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Towards Optimal Placement of Bio-Weapon Detectors

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***Abstract*—Biological weapons are difficult and expensive to detect. Within a limited budget, we can afford a limited number of bio-weapon detector stations. It is therefore important to find the optimal locations for such stations. A natural idea is to place more detectors in the areas with more population – and fewer in desert areas, with fewer people. However, such a commonsense analysis does not tell us how many detectors to place where. To decide on the exact placement of bio-weapon detectors, we formulate the placement problem in precise terms, and come up with an (almost) explicit solution to the resulting optimization problem.**

Formulation of the practical problem. Biological weapons are difficult and expensive to detect. Within a limited budget, we can afford a limited number of bio-weapon detector stations. It is therefore important to find the optimal locations for such stations.

Commonsense analysis of the problem. A natural idea is to place more detectors in the areas with more population – and fewer in areas with fewer people, e.g., in the desert areas. However, such a commonsense analysis does not tell us how many detectors to place where. To decide on the exact placement of bio-weapon detectors, we must formulate the placement problem in precise terms.

Objective function. The above commonsense idea is based on a (reasonable) assumption that the adversary's objective is to kill as many people as possible. Vice versa, our objective is to minimize the potential effect of a bio-weapon attack.

Comment. In this paper, we mainly concentrate on the above objective function. This objective function may not always fully describe the adversary's objectives. For example, one of the objectives of political terrorism may be extra publicity for the cause. From this viewpoint, an adversary may prefer a scenario with a smaller number of victims if several of these victims are well-known. It is therefore desirable to formulate the objective functions that describe this (and similar) approaches, and extend our optimization analysis to the case of such more complex objective functions.

Towards precise formulation of the problem: what is known. Since the objective is to target as many people as possible, to analyze this situation, we need to know how many people live at different locations. In precise terms, we assume that we know, for every possible location x, the population density $\rho(x)$ in the vicinity of this location.

We assume that we know the number N of detectors that we can afford to place in the given territory.

We also assume that we know the efficiency of a bio-weapons detector station. We will estimate this efficiency by the distance d_0 at which this station can detect an outbreak of a disease.

For many diseases, $d_0 = 0$ – we can only detect a disease when the sources of this disease reach the detecting station.

However, it is quite possible that for some diseases, we have a super-sensitive equipment that is able to detect the concentration of the bio-weapons agent at a level below the threshold that makes this agent dangerous to the population. In this case, we can detect the coming disease before it starts affecting people in the direct vicinity of the station – i.e., in effect, we have $d_0 > 0$.

For simplicity, we assume that the disease spreads equally fast in all directions.

Comment. This is also a somewhat simplifying assumption, since in reality, a disease spreads

- either with human movements – in which case in the vicinity of an interstate it spreads faster in the direction of the interstate,
- or with wind – in which case it spreads faster in the direction of the prevailing winds.

How we can describe the detector placement. On a large-scale basis, we need to decide how many detectors to place in different areas. In other words, we need to find the *density* $\rho_d(x)$ of detector placement – the number of detectors per unit of area (e.g., a square mile).

Under this description, the number of detectors in an area of size Δx is approximately equal to $\rho_d(x) \cdot \Delta x$, so the overall number of detectors can be obtained by adding these amounts, as $\int \rho_d(x)\,dx$. Thus, the constraint that we have exactly N detecting stations can be described as

$$\int \rho_d(x)\,dx = N. \quad (1)$$

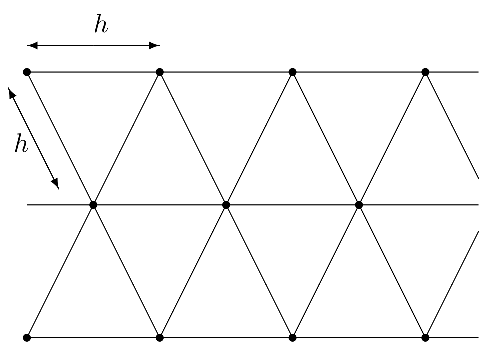
Optimal placement of sensors: at the vertices of a hexagonal grid. We want to place the sensors in such a way that the largest distance D to a sensor is as small as possible. Alternatively, if D is fixed, we want to minimize the number of sensors for which every point is at a distance $\leq D$ from one of the sensors. In geometric terms, this means that every point on a plane belongs to a circle of radius D centered on one the sensors – and thus, the whole plane is covered by such circles. Out of all such coverings, we want to find the covering with the smallest possible number of sensors.

