma^\mu\partial_\mu - m - K_N\sigma_{\mu\nu}F^{\mu\nu}\Big)\Psi \tag{2.65}$$

2.6 Conservation Laws and Energy-Momentum Tensor

In physics, conservation laws play a very important role. According to Neother's theorem, every conservation law results from symmetries of the given system. For example, translational symmetry results in the conservation of linear momentum, and temporal symmetry gives us energy conservation, etc. So, if a physical system has symmetry with respect to a quantity, it means that physical quantity remains the same as the system evolves,the canonical conjugate implying that related to that symmetry theire is a conservation law.
In the quantum field theories, the role of symmetries is possibly even more important than in particles mechanics. There are different types of symmetries, i.e. Lorentz symmetry, gauge symmetries, which are internal symmetries, supersymmetry, etc.

2.6.1 Neother's Theorem and The Energy-Momentum

In classical mechanics, we see that invariance in the spatial translation gives rise to the conservation of linear momentum, while invariance under time translations is responsible for the conservation of energy. We will now apply these invariance in the context of field theories.
The Hamilton's action principle in the function of scalar field ϕ is

$$S = \int d^4x \mathcal{L}\Big(\phi(x^\mu, \partial_\mu \phi(x^\mu))\Big) \tag{2.66}$$

Under the infinitesimal translation

$$x'^\mu = x^\mu - \epsilon^\mu \tag{2.67}$$

$$\tag{2.68}$$

the scalar field ϕ does not change

$$\phi'(x) = \phi(x) \tag{2.69}$$

Using Neother's theorem and the Hamilton's least action principle (1.66), we get,

$$\delta S = \delta\mathcal{L} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial\phi - \delta^\mu_\nu \mathcal{L} \right] = 0 \tag{2.70}$$

$$\delta\mathcal{L}' = \partial_\mu \left[T^{\mu\nu} \right] = 0 \tag{2.71}$$

where

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial\phi - \delta^\mu_\nu \mathcal{L} \tag{2.72}$$

and $T^{\mu\nu}$ is two rank energy-momentum tensor. The four conserved quantities are given by

$$E = \int d^3x T^{00} \text{ and } P^i = \int d^3x T^{0i} \tag{2.73}$$

To turn the energy-momentum tensor,$T^{\mu\nu}$ into symmetric, we add an extra term as

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho \Gamma^{\rho\mu\nu} \tag{2.74}$$

Where $\Gamma^{\rho\mu\nu}$ is a function of the fields, which is anti-symmetric in the first two indices so $\Gamma^{\rho\mu\nu} = -\Gamma^{\mu\rho\nu}$. This guarantees that $\partial_\mu \partial_\rho \Gamma^{\rho\mu\nu} = 0$ so that the new energy-momentum tensor is also conserved.

2.7 Statistical Mechanics

In our study, we will deal with a system of neutrons, which are in particular the main components neutron stars. That system practically consists of trillion of neutron particles. To describe this type of system it obviously requires a statistical thermodynamics approach. Statistical thermodynamics deals with three types of system- 1) micro-canonical ensemble 2) canonical 3) grand-canonical ensemble. The micro-canonical system refers to an isolated system that has a fixed energy E, fixed particle number N, and fixed volume. The canonical-ensemble describes a system in contact with heat reservoir at constant temperature T where the system can freely exchange energy with the reservoir. Finally the grand-canonical ensemble, where the number of particles, N, as well as the energy can be exchanged with the reservoir, but the temperature T, volume V and the chemical potential μ are fixed quantities. Consider a system which can exchange energy and particles with a reservoir, and that the volume of which is also variable. If the system is heated, the system's internal energy will be increased by dU, which equals TdS. Some of portion of that energy, TdS, will be used to make work to increase the volume as pdV. If the given system gains dN particles, then it will increase energy by μdN. So, the net change of energy of this system is given by the first law of thermodynamics

$$dU = TdS - pdV + \mu dN \tag{2.75}$$

In our study, we will keep entropy S and volume fixed. Keeping these quantities unchanged, the chemical potential can be expressed by

$$\mu = \frac{\partial U}{\partial N}|_{S,V} \tag{2.76}$$

A quantum mechanical system in a pure state Ψ can be represented by the density matrix operator as

$$\rho = |\Psi >< \Psi| \tag{2.77}$$

Considering a system of mixed states and with Hamiltonian H, the system also can exchange particles and energy with its reservoir. This system, so-called grand-canonical ensemble, is described by statistical density matrix

$$\hat{\rho} = \exp^{-\beta(H-\mu N)} \tag{2.78}$$

where $\beta = 1/T$ is the inverse absolute temperature. Hence, the statistical average of a qunatity A can be calculated by

$$< A >= \frac{Tr(A\rho)}{Z} \tag{2.79}$$

where Z is the partition function of that given system expressed by $Z = Tr(\rho)$. Once we know the partition function of a given system, we can obtain all thermodynamical properties like pressure, P, entropy, S, parties number, N, and energy density, $\mathcal{E}$ in the following way, respectively.

