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How to Bargain: An Interval Approach

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Abstract

In many real-life situations, we need to bargain. What is the best bargaining strategy? If you are already in a negotiating process, your previous offer was $\underline{a}$, the seller's last offer was $\overline{a} > \underline{a}$, what next offer a should you make? A usual commonsense recommendation is to "split the difference", i.e., to offer $a = (\underline{a} + \overline{a})/2$, or, more generally, to offer a linear combination $a = k \cdot \overline{a} + (1 - k) \cdot \underline{a}$ (for some parameter $k \in (0, 1)$).

The bargaining problem falls under the scope of the theory of cooperative games. In cooperative games, there are many reasonable solution concepts. Some of these solution concepts – like Nash's bargaining solution that recommends maximizing the product of utility gains – lead to offers that linearly depend on $\underline{a}$ and $\overline{a}$; other concepts lead to non-linear dependence. From the practical viewpoint, it is desirable to come up with a recommendation that would not depend on a specific selection of the solution concept – and on specific difficult-to-verify assumptions about the utility function etc.

In this paper, we deliver such a recommendation: specifically, we show that under reasonable assumption, we should always select an offer that linearly depends on $\underline{a}$ and $\overline{a}$.

1 Introduction

Formulation of the practical problem. In many real-life situations, we need to negotiate. For example, if you want to buy a house, then:

- If you want to buy a house with a list price $\overline{a}$, what offer a should you make?
- If you are already in a negotiating process, your previous offer was $\underline{a}$, the seller's last offer was $\overline{a} > \underline{a}$, what next offer a should you make?

From the viewpoint of a seller, there are similar problems:

- If you want to sell a house, and a potential buyer made an offer $\underline{a}$, what counter-offer a should you make?
- If you are already in a negotiating process, your previous offer was $\overline{a}$, the buyer's related counter-offer was $\underline{a} < \overline{a}$, what next offer a should you make?

A similar problem occurs if instead of negotiating a purchase, a person or a company in bad financial shape (e.g., in bankruptcy) is negotiating a deal on its debts. If the original debt was $\overline{a}$, what offer should we make? If you agreed that it pays $\overline{a}$, and the company's counter-offer is to pay $\underline{a} < \overline{a}$, what is your reasonable next step?

Another similar negotiation cases are negotiating for a salary with a new hire, or negotiating between an employer and an insurance company for the best way to provide insurance to the company's employers.

A similar problem occurs in an auction: when the previous bid was $\underline{a}$, what next bid should you make?

Commonsense solutions. A usual advise for the first offer is to offer a certain portion of the asking price, i.e., to offer $a = k \cdot \overline{a}$ for some coefficient $k \in (0, 1)$. The exact value of the coefficient k depends on the situation: when buying a house in the US, 70-80% is usually appropriate, while in some places, when bargaining for a tourist souvenir in an oriental bazaar, it is recommended to offer 1/3 or even 1/4 of the asking price.

A usual advise to use in the middle of negotiations is, e.g., to split the difference, i.e., to select $a = (\underline{a} + \overline{a})/2$.

In most cases, the recommended offer a is a linear function of the bounds $\underline{a}$ and $\overline{a}$.

The usual game theory approach to bargaining and negotiations: successes and limitations. In general, economic situations like this, with conflict of interest, are handled by game theory. Some game theory concepts (see, e.g., [6]) simply select the range of reasonable outcomes and to state that the exact choice of an outcome from this range is up to the participants' bargaining skills.

The first game-theoretic approach to bargaining was proposed by the Nobelist John Nash in [7, 8]; his *bargaining solution* is to select an alternative for which the product of utility gains is the largest possible. As shown in [4, 5], under reasonable assumptions about the utility function, this idea leads to the negotiation result which is a linear function of the bounds $\underline{a}$ and $\overline{a}$.

Nash's bargaining solution, by itself, does not explain how exactly we should bargain, but a more sophisticated game-theoretic analysis of the bargaining game does lead to recommendations, and under reasonable conditions, the recommended offer a is a linear function of the bounds $\underline{a}$ and $\overline{a}$; see [4, 5, 11].

However, it is well known that in cooperative game theory, there are many different solution concepts; see, e.g., [6]. As shown in [4, 5], the linearity conclusion strongly depend on the game solution concept and on assumptions about the utility function; under some other concepts and/or assumptions, the optimal offer non-linearly depends on $\underline{a}$ and $\overline{a}$.

From the practical viewpoint, it is therefore desirable to come up with a recommendations which would not depend on the specific (and somewhat arbitrary) choice of a solution concept and/or on difficult-to-verify assumptions about utility functions. In other words, it is desirable to come up with recommendations which would follow from general fundamental ideas behind the bargaining process.

Such recommendations are given in this paper.

What we do in this paper. We provide a new theoretical analysis of the negotiations situations. Specifically, we show that under reasonable assumptions, the recommended next offer/counter-offer a should indeed linearly depend on the given data.