$$P = \frac{\partial T lnZ}{\partial V} \tag{2.80}$$

$$S = \frac{\partial lnZ}{\partial T} \tag{2.81}$$

$$N = \frac{\partial TlnZ}{\partial \mu} \tag{2.82}$$

$$E = TS - PV + \mu N \tag{2.83}$$

On the other hand, introducing the grand potential

$$\Omega = -T \ln Z \tag{2.84}$$

we can write the previous equations as

$$P = -\Omega \tag{2.85}$$

$$S = -\frac{\partial \Omega}{\partial T} \tag{2.86}$$

$$\mathcal{N} = \frac{\partial \Omega}{\partial \mu} \tag{2.87}$$

$$\mathcal{E} = \Omega - \mu N \tag{2.88}$$

Chapter 3

Energy-Momentum Tensor:Equation of State

3.1 Energy-Momentum Tensor

The energy-momentum tensor as we have derived in section (2.6) in terms of the Lagrangian density ($\mathcal{L}$) is given by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial\phi - \delta^\mu_\nu \mathcal{L} \tag{3.1}$$

Adding to the tensor an arbitrary divergence less field such that it preserves the condition

$$\partial_\mu T^{\mu\nu} = 0 \tag{3.2}$$

Which means that the energy tensor is symmetric and guage invariant. The gauge invariance of the Maxwell energy-momentum tensor also confirms that the photon is a massless particle. The action of the gravitational field in curvilinear coordinates is given by

$$S = \int \mathcal{L}_G \; d\Omega = -\int \sqrt{-g} R \; d^4x \tag{3.3}$$

where R is a scalar, and $g = detg_{\mu\nu}$, which is $g = -1$ for flat space. In the presence of pure matter field and pure Maxwell field, the least action principle can be written in the following way [14]

$$\delta S = \int \delta\mathcal{L}_{int}(F^{\mu\nu}, g_{\mu\nu}) d^4x = 0 \tag{3.4}$$

$$= -\frac{1}{2}\int T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}d^4x = 0 \tag{3.5}$$

where $\mathcal{L}_{int} = -\frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}$ is the Lagrangian for the interaction between gravity and electromagnetic field.

and the generalized Lagrangian $\mathcal{L}$ is expressed as the sum of Maxwell field and matter fields.

$$\mathcal{L}(\Psi, F^{\mu\nu}) = \mathcal{L}_{A_\mu}(F^{\mu\nu}) + \mathcal{L}_\Psi(\Psi, F^{\mu\nu}) \tag{3.6}$$

$$= -\frac{1}{4}F^{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi + \bar{\Psi}\gamma^\mu\Psi A_\mu \tag{3.7}$$

where

$$\mathcal{L}_{A_\mu}(F^{\mu\nu}) = -\frac{1}{4}F^{\mu\nu}F^{\mu\nu} \tag{3.8}$$

and

$$\mathcal{L}_\Psi(\Psi, F^{\mu\nu}) = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi + \bar{\Psi}\gamma^\mu\Psi A_\mu \tag{3.9}$$

The last term of the above equation, $\bar{\Psi}\gamma^\mu\Psi A_\mu$, is responsible for the interaction charged fermions with the electromagnetic field.

3.2 Energy - Momentum Tensor of Maxwell Field

The Lagrangian of pure Maxwell field in general relativity is given by

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \tag{3.10}$$

Using equations (3.5) and (3.10), we find the energy-momentum tensor

$$T^{\rho\sigma}_{A_\mu} = \frac{-2}{\sqrt{-g}} \frac{\delta}{\delta g_{\rho\sigma}} (S_{EM}) = \frac{-2}{\sqrt{-g}} \frac{\delta}{\delta g_{\rho\sigma}} (\sqrt{-g} \mathcal{L}_{EM}) \tag{3.11}$$

or

$$T^{\mu\nu}_{A_\mu} = \frac{1}{2\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} (\sqrt{-g} F_{\lambda\tau} F_\rho \sigma g^{\rho\lambda} g^{\tau\sigma}) \tag{3.12}$$

$$= -F^{\mu\rho} F^{\nu}_{\rho} - \frac{1}{4} g^{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} \tag{3.13}$$

The components of the energy-momentum tensor are given by

$$T^{\mu\nu} = (p + \mathcal{E}) u^\mu u^\nu + p g_{\mu\nu} \tag{3.14}$$

where u^μ is the four-velocity, given in the rest frame as $u_\mu = (1, \vec{0})$. Transforming it into Minkowskian space and applying a uniform magnetic field along the x_3 direction, the energy-momentum tensor from (3.14) is [6]

$$T^{\mu\nu}_{EM} = (p + \mathcal{E}) u^\mu u^\nu + p(\eta^{\mu\nu}_{\parallel} - \eta^{\mu\nu}_{\perp}) \tag{3.15}$$

where $\mathcal{E} = -p = H^2/2$, $\eta_{\parallel}^{\mu\nu}$ is logitudinal Minkowskian metric tensor with $\mu, \nu = 0, 3$, and $\eta_{\perp}^{\mu\nu}$ is transverse Minkowskian metric tensor with $\mu, \nu = 1, 2$.

In the presence of a uniform magnetic field the energy-momentum tensor becomes anisotropic due to $O(3)$ symmetry breaking, which provides the splitting into different pressures components such as $p_{\parallel}$ and $p_{\perp}$ to the field direction.

3.3 Energy-Momentum Tensor of Free Dirac Field

We start from the Lagrangian for the free Dirac field given in (3.9), for uncharged particle, for example, neutrons

$$\mathcal{L}_{\Psi} = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - m)\Psi \tag{3.16}$$

where $\bar{\Psi} = \Psi^{\dagger}\gamma^0$.

Under the space-time transformation, the 1/2 fermionic field Ψ transforms as

$$\delta\Psi = \epsilon^{\mu}\partial_{\mu}\Psi \tag{3.17}$$

Using the condition in (2.75), the energy-momentum can be written for the Dirac Lagrangian as

$$T_{\Psi}^{\mu\nu} = i\bar{\Psi}\gamma^{\mu}\partial^{\nu}\Psi - \eta^{\mu\nu}\mathcal{L}_{\Psi} \tag{3.18}$$

We find that

$$T_{\Psi}^{00} = i\bar{\Psi}\gamma^0\partial^0\Psi - \mathcal{L}_{\Psi} \tag{3.19}$$

$$T_{\Psi}^{33} = i\bar{\Psi}\gamma^3\partial^3\Psi + \mathcal{L}_{\Psi} \tag{3.20}$$

and

$$T_{\Psi}^{jj} = i\bar{\Psi}\gamma^j\partial^j\Psi + \mathcal{L}_{\Psi}, \quad j = 1, 2 \tag{3.21}$$

3.4 Energy and Pressures of the Dense and Magnetized System

Let's assume a system with Hamiltonian H , particles number N, and chemical potential μ. We assume furthermore that the system is in thermodynamical equilibrium. The quantum-statistical average of the energy-momentum tensor of this system is given by

$$\langle T^{\mu\nu} \rangle = \frac{Tr\left[T^{\mu\nu}\exp^{-\beta(H-\mu N)}\right]}{Z} \tag{3.22}$$

where

$$T^{\mu\nu} = \int d^4x \left[T_E^{\mu\nu}M + T_{\Psi}^{\mu\nu}\right] = \int d\tau \int d^3x \left[T_E^{\mu\nu}M + T_{\Psi}^{\mu\nu}\right] \tag{3.23}$$

and Z is the partition function of the grand canonical ensemble given by

$$Z = Tr \exp^{-\beta(H-\mu N)} \tag{3.24}$$

This function can be rewritten in terms of functional integral over the set of fields ϕ as

$$Z = \int [d\phi] \exp^{-S[\phi]} \tag{3.25}$$

Where $[d\phi]$ denfines all possible field configurations, and $S[\phi] = \int d\tau d^3x \mathcal{L}[\phi]$ is the *Eulcedian* action with τ equals it. So, the partition function can be rewritten as a path integral containing the Lagrangian density as

$$Z = \int [d\phi] \exp^{\int d\tau d^3x \mathcal{L}[\phi]} = \int [d\phi] \exp^{\int_0^\beta d\tau \int d^3x \mathcal{L}(\tau,x)} \tag{3.26}$$