2 Definitions and the Main Results

First offer problem: discussion. Both for the buyer and for the seller, in the first offer problem, our objective is to come up with an offer $a' \geq 0$ based on the asking price $a \geq 0$. In other words, our objective is to produce a function that, given a, generates a value a'. We will denote this function by $f(a)$.

What are the reasonable properties of this function $f(a)$? Suppose that we want to buy (or sell) two houses at the same time, with asking prices a and b, and suppose that these two houses are sold by the same seller (or buyer) company. If we treat these houses as separate purchases, then, according to the recommendation function f, we should offer $f(a)$ for the first house and $f(b)$ for the second house. Thus, the total amount of the offer is $f(a) + f(b)$.

On the other hand, if we view the two houses as a single purchase, this is equivalent to a seller offering an initial price $t = a + b$ for both houses. In this case, a reasonable thing is to offer the amount $f(t) = f(a + b)$.

It makes sense to require that the total amount offered for both houses should not depend on whether we treat these two houses separately or as a

single purchase. In other words, we require that $f(a+b) = f(a) + f(b)$, i.e., in mathematical terms, that the function $f(a)$ is *additive*.

Definition 1. *By a* unary recommendation function, *we mean a function*

$$f : \mathbb{R} \to \mathbb{R}$$

that maps every non-negative number a into a non-negative value $f(a)$ for which $f(a+b) = f(a) + f(b)$.

Proposition 1. *Every unary recommendation function has the form $f(a) = k \cdot a$ for some real number $k \geq 0$.*

Comments.

- The proofs of all the results are given in the Proofs section.
- The choice of k depends on whether we consider a buyer or a seller problem.
  - When a buyer decides on a counter-offer $f(\overline{a})$ to the original seller's price $\overline{a}$, this counteroffer should not exceed the seller's asking price, so we should have $f(\overline{a}) \leq \overline{a}$. For the linear function $f(\overline{a}) = k \cdot \overline{a}$, this requirement is equivalent to $k \leq 1$.
  - When a seller decides on a counter-offer $f(\underline{a})$ to the original buyer's proposed price $\underline{a}$, this counteroffer should not be smaller than the buyer's asking price, so we should have $f(\underline{a}) \geq \underline{a}$. For the linear function $f(\underline{a}) = k \cdot \underline{a}$, this requirement is equivalent to $k \geq 1$.

Application to auctions. In an auction, wee need to come up with a next bid $a' \geq 0$ based on the previous bid $a \geq 0$, i.e., to produce a function $f(a)$ that, given a, generates a value $a' = f(a)$.

Suppose that we participate in two auctions at the same time, with current bids a and b. If we treat these auctions as separate events, then, according to the recommendation function f, we should bid $f(a)$ in the first auction and $f(b)$ in the second auction. Thus, the total amount of our next bid is $f(a) + f(b)$.

On the other hand, if we view the two auctions as a single event, this is equivalent to a previous bid $t = a + b$ for both auctioned objects. In this case, a reasonable thing is to bid the amount $f(t) = f(a+b)$.

It makes sense to require that the total bid should not depend on whether we treat these two auctions separately or as a single event. In other words, we require that $f(a+b) = f(a)+f(b)$, i.e., in mathematical terms, that the function $f(a)$ is *additive*. Due to Proposition 1, we now conclude that $f(a) = k \cdot a$ for some $k \geq 1$.

The next bid cannot be smaller than the previous bid, so we have $k \geq 1$.

Selecting an offer in the middle of a bargaining process: discussion. In this problem, we need to come with an offer a based on the current offers $\underline{a} < \overline{a}$. In other words, we need to produce a function that, given the two non-negative numbers $\underline{a}$ and $\overline{a}$ for which $\underline{a} \le \overline{a}$, generates a value $a \in [\underline{a}, \overline{a}]$.

What are the reasonable properties of this function $f(\underline{a}, \overline{a})$? Suppose that we want to buy (or sell) two houses at the same time, with correspondingly,

- offers $\underline{a}$ and $\overline{a}$ for the first house, and
- offers $\underline{b}$ and $\overline{b}$ for the second house.

Suppose also that these two houses are sold by the same seller (or buyer) company. If we treat these houses as separate purchases, then, according to the recommendation function f, we should offer $f(\underline{a}, \overline{a})$ for the first house and $f(\underline{b}, \overline{b})$ for the second house. Thus, the total amount of the offer is $f(\underline{a}, \overline{a}) + f(\underline{b}, \overline{b})$.

On the other hand, if we view the two houses as a single purchase, this is equivalent, e.g., to the seller offering a price $\overline{t} = \overline{a} + \overline{b}$ for both houses and the buyer to buy both houses for the amount $\underline{t} = \underline{a} + \underline{b}$. In this case, a reasonable thing is to offer the amount $f(\underline{t}, \overline{t}) = f(\underline{a} + \underline{b}, \overline{a} + \overline{b})$.