3.4.1 Energy Density

The energy of the system is average of sum $T_E^{00}M$ and T_Ψ^{00} equals

$$\langle T^{00} \rangle = \langle T_{EM}^{00} + T_\Psi^{00} \rangle \tag{3.27}$$

where $\langle T_{EM}^{00} \rangle = \beta V \frac{H^2}{2}$ comes from the contribution of pure classical electromagnetic field, and $\langle T_\Psi^{00} \rangle$ is responsible for the Dirac matter fields.

$$\langle T_\Psi^{00} \rangle = \frac{\int [d\Psi][\bar{\Psi}] T_\Psi^{00} e^{\int_0^\beta d\tau \int d^3x \mathcal{L}_\Psi(\tau,x)}}{Z} \tag{3.28}$$

where

$$T_{\Psi}^{00} = \int_0^{\beta} d\tau \int d^3x T_{\Psi}^{00}(\tau, x) \tag{3.29}$$

$$T_{\Psi}^{00} = i\bar{\Psi}\gamma^0\partial^0\Psi - \mathcal{L}_{\Psi} \tag{3.30}$$

and

$$\mathcal{L}_{\Psi} = \bar{\Psi}[o\gamma^0(\partial^0 - i\mu) - i\gamma^1\partial^1 - o\gamma^2\partial^2 - i\gamma^3\partial^3 - m]\Psi \tag{3.31}$$

is the many-particle Lagrangian density for uncharged particles.

$$T_{\Psi}^{00} = \int_0^{\beta} d\tau \int d^3x(i\bar{\Psi}\gamma^0\partial^0\Psi) - \int_0^{\beta} d\tau \int d^3x\bar{\Psi}[o\gamma^0\partial^0 - i\gamma^1\partial^1 - o\gamma^2\partial^2 - i\gamma^3\partial^3 - m]\Psi \tag{3.32}$$

For the partition function, we make the variable change, $\tau -> \beta\tau$

$$Z = \int [d\Psi][\bar{\Psi}]e^{\beta \int_0^1 d\tau \int d^3x \mathcal{L}'_{\Psi}} \tag{3.33}$$

Finding $\beta\frac{dZ}{d\beta}$, we can calculate the quantum-statistical average of T^{00} as

$$\langle T_{\Psi}^{00} \rangle = -\beta\left[\frac{dZ/d\beta}{Z} - \mu N\right] = -\frac{\partial\Phi}{\partial T} + \beta\Phi - \beta\mu\frac{\partial\Phi}{\partial\mu} \tag{3.34}$$

and

$$\mathcal{E} = \frac{1}{\beta V}\langle T^{00}_{\Psi}\rangle = -\frac{T}{V}\Big(\frac{\partial \Phi}{\partial T}\Big) + \frac{1}{V}\Phi - \frac{\mu}{V}\Big(\frac{\partial \Phi}{\partial \mu}\Big) \tag{3.35}$$

The grand canonical potential is related to the thermodynamical potential as $\Phi = V\Omega$, the particle density is defined as $N/V = -\frac{\partial \Phi}{\partial \mu}$, and the system's entropy as , $S = -\frac{\partial \Omega}{\partial T}$. So, the energy density can be rewritten as

$$\mathcal{E} = \Omega_f + TS + \mu\mathcal{N} + \frac{H^2}{2} \tag{3.36}$$

3.4.2 Longitudinal Pressure

The parallel pressure is calculated taking quantum-statistical average of T^{33} using the same functional integral

$$\langle T^{33}\rangle = \langle T^{33}_{EM} + T^{33}_{\Psi}\rangle = \frac{\int [d\Psi][\bar{\Psi}] T^{33} e^{\int_0^\beta d\tau \int d^3x \mathcal{L}_{\Psi}(\tau,x)}}{Z} \tag{3.37}$$

where

$$T^{33} = \int_0^\beta d\tau \int d^3x T^{33}(\tau, x) \tag{3.38}$$

$$T^{33}_{\Psi} = \int_0^\beta d\tau \int d^3x (i\bar{\Psi}\gamma^3\partial^3\Psi) + \int_0^\beta d\tau \int d^3x \bar{\Psi}[\gamma^0\partial^0 - i\gamma^1\partial^1 - \gamma^2\partial^2 - i\gamma^3\partial^3 - m]\Psi \tag{3.39}$$

and

$$\mathcal{L}_\Psi = \bar{\Psi}[i\gamma^0(\partial^0 - i\mu) - i\gamma^1\partial^1 - o\gamma^2\partial^2 - i\gamma^3\partial^3 - m]\Psi \tag{3.40}$$