It makes sense to require that the total amount offered for both houses should not depend on whether we treat these two houses separately or as a single purchase. In other words, we require that $f(\underline{a} + \underline{b}, \overline{a} + \overline{b}) = f(\underline{a}, \overline{a}) + f(\underline{b}, \overline{b})$, i.e., that the function $f(\underline{a}, \overline{a})$ is additive.

Definition 2. *By a* binary recommendation function*, we mean a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that maps every pair $(\underline{a}, \overline{a})$ of non-negative numbers with $\underline{a} \le \overline{a}$ into a non-negative value $f(\underline{a}, \overline{a}) \in [\underline{a}, \overline{a}]$ and for which $f(\underline{a} + \underline{b}, \overline{a} + \overline{b}) = f(\underline{a}, \overline{a}) + f(\underline{b}, \overline{b})$.*

Comment. The pair $(\underline{a}, \overline{a})$ of non-negative numbers with $\underline{a} \le \overline{a}$ represents an *interval* $\mathbf{a} = [\underline{a}, \overline{a}]$. We can say that an interval $\mathbf{a} = [\underline{a}, \overline{a}]$ is *non-negative* if $\underline{a} \ge 0$. Thus, f can be viewed as a function $f(\mathbf{a})$ from non-negative intervals to non-negative real numbers for which $f(\mathbf{a}) \in \mathbf{a}$ for all intervals $\mathbf{a}$.

For intervals, addition can be defined in a usual way (see, e.g., [3]), as the range of the sum $a + b$ when a is in $\mathbf{a}$ and b is in $\mathbf{b}$:

$$\mathbf{a} + \mathbf{b} \stackrel{\text{def}}{=} \{a + b : a \in \mathbf{a}, b \in \mathbf{b}\}.$$

Since addition is an increasing function of both variables, its largest value is attained when both a and b attain their largest values $\overline{a}$ and $\overline{b}$, and its smallest value is attained when both a and b attain their smallest values $\underline{a}$ and $\underline{b}$. Thus, the sum of the two interval takes the form

$$[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}].$$

Thus, in interval terms, the additivity requirement takes the form $f(\mathbf{a} + \mathbf{b}) = f(\mathbf{a}) + f(\mathbf{b})$.

Proposition 2. *Every binary recommendation function has the form* $f(\underline{a}, \overline{a}) = k \cdot \overline{a} + (1-k) \cdot \underline{a}$ *for some real number* $k \in [0,1]$.

Comment. This mathematical result is also applicable to a different practical problem: of decision making under uncertainty. Suppose that we have an object whose exact price is unknown, we only know that this price is between $\underline{a}$ and $\overline{a}$. What is a reasonable price to pay for this object? This reasonable price a should be a function of the two bounds $\underline{a}$ and $\overline{a}$; let us denote this function by $f(\underline{a}, \overline{a})$.

Since we know that the object is worth at least $\underline{a}$, the fair price must be at least $\underline{a}$: $\underline{a} \le f(\underline{a}, \overline{a})$. Similarly, since we know that the object is worth at most $\overline{a}$, the fair price must be at most $\overline{a}$: $f(\underline{a}, \overline{a}) \le \overline{a}$. Thus, we must have $f(\underline{a}, \overline{a}) \in [\underline{a}, \overline{a}]$.

If we buy two objects at the same time, then the fair price should not depend on whether we consider these objects separately or together. Now the argument similar to the one above leads to additivity $f(\mathbf{a} + \mathbf{b}) = f(\mathbf{a}) + f(\mathbf{b})$.

Thus, Proposition 2 is applicable, and according to this proposition, the fair price is $f(\underline{a}, \overline{a}) = k \cdot \overline{a} + (1-k) \cdot \underline{a}$. This formula is well known in decision making under uncertainty – it is the optimism-pessimism criterion proposed by another Nobelist L. Hurwicz [2, 6].

According to Hurwicz, the case $k = 1$, when the fair price is equal to $\overline{a}$, is the most optimistic case, when we assume that the actual value of the object is described by the largest possible value $\overline{a}$. Similarly, the case $k = 0$, when the fair price is equal to $\underline{a}$, is the most pessimistic case, when we assume that the actual value of the object is described by the smallest possible value $\underline{a}$.

3 Should the Recommendation Depend on Pre-History?

In the previous section, we assumed that the recommended offer a depends only on the bounds $\mathbf{a}$ from the previous iteration. A (seemingly) reasonable idea is to take into account the entire history of negotiations, not just the last step.

Let us describe this idea in precise mathematical terms.

- Let $\mathbf{a}_0 = [\underline{a}_0, \overline{a}_0]$ be the interval formed by the initial situation.
- Let the interval $\mathbf{a}_1 = [\underline{a}_1, \overline{a}_1]$ describes the offer and counter-offer on the 1st iteration.
- ...
- Let the interval $\mathbf{a}_t = [\underline{a}_t, \overline{a}_t]$ describes the offer and counter-offer on the t-th iteration.
- ...