We can write down the partition function, making the change of variable $x_3 - > Lx_3$ as

$$Z = \int [d\Psi][\bar{\Psi}] e^{\beta \int_0^1 d\tau \int d^3x \mathcal{L}'_\Psi} \tag{3.41}$$

where

$$\mathcal{L}'_\Psi = \bar{\Psi}[\gamma^0\partial^0 + i\mu\gamma^0 - i\gamma^1\partial^1 - o\gamma^2\partial^2 - \frac{i}{L}\gamma^3\partial^3 - m]\Psi \tag{3.42}$$

Once we have $L\frac{dZ}{dL}$, we can obtain

$$\langle T^{00}_\Psi \rangle = L\frac{(dZ/dL)}{Z} = -\frac{L}{T}\Big(\frac{d\Phi}{dL}\Big) = -\Omega_f \tag{3.43}$$

After similar calculations the contribution from T^{33}_{EM} equals $\frac{H^2}{2}$ Hence, the parallel pressure is given by

$$P_\| = -\Omega_f - \frac{H^2}{2} \tag{3.44}$$

3.5 Transverse Pressure

To find the matter contribution to the transverse pressure we have

$$\langle T^{jj}_\Psi \rangle = \frac{\int [d\Psi][\bar{\Psi}] T^{jj}_\Psi e^{\int_0^\beta d\tau \int d^3x \mathcal{L}_\Psi(\tau,x)}}{Z} \tag{3.45}$$

where

$$T_{\Psi}^{jj} = \int_0^{\beta} d\tau \int d^3x T_{\Psi}^{jj}(\tau, x), \text{with } j = 1, 2 \tag{3.46}$$

The magnetic field is applied along x_3 direction, so there is an $O(2)$ symmetry in the $x_1 - x_2$ plane. Thus, the tensor components along x_1 and x_2 direction are equivalent, that is $\langle T_{\Psi}^{11} \rangle = \langle T_{\Psi}^{22} \rangle$. Therefore, the transverse pressure, perpendicular to the magnetic field direction, is given by the average

$$\langle T_{\Psi}^{\perp\perp} \rangle = \frac{1}{2}\Big(\langle T_{\Psi}^{11} \rangle + \langle T_{\Psi}^{22} \rangle\Big) \tag{3.47}$$

With the help of equation (3.21) and making some variable change [6], we write the transverse pressure as

$$P_{\perp} = \frac{1}{\beta V}\langle T_{\Psi}^{\perp\perp} + T_{EM}^{\perp\perp} \rangle = -\Omega_f + H\frac{\partial \Omega_f}{\partial H} + \frac{H^2}{2} \tag{3.48}$$

where $\frac{\partial \Omega_f}{\partial H} = -M_f$, M_f is the magnetization of the given system. Therefore, the transverse pressure is

$$P_{\perp} = -\Omega_f - HM_f + \frac{H^2}{2} \tag{3.49}$$

3.6 Themodynamical Potential of a Dense and Magnetized Neutron System

The Lagrangian density for a neutron-particles system in the presence of a constant magnetic field is given from (2.65) by

$$\mathcal{L}_N = \bar{\Psi}\Big(i\gamma^\mu\partial_\mu - m_N - K_N\sigma_{\mu\nu}F^{\mu\nu}\Big)\Psi \tag{3.50}$$

The corresponding energy spectrum is given by

$$\mathcal{E}_{\eta,\sigma} = \eta\sqrt{p_3^2 + (\sqrt{m_N^2 + p_1^2 + p_2^2} + \sigma K_N B)^2} \tag{3.51}$$

The grand canonical partition function Z, can be written in terms of the Lagrangian (3.50) is given by

$$Z = \int [d\Psi][d\bar{\Psi}]e^{\int d^4x\mathcal{L}_N(\Psi)} \tag{3.52}$$

By definition, the thermodynamic potential Ω is given in term of the partition function as

$$\Omega_N = -T\ln Z \tag{3.53}$$

Hence, the one-loop thermodynamic potential in a constant magnetic field is given by

$$\Omega_N = -\sum_{\eta,\sigma}\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\left[\mathcal{E}_{\eta,\sigma} + \frac{1}{\beta}\ln\left[1 + e^{-\beta(\mathcal{E}_{\eta\mu}\pm\mu)}\right]\right] \tag{3.54}$$

The thermodynamic potential ,Ω_N, can be decomposed into

$$\Omega_N = \Omega_{vac} + \Omega_\mu + \Omega_\beta \tag{3.55}$$

where

$$\Omega_{vac} \equiv \Omega_N(B,0,0) = -\sum_{\eta,\sigma} \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3} |\mathcal{E}_{\eta,\sigma}|, \tag{3.56}$$

$$\Omega_\mu \equiv \Omega_N(B,\mu,0) = -\sum_{\eta,\sigma} \frac{1}{2} \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3} \Big(|\mathcal{E}_{\eta,\sigma} - \mu| - |\mathcal{E}_{\eta,\sigma}| \Big), \tag{3.57}$$

and

$$\Omega_\beta \equiv \Omega_N(B,\mu,T) = -\frac{1}{\beta} \sum_{\eta,\sigma} \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3} \ln \left(1 + e^{-\beta|\mathcal{E}_{\eta,\sigma} - \mu|} \right) \tag{3.58}$$

are the vacuum, many-particle and thermal contributions respectively.

Chapter 4

Equation of State:Numerical Results

Taking into account the $\mathcal{O}(2)$ symmetry, which remains in the presence of a uniform magnetic field, we write Eq. (3.58) in polar coordinates as

$$\Omega_\mu = -\frac{1}{4\pi^2}\sum_{\eta,\sigma}\int_0^\infty dp_3\int_0^\infty p_\perp dp_\perp\Big(|\mathcal{E}_{\eta,\sigma}-\mu| - |\mathcal{E}_{\eta,\sigma}|\Big) \tag{4.1}$$

Now, taking into account that

$$|x-\mu| = \begin{cases} x>0, & \begin{cases} x-\mu, & if\ x>\mu \\ \mu - x, & if\ x<\mu \end{cases} \\ x<0, & |x|+\mu \end{cases} \tag{4.2}$$

Then,

$$\Omega_\mu = -\frac{1}{2\pi^2}\sum_{\sigma}\int_0^\infty dp_3\int_0^\infty p_\perp dp_\perp\Big[(\mu-\mathcal{E}_{+,\sigma})\Theta(\mu-\mathcal{E}_{+,\sigma})\Big] \tag{4.3}$$

Making explicit use use of $\Theta(\mu-\mathcal{E}_{+,\sigma})$, we can write

$$\Omega_\mu = -\frac{1}{2\pi^2}\sum_{\sigma}\int_0^{\sqrt{(\mu-\sigma k_N B)^2-m_N^2}} p_\perp dp_\perp\int_0^{\sqrt{\mu^2-[(m_N^2+p_\perp)^{\frac{1}{2}}+\sigma K_N B]^2}} dp_3\ (\mu-\mathcal{E}_{+,\sigma}) \tag{4.4}$$

Now we define the different thermodynamic quantities entering in the system EoS. They are defined from (4.4)

The system magnetization can be found from

$$M = -\frac{\partial \Omega_N}{\partial B} \tag{4.5}$$

and the particle -number density as

$$\mathcal{N} = -\frac{\partial \Omega_N}{\partial \mu} \tag{4.6}$$

The energy density of the system comes from the contributions of quantum statistical average of Dirac and Maxwell field tensor. As it is demonstrated in the previous chapter, the breaking of rotational symmetry due to the applied uniform magnetic field gives rise to an anisotropy in the energy-momentum tensor. As a result, we can write the energy density $\mathcal{E}$ and pressure anisotropy at finite density μ, and temperature T = 0 as

Energy density:

$$\mathcal{E} = \Omega_N + \mu\mathcal{N} + \frac{B^2}{2} \tag{4.7}$$

Parallel pressure:

$$P_{\parallel} = -\Omega_N - \frac{B^2}{2} \tag{4.8}$$

and transverse pressure:

$$P_\perp = -\Omega_N - MB + \frac{B^2}{2} \tag{4.9}$$

which are the EoS of the dense and magnetized neutron system obtained from Eq (3.36),Eq (3.44) and Eq (3.49) in previous chapter.

In our case, we consider the ranges of applied constant magnetic field from $10^{14}G$ to $10^{18}G$.

In Fig. 4.1, we show how the system's magnetization changes as the magnetic field is increased. It is found that the system becomes completely magnetized around $B \approx 10^{15}G$ for μ.

From Fig 4.2 (a), it is obvious that neglecting Maxwell term ($B^2/2$) the value of the critical magnetic field producing the splitting is the pressures coincides with that producing the significant increase of the magnetization. Thus, in both cases, without and with Maxwell contribution, the pressure splitting starts to be noticeable closely around $B \sim 5 \times 10^{14}G$.

In Fig. 4.3, the pressure splitting between parallel and transverse pressures including Maxwell pressure for a system of charged fermion is taken from ref [7]. In this case, the applied magnetic field which makes the splitting significantly noticeable is about 10 times greater in magnitude than that found for the system of neutral fermions.