- Let the interval $\mathbf{a}_T = [\underline{a}_T, \overline{a}_T]$ describes the offer and counter-offer on the last (T-th) iteration.

At each iteration, the interval becomes narrower, i.e., $\mathbf{a}_{t+1} \subseteq \mathbf{a}_t$ for all t.

Based on these $T+1$ intervals $\mathbf{a}_t$, we want to generate a recommendation $f(\mathbf{a}_0, \ldots, \mathbf{a}_T) \in \mathbf{a}_T$. As usual, the idea that we should generate the same recommendation whether we consider the two purchases separately or as a single purchase leads to the additivity requirement. Thus, we arrive at the following problem.

Definition 3. *Let T be a positive integer. By a T-ary* recommendation function*, we mean a function f that maps every tuple $(\mathbf{a}_0, \ldots, \mathbf{a}_T)$ of $T+1$ non-negative intervals satisfying the condition $\mathbf{a}_{t+1} \subseteq \mathbf{a}_t$ into a non-negative value*

$$f(\mathbf{a}_0, \ldots, \mathbf{a}_T) \in \mathbf{a}_T$$

for which

$$f(\mathbf{a}_0 + \mathbf{b}_0, \ldots, \mathbf{a}_T + \mathbf{b}_T) = f(\mathbf{a}_0, \ldots, \mathbf{a}_T) + f(\mathbf{b}_0, \ldots, \mathbf{b}_T).$$

Proposition 3. *For every T, every T-ary recommendation function depends only on $\mathbf{a}_T$: $f(\mathbf{a}_0, \ldots, \mathbf{a}_T) = f(\mathbf{a}_T)$.*

Due to Proposition 2, we thus have $f(\mathbf{a}_0, \ldots, \mathbf{a}_T) = k \cdot \overline{a}_T + (1-k) \cdot \underline{a}_T$ for some real number $k \in [0, 1]$.

Comment. Somewhat surprisingly, it turns out that the the resulting recommendation depends only on the latest offer $\mathbf{a}_T$. In other words, the corresponding negotiations are similar to Markov processes in statistics – process in which the next state depends only on the previous state but not on the pre-history.

Mathematical comment. From the mathematical viewpoint, Proposition 2 is easy to explain. Crudely speaking, due to additivity, the function $f(\mathbf{a}_0, \ldots, \mathbf{a}_T)$ linearly depends on all its variables, so

$$f(\mathbf{a}_0, \ldots, \mathbf{a}_T) = \sum_{t=0}^{T} \ell_t \cdot \underline{a}_t + \sum_{t=0}^{T} u_t \cdot \overline{a}_t$$

for some coefficients ℓ_t and u_t. For the case when $\mathbf{a}_T$ is a degenerate interval $\mathbf{a}_T = [a_T, a_T]$, due to $f(\mathbf{a}_0, \ldots, \mathbf{a}_T) \in \mathbf{a}_T$, we should have $f(\mathbf{a}_0, \ldots, \mathbf{a}_T) = a_T$. Thus, the coefficients ℓ_t and u_t corresponding to $t < T$ should be equal to 0. This is *not* a proof – a proof is given below – but it is a plausible explanation of the result.

Comment. In [10] (see also [9]), the Hurwicz formula is justified based on a different assumption: that the recommended value $f(\underline{a}, \overline{a})$ be shift- and scale-invariant. Shift-invariance means that if we add a thing of a fixed cost c to the object, the recommended price should increase by this price c: $f(\underline{a}+c, \overline{a}+c) = f(\underline{a}, \overline{a}) + c$; this is, in effect, a particular case of additivity.

Scale-invariance means that if we use different monetary units, e.g., units which are λ times smaller than the original ones, recommendations should stay the same. The original offer $\overline{a}$ in the new unit takes the form $\lambda \cdot \overline{a}$, similarly for $\underline{a}$ and for the resulting recommendation a, so scale-invariance means that

$$f(\lambda \cdot \underline{a}, \lambda \cdot \overline{a}) = \lambda \cdot f(\underline{a}, \overline{a}).$$

These two requirements justify the Hurwicz formula and, thus, a similar formula for the binary recommendation. However, they are not sufficient to analyze the dependence on pre-history.

For example, for $T = 1$, the recommendation

$$f([\underline{a}_0, \overline{a}_0], [\underline{a}_1, \overline{a}_1]) = \max\left(\underline{a}_1, \min\left(\overline{a}_1, \frac{\underline{a}_0 + \overline{a}_0 + \underline{a}_1 + \overline{a}_1}{4}\right)\right)$$

is shift- and scale-invariant, but this recommendation depends on $\mathbf{a}_0$ as well. For example,

$$f([0,3],[1,2]) = 1.5 \neq f([0,4],[1,2]) = 1.75.$$

Application to auctions. In the auction case, we can similarly consider the possible dependence of the next bid on the sequence of previous bids $a_0 \le a_1 \le a_2 \le \ldots \le a_T$. Here, the result is different.