The percentage of pressure splitting between parallel and transverse pressures shows that the pressure anisotropies become significant as magnetic field increases,

$$\Delta[\%] = \frac{|P_\parallel - P_\perp|}{|P_\perp|} \times 100 \tag{4.10}$$

In figure 4.4, the pressure splitting percentage versus magnetic field including Maxwell

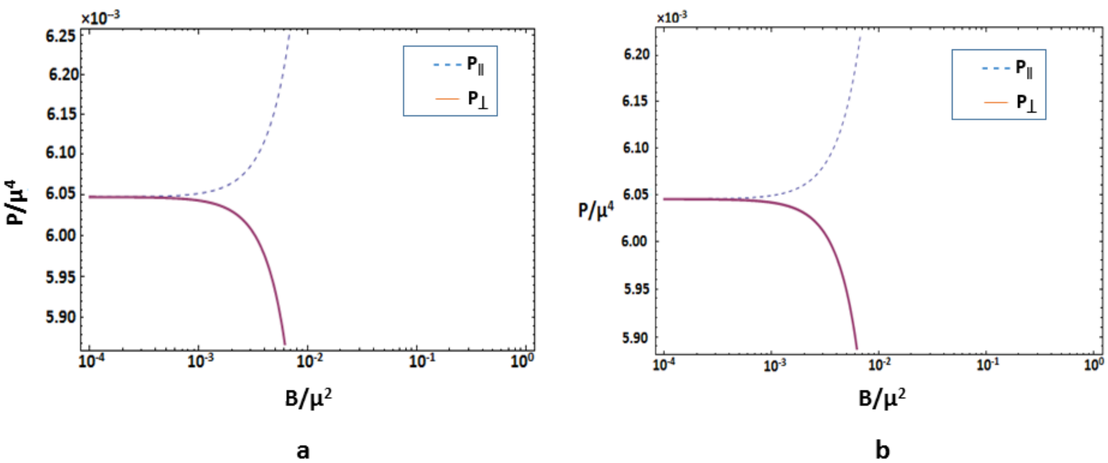

Figure 4.1: a) Effects without Maxwell term $B^2/2$, and b) with Maxwell term on parallel and transverse pressures at fixed $\mu = 1000 MeV and m_N/\mu = 0.9395$

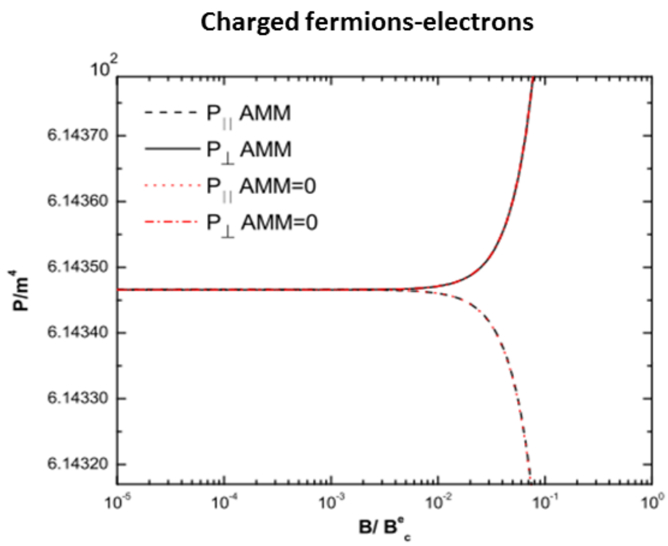

Figure 4.2: Effects of Maxwell term $B^2/2$ on the parallel and transverse pressures for charged fermions at fixed values $\mu = 10.0MeV$ and $m_N/\mu = 0.051$ [7].

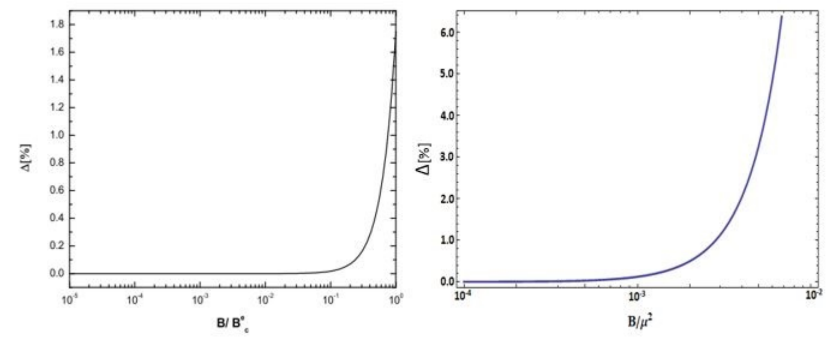

Figure 4.3: The percentage of pressure splitting for charged fermions-electrons and uncharged fermions-neutrons.

contribution. The splitting percentage for charged fermions becomes significant at around $B \sim (10^{-1}\mu^2)$. For neutral fermions the percentage of pressure splitting becomes noticeable at around $B \sim (10^{-2})\mu^2$ G.