Definition 4. *Let T be a positive integer. By a T-ary* auction recommendation function*, we mean a function f that maps every tuple $(a_0, \ldots, a_T)$ of $T+1$ non-negative numbers satisfying the condition $a_t \le a_{t+1}$ into a non-negative value $f(a_0, \ldots, a_T) \ge a_T$ for which*

$$f(a_0 + b_0, \ldots, a_T + b_T) = f(a_0, \ldots, a_T) + f(b_0, \ldots, b_T).$$

Proposition 4. *For every T, every T-ary recommendation function has the form*

$$f(a_0, \ldots, a_T) = c_0 \cdot a_0 + \sum_{t=1}^{T} c_t \cdot (a_t - a_{t-1})$$

for some values $c_0 \ge 1$, $c_1 \ge 1$, $\ldots$, $c_T \ge 1$.

4 How to Find the Equilibrium

Need to compute the equilibrium. As a result of the negotiations process, we get converging offers and counter-offers and, in the limit, we reach an "equilibrium" – the price to which both the buyer and the seller agree. What is this final price for the above linear recommendation function?

We consider the cases that start with the seller's offer. Without losing generality, let us consider the case when the negotiation process starts with the seller's offer $\overline{a}_0$; the case when it starts with the buyer's offer can be treated similarly. In this case, we start with an interval $[\underline{a}(0), \overline{a}(0)] = [0, \overline{a}_0]$.

Analysis of each iteration. Let $\underline{a}_t$ and $\overline{a}_t$ denote the values corresponding to the t-th iteration. On the previous iteration, we have an interval $[\underline{a}_{t-1}, \overline{a}_{t-1}]$. Each iteration starts with the buyer's new offer. According to Propositions 2 and 3, this offer has the form

$$\underline{a}_t = k_b \cdot \overline{a}_{t-1} + (1 - k_b) \cdot \underline{a}_{t-1}, \tag{1}$$

where k_b is a coefficient corresponding to the buyer's strategy.

Now, the seller has an interval $[\underline{a}_t, \overline{a}_{t-1}]$. On this interval, according to Propositions 2 and 3, the seller selects the value

$$\overline{a}_t = k_s \cdot \overline{a}_{t-1} + (1 - k_s) \cdot \underline{a}_t, \tag{2}$$

where k_s is a coefficient corresponding to the seller's strategy. Substituting the expression (1) for $\underline{a}_t$ into this formula, we get an expression for $\overline{a}_t$ in terms of the bounds $\underline{a}_{t-1}$ and $\overline{a}_{t-1}$ obtained on the previous iteration:

$$\overline{a}_t = k \cdot \overline{a}_{t-1} + (1 - k) \cdot \underline{a}_{t-1}, \tag{3}$$

where

$$k \stackrel{\text{def}}{=} k_s + (1 - k_s) \cdot k_b = k_s + k_b - k_s \cdot k_b. \tag{4}$$

Proof of convergence. Due to (1) and (3), the width $\overline{a}_t - \underline{a}_t$ of the new interval $[\underline{a}_t, \overline{a}_t]$ is equal to

$$\overline{a}_t - \underline{a}_t = (k - k_b) \cdot (\overline{a}_{t-1} - \underline{a}_{t-1}), \tag{5}$$

where $k - k_b = (k_s + k_b - k_s \cdot k_b) - k_b = k_s \cdot (1 - k_b)$. Thus, the width of the interval decreases in the geometric progression:

$$\overline{a}_t - \underline{a}_t = (k - k_b)^t \cdot (\overline{a}_0 - \underline{a}_0). \tag{6}$$

The width decrease to 0, and therefore, both bounds $\underline{a}_t$ and $\overline{a}_t$ tends to the same limit. Let us denote this limit by a.

Computing the equilibrium. At each stage, the lower bound increases by a value

$$\underline{a}_t - \underline{a}_{t-1} = k_b \cdot (\overline{a}_{t-1} - \underline{a}_{t-1})$$

and the upper bound decreases by the amount

$$\overline{a}_{t-1} - \underline{a}_t = (1-k) \cdot (\overline{a}_{t-1} - \underline{a}_{t-1}).$$

Thus, at each iteration, the change in the lower bound is equal to $k_b/(1-k)$ times the change in the upper bound. In the limit, when both bounds tend to the same "equilibrium" value a, we thus have the same ratio

$$a - \underline{a}_0 = \frac{k_b}{1-k} \cdot (\overline{a}_0 - a).$$

In other words, the point a divides the original interval $[\underline{a}_0, \overline{a}_0]$ in proportion $k_b/(1-k)$. Thus, the resulting equilibrium point has the form

$$a = \underline{a}_0 + \frac{1-k}{1-k+k_b} \cdot (\overline{a}_0 - \underline{a}_0) = \frac{1-k}{1-k+k_b} \cdot \overline{a}_0 + \frac{k_b}{1-k+k_b} \cdot \underline{a}_0. \quad (7)$$

Prices at different moments of time reformulated in terms of the equilibrium value. One can check that the seller's price $\overline{a}_t$ at the t-th iteration is equal to

$$a + (\overline{a}_0 - a) \cdot (k - k_b)^t. \quad (8)$$

5 What is the Purpose of Negotiations in the First Place?

Problem: why negotiations? In the above text, we simply analyzed negotiations. However, as we have mentioned in the previous section, the long negotiation process ends up with a single value a anyway. So why not come up with this value anyway? Why do we need a long negotiation process?

Idea of an explanation. The example of a souvenir seller may explain why negotiations are reasonable. While some buyers do go through the whole negotiation process and get the equilibrium price, other may value their time more and stop negotiations earlier – or even buy at the original asking price.

This idea is describe, in detail, in [4, 5].

How to formalize this idea. Each iteration of the negotiation process requires a certain time. Let us denote by w_b the amount of money by which the buyer values the time needed for a single iteration. In these terms, for the buyer, spending time on t iterations is equivalent to losing the amount $w_b \cdot t$.

Similarly, let us denote by w_s the amount of money by which the seller values the time needed for a single iteration. In these terms, for the seller, spending time on t iterations is equivalent to losing the amount $w_s \cdot t$.

If the buyer waits until t iterations, then, in accordance to the formula (8) derived in the previous section, the buyer overpays the amount $\Delta \cdot k_0^t$, where $\Delta \stackrel{\text{def}}{=} \overline{a}_0 - a$ denotes the overpayment of the original asking price $\overline{a}_0$, and $k_0 \stackrel{\text{def}}{=} k - k_b$. In addition to paying this price, the buyer also loses the amount $w_b \cdot t$ proportional to the negotiations time. Thus, the overall loss to the buyer is

$$\Delta \cdot k_0^t + w_b \cdot t. \tag{9}$$

The seller gains the overpricing amount $\Delta \cdot k_0^t$ but loses the amount $w_s \cdot t$ proportional to the negotiations time. Thus, the overall gain to the seller is

$$\Delta \cdot k_0^t - w_s \cdot t. \tag{10}$$

When to stop negotiations? When $t \to \infty$, the buyer's loss tends to ∞ and the seller's gain becomes loss. Thus, both sides are interested in stopping the negotiation process. For the buyer, the best time to stop is when the overall loss (9) is the smallest. Differentiating the expression (9) with respect to t and equating the derivative to 0, we conclude that

$$\Delta \cdot k_0^t \cdot \ln(k_0) + w_b = 0,$$

i.e., that

$$\Delta \cdot k_0^t = \frac{w_b}{|\ln(k_0)|},$$

$$k_0^t = \frac{w_b}{\Delta \cdot |\ln(k_0)|},$$

and thus, that

$$t = \frac{\ln(w_b) - \ln(\Delta) - \ln(|\ln(k_0)|)}{\ln(k_0)}.$$

For the seller, the longer the negotiations hold, the more he or she loses. When the gain (10) becomes 0, it is time to stop and accept the buyer's offer. Thus, the seller can continue negotiations until the iteration t for which

$$\Delta \cdot k_0^t = w_s \cdot t. \tag{11}$$

6 Proofs

Proof of Proposition 1. This proof easily follows from the general results about additive functions from [1]. For completeness, we can reproduce the proof here. Let us denote $f(1)$ by k. We want to prove that $f(a) = k \cdot a$ for all a.

1°. Let us first prove that $f(n) = k \cdot n$ for all integers n.

Indeed, due to additivity, from $n = 1 + \ldots + 1$ (n times), we conclude that

$$f(n) = f(1) + \ldots + f(1) \ (n \text{ times}),$$

i.e., that $f(n) = n \cdot f(1) = k \cdot n$.

2°. Let us now prove that $f(r) = k \cdot r$ for all rational numbers $r = p/q$.

Indeed, due to additivity, from $p = r \cdot q = r + \ldots + r$ (q times), we conclude that

$$f(p) = f(r) + \ldots + f(r) \text{ (q times)},$$

i.e., that $f(p) = q \cdot f(r)$. Therefore, $f(r) = f(p)/q$. From Part 1, we already know that $f(p) = k \cdot p$, hence $f(p/q) = f(r) = (k \cdot p)/q = k \cdot (p/q) = k \cdot r$. The statement is proven.

3°. Finally, let us prove that $f(a) = k \cdot a$ for all non-negative real numbers a.

Indeed, an arbitrary non-negative real number a can be, with an arbitrary accuracy 2^{-n}, approximated by non-negative rational numbers $l_n \le a \le u_n$ for which $l_n \to a$ and $u_n \to a$ as $n \to \infty$.