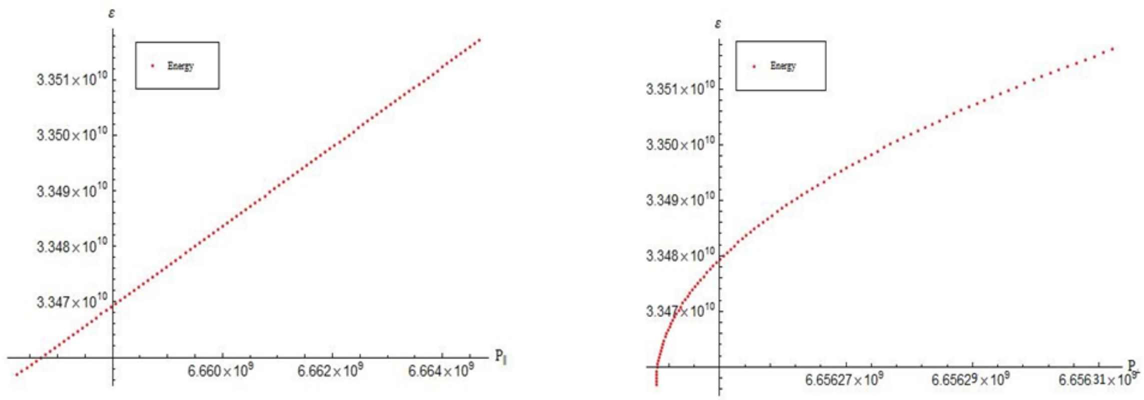

Figure 4.4: The graph shows how the energy changes with parallel and transverse pressure.

Chapter 5

Concluding Remarks

In this works, we investigate the EoS of a dense and magnetized neutron system using a relativistic field theory approach. We calculated the different statistical quantities that characterized the system EoS, and the magnetization, energy density and the parallel and transverse pressures. It was shown that the magnitude of all the quantities increase with the applied magnetic field. We found that the system in the presence of strong magnetic field exhibits as anisotropy in the pressure. The effect of pressure anisotropy becomes noticeable for the field 3 order smaller than μ^2 in magnitude. Once the field strength is increased 10 times from that value, the pressure anisotropy changes hugely into a considerable amount. The implication of this significant change in the pressure anisotropy is mainly important, in the context of astrophysical compact objects, it may lead to the deformation of the neutron star, or even enough to destroy it. Furthermore, in our the investigation the pressure anisotropy, we found that at moderated fields the contribution of matter field is more significant that that of the photon field.

The result for the neutron system (neutral spin-1/2 fermions) under consideration has been compared to those for a charged fermionic system [7]. In the comparison of these two systems, it is found that charged fermions require approximately 10 times greater fields than neutrons to have a noticeable effect in the pressure anisotropy including the Maxwell contribution.

Finally, we can conclude that the effects of pressure anisotropies are fermions both neutral

and charged. Moreover, the effect for neutral fermions is more significant, since they are free from the Landau quantization effect induced by the interaction of the applied field with the charged particles, which is known to soften the system EoS.

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Curriculum Vitae

Md Masud was born on November 10, 1986 in Narayanganj, Dhaka, Bangladesh. After completing secondary and higher secondary level majoring in science, he admitted in physics program at the "University of Dhaka ", the most prestigious university in his country, to peruse bachelor degree in physics. He continued his study in theoretical physics in master's program at the same university after the successful completion of bachelor degree in physics. In the meantime, he got a chance to excel in physics further from University of Texas at El Paso (UTEP) for studying in master's program with more competitive environment.

During this period of his master's program in UTEP, he worked on a research project as well as a teaching assistant until July 30. On his research project, he worked under his supervisor Dr. Effrain J. Ferrer, while he studied the system of neutral particles under extreme conditions with the title : "Equation of State of a Dense and Magnetized System".

Md Masud will continue his Ph.D studies at UTEP in the program of Computational Science. This interdisciplinary program will enable him to enhance computational skills as well as the knowledge of science.

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