Due to additivity, we have $f(a) = f(l_n) + f(a - l_n)$ and since all the values of the function $f(x)$ are non-negative, we conclude that $f(l_n) \le f(a)$. Similarly, we can prove that $f(a) \le f(u_n)$ and thus, $f(l_n) \le f(a) \le f(u_n)$. For rational numbers l_n and u_n, we already know that $f(l_n) = k \cdot l_n$ and $f(u_n) = k \cdot u_n$, hence we get $k \cdot l_n \le f(a) \le k \cdot u_n$. In the limit $n \to \infty$, when $l_n \to a$ and $u_n \to a$, we get $k \cdot a \le f(a) \le k \cdot a$, i.e., the desired equality $f(a) = k \cdot a$. The proposition is proven.

Proof of Proposition 2. In view of the comment right before the formulation of Proposition 2, let us reformulate the problem in terms of intervals.

Every interval $[\underline{a}, \overline{a}]$ can be represented as a sum of two interval: a degenerate interval $[\underline{a}, \underline{a}]$ and an interval $[0, \overline{a} - \underline{a}]$ with a zero left endpoints:

$$[\underline{a}, \overline{a}] = [\underline{a}, \underline{a}] + [0, \overline{a} - \underline{a}].$$

Due to additivity, we have

$$f([\underline{a}, \overline{a}]) = f([\underline{a}, \underline{a}]) + f([0, \overline{a} - \underline{a}]).$$

Thus, to find the value $f([\underline{a}, \overline{a}])$ for a general interval, it is sufficient to find the values $f([\underline{a}, \underline{a}])$ for degenerate intervals and the values $f([0, \overline{a} - \underline{a}])$ for intervals with zero left endpoints.

For a degenerate interval $[\underline{a}, \underline{a}]$ that contains only one point $\underline{a}$, the requirement that $f(\mathbf{a}) \in \mathbf{a}$ implies that $f([\underline{a}, \underline{a}]) = \underline{a}$.

For intervals of the type $[0, a]$, we have $[0, a] + [0, b] = [0, a + b]$. Thus, due to additivity, the function $f([0, a])$ satisfies the conditions of Proposition 1, and hence, $f([0, a]) = k \cdot a$ for some $k \ge 0$. The requirement that $f([0, a]) \in [0, a]$, i.e., that $f([0, a]) \le a$, implies that $k \le 1$. Thus,

$$f([\underline{a}, \overline{a}]) = f([\underline{a}, \underline{a}]) + f([0, \overline{a} - \underline{a}]) = \underline{a} + k \cdot (\overline{a} - \underline{a}),$$

hence

$$f([\underline{a}, \overline{a}]) = k \cdot \overline{a} + (1 - k) \cdot \underline{a}.$$

The proposition is proven.

Proof of Proposition 3. Let $(\mathbf{a}_0, \ldots, \mathbf{a}_T)$ be an arbitrary tuple of non-negative intervals with the inclusion property $\mathbf{a}_{t+1} \subseteq \mathbf{a}_t$.

Due to the inclusion property, we have $\mathbf{a}_T \subseteq \mathbf{a}_0$ and thus, $\underline{a}_T \geq \underline{a}_0$. Let us denote the corresponding non-negative difference $\underline{a}_T - \underline{a}_0$ by d.

Due to additivity, for the intervals

$$\mathbf{b}_t \stackrel{\text{def}}{=} \mathbf{a}_t + [d, d] = [\underline{a}_t + d, \overline{a}_t + d],$$

we have

$$f(\mathbf{b}_0, \ldots, \mathbf{b}_T) = f(\mathbf{a}_0, \ldots, \mathbf{a}_T) + f([d, d], \ldots, [d, d]). \tag{12}$$

Due to the requirement that $f(\mathbf{a}_0, \ldots, \mathbf{a}_T) \in \mathbf{a}_T$, we have $f([d, d], \ldots, [d, d]) = d$, hence (12) implies

$$f(\mathbf{b}_0, \ldots, \mathbf{b}_T) = f(\mathbf{a}_0, \ldots, \mathbf{a}_T) + d. \tag{13}$$

Now, we will represent each interval $\mathbf{b}_t$ as a sum $\mathbf{a}_T + \mathbf{c}_t$, where $\underline{c}_t \stackrel{\text{def}}{=} \underline{b}_t - \underline{a}_T$ and $\overline{c}_t \stackrel{\text{def}}{=} \overline{b}_t - \overline{a}_T$. Here, $\underline{c}_t = \underline{b}_t - \underline{a}_T = \underline{a}_t - \underline{a}_T + d$. Due to inclusion property, we have $\mathbf{a}_t \subseteq \mathbf{a}_0$ hence $\underline{a}_t \geq \underline{a}_0$ thence, by definition of the difference d,

$$\underline{a}_t - \underline{a}_T + d = \underline{a}_t - \underline{a}_T + \underline{a}_T - \underline{a}_0 = \underline{a}_t - \underline{a}_0.$$

Thus, all the values $\underline{c}_t$ are non-negative, so we have a tuple of non-negative intervals. It is easy to see that this tuple satisfies the property $\mathbf{c}_{t+1} \subseteq \mathbf{c}_t$. Thus, by additivity,

$$f(\mathbf{b}_0, \ldots, \mathbf{b}_T) = f(\mathbf{a}_T, \ldots, \mathbf{a}_T) + f(\mathbf{c}_0, \ldots, \mathbf{c}_T). \tag{14}$$

Here, $\underline{c}_T = \underline{b}_T - \underline{a}_T = d$ and $\overline{c}_T = \overline{b}_T - \overline{a}_T = d$, so $\mathbf{c}_T = [d, d]$ and thus, due to the property $f(\mathbf{c}_0, \ldots, \mathbf{c}_T) \in \mathbf{c}_T$, we conclude that $f(\mathbf{c}_0, \ldots, \mathbf{c}_T) = d$. Thus, the formula (14) takes the form

$$f(\mathbf{b}_0, \ldots, \mathbf{b}_T) = f(\mathbf{a}_T, \ldots, \mathbf{a}_T) + d. \tag{15}$$

Substituting the expression (13) for $f(\mathbf{b}_0, \ldots, \mathbf{b}_T)$ into the left-hand side of the formula (15), we conclude that

$$f(\mathbf{a}_0, \ldots, \mathbf{a}_T) + d = f(\mathbf{a}_T, \ldots, \mathbf{a}_T) + d,$$

hence

$$f(\mathbf{a}_0, \ldots, \mathbf{a}_T) = f(\mathbf{a}_T, \ldots, \mathbf{a}_T).$$

Thus, the recommendation indeed depends only on the latest interval. The proposition is proven.

Proof of Proposition 4. Let us define the differences Δ_t as follows: $\Delta_0 = a_0$ and $\Delta_t = a_t - a_{t-1}$ for $t \geq 1$. Once we know the differences, we can reconstruct the original values a_t as $a_t = \sum_{s=0}^{t} \Delta_s$. In terms of the differences, the original monotonicity condition $a_0 \leq a_1 \leq \ldots \leq a_T$ takes the simplified form $\Delta_t \geq 0$ for all t.

Every function $f(a_0, \ldots, a_T)$ can be reformulated in terms of the differences Δ_t as $f(a_0, a_1, \ldots, a_T) = F(\Delta_0, \Delta_1, \ldots, \Delta_T)$, where

$$F(\Delta_0, \ldots, \Delta_T) \stackrel{\text{def}}{=} f(\Delta_0, \Delta_0 + \Delta_1, \ldots, \Delta_0 + \ldots + \Delta_T).$$

Thus, to find the desired function $f(a_0, a_1, \ldots, a_T)$, it is sufficient to find the new function $F(\Delta_0, \Delta_1, \ldots, \Delta_T)$.

In terms of the new function, the conditions on the function F are additivity and $F(\Delta_0, \Delta_1, \ldots, \Delta_T) \geq \Delta_0 + \ldots + \Delta_T$ for all Δ_t. Due to additivity, the fact that

$$(\Delta_0, \Delta_1, \ldots, \Delta_T) = (\Delta_0, 0, \ldots, 0) + (0, \Delta_1, 0, \ldots, 0) + \ldots + (0, 0, \ldots, 0, \Delta_T)$$

implies that

$$F(\Delta_0, \Delta_1, \ldots, \Delta_T) = F_0(\Delta_0) + F_1(\Delta_1) + \ldots + F_T(\Delta_T),$$

where $F_0(\Delta_0) \stackrel{\text{def}}{=} F(\Delta_0, 0, \ldots, 0)$, $F_1(\Delta_1) \stackrel{\text{def}}{=} F(0, \Delta_1, 0, \ldots, 0)$, ..., $F_T(\Delta_T) \stackrel{\text{def}}{=} F(0, 0, \ldots, 0, \Delta_T)$ and, in general, $F_t(\Delta_t) \stackrel{\text{def}}{=} F(0, 0 \ldots, 0, \Delta_t, 0, \ldots, 0)$. For each of the functions $F_t(\Delta_t)$, additivity (due to Proposition 1) implies that $F_t(\Delta_t) = c_t \cdot \Delta_t$ for some $c_t \geq 0$, hence

$$F(\Delta_0, \Delta_1, \ldots, \Delta_T) = \sum_{t=0}^{T} c_t \cdot \Delta_t.$$

The condition that

$$F(\Delta_0, \Delta_1, \ldots, \Delta_T) = \sum_{t=0}^{T} c_t \cdot \Delta_t \geq \sum_{t=0}^{T} \Delta_t$$

implies, for the case when only one value Δ_t is different from 0, that $c_t \geq 1$. Vice versa, once $c_t \geq 1$ for all t, the condition $F(\Delta_0, \Delta_1, \ldots, \Delta_T) \geq \sum_{t=0}^{T} \Delta_t$ is satisfied.

The proposition is proven.

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