It is known that the smallest such number is provided by an equilateral triangle grid, i.e., a grid formed by equilateral triangles; see, e.g., [1]. Hence, in this paper, we will select such a grid.

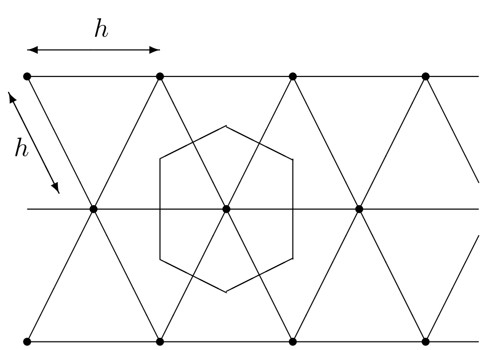
Locations of detector stations are assumed to be known to the adversary. Bio-weapon detector stations are not easily concealable. Thus, we assume that the adversary knows the locations of different stations.

How to estimate the effect of placing bio-weapons at a location x. Let us assume that we have already decided how many detectors to place in different regions, i.e., that we have already selected the density function $\rho_d(x)$.

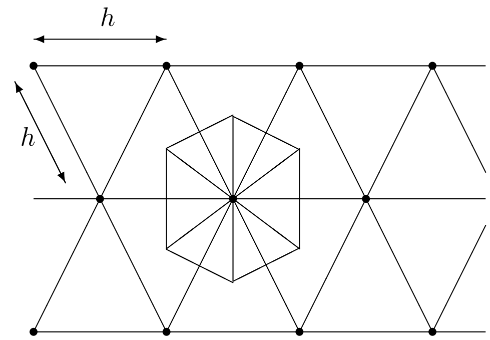
Within a small region of area A, we have $A \cdot \rho_d(x)$ detectors. Thus, if we, e.g., place these detectors on a grid with distance h between the two neighboring ones in each direction, we have:



For this placement, the set of all the points which are closest to a given detector forms a hexagonal area:



This hexagonal area consists of 6 equilateral triangles with height $h/2$:



In each triangle, the height $h/2$ is related to the size s by the formula

$$\frac{h}{2} = s \cdot \cos(60^\circ) = s \cdot \frac{\sqrt{3}}{2}, \tag{2}$$

hence

$$s = \frac{h}{\sqrt{3}} = h \cdot \frac{\sqrt{3}}{3}. \tag{3}$$

Thus, the area A_t of each triangle is equal to

$$A_t = \frac{1}{2} \cdot s \cdot \frac{h}{2} = \frac{1}{2} \cdot \frac{\sqrt{3}}{3} \cdot \frac{1}{2} \cdot h^2 = \frac{\sqrt{3}}{12} \cdot h^2. \tag{4}$$

So, the area A_s of the whole set is equal to 6 times the triangle area:

$$A_s = 6 \cdot A_t = \frac{\sqrt{3}}{2} \cdot h^2. \tag{5}$$

Each point from the region is the closest to one of the points from the detector grid, so the region of area A is thus divided into $A \cdot \rho_d(x)$ (practically) disjoint sets of area $\frac{\sqrt{3}}{2} \cdot h^2$. So, the area of the region is equal to the sum of the areas of these sets:

$$A = (A \cdot \rho_d(x)) \cdot \frac{\sqrt{3}}{2} \cdot h^2. \tag{6}$$

Dividing both sides of this equality by A, we conclude that

$$1 = \rho_d(x) \cdot \frac{\sqrt{3}}{2} \cdot h^2, \tag{7}$$

and hence, that

$$h = \frac{c_0}{\sqrt{\rho_d(x)}}, \tag{8}$$

where we denote

$$c_0 \stackrel{\text{def}}{=} \sqrt{\frac{2}{\sqrt{3}}}. \tag{9}$$

From the viewpoint of the adversary, it is desirable to place the bio-weapon at a location which is the farthest away from the detectors – so that it will take the longest time to be detected. For the grid placement, this location is at one of the vertices of the hexagonal zone – at which the distance from each neighboring detector is equal to $s = h \cdot \frac{\sqrt{3}}{3}$. By using formula (8), we can determine s in terms of $\rho_d(x)$, as

$$s = \frac{c_1}{\sqrt{\rho_d(x)}}, \tag{10}$$

where we denote

$$c_1 = \frac{\sqrt{3}}{3} \cdot c_0 = \frac{\sqrt[4]{3} \cdot \sqrt{2}}{3}. \quad (11)$$

Once the bio-weapon is placed at this location, it starts spreading until its spread area reaches the threshold distance d_0 from the detector. In other words, it spreads for the distance $s - d_0$. During this spread, the disease covers the circle of radius $s - d_0$ and area $\pi \cdot (s - d_0)^2$.

By using the known population density $\rho(x)$, we can conclude that the number of affected people $n(x)$ is equal to

$$n(x) = \pi \cdot (s - d_0)^2 \cdot \rho(x). \quad (12)$$

Substituting the expression (10) into this formula, we conclude that

$$n(x) = \pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x). \quad (13)$$

Adversary's choice of the location. According to our assumption about the adversary's objective function, the adversary wants to maximize the number of affected people. Thus, the adversary will select a location x for which this number $n(x)$ (as described by the expression (13)) is the largest possible. The resulting damage n is thus equal to the largest of the values $n(x)$:

$$n = \max_x \left(\pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x) \right). \quad (14)$$

Our objective. Our objective is to minimize this overall damage, i.e., to select the detector placement $\rho_d(x)$ so as to minimize this value n.

In other words, we want to *minimize* the worst-possible (maximal) damage. This *minimax* formulation is typical for *zero-sum games*, in which the interests of the two sides are exactly opposite; see, e.g., [3].

Thus, we arrive at the following problem:

Resulting formulation of the problem in precise terms. We are given the population density $\rho(x)$, the value d_0, and the total number of detectors N. We want to find a function $\rho_d(x)$ that minimizes the expression (14) under the constraint $\int \rho_d(x)\, dx = N$.

Analysis of the resulting optimization problem. The damage is determined by the maximum n of the function $n(x)$. Let us assume that we have already selected the optimal detector density function, i.e., the function $\rho_d(x)$ that minimizes the desired objective function n.

Let us show that the damage function $n(x)$ corresponding to this selection is constant. We will prove this by contradiction. If the function $n(x)$ is not constant, this means that at some locations x, the values $n(x)$ are smaller than the maximum n. In this case, we can slightly increase the detector density at the locations where $n(x) = n$, at the expense of slightly decreasing the location density at locations where $n(x) < x$.

The value of the expected damage $n(x)$ monotonically decreases with the detector density $\rho_d(x)$. This mathematical observation is in perfect accordance with common sense: the more detectors we place at some location, the earlier we will be able to detect bio-weapons and thus, the smaller will be the resulting damage.

Thus, the above re-arrangement of detectors will decrease the value of $n(x)$ at all locations where $n(x) = n$ – and slightly increase at all other locations. As a result, after this detector relocation, the overall maximum $n = \max\limits_x n(x)$ will decrease. This possibility contradicts to our initial assumption that the value n is the smallest possible. Thus, the function $n(x)$ is indeed constant.

Let us denote this constant by n_0. Then, from the formula (13) for $n(x)$, we conclude that

$$n_0 = \pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x). \quad (15)$$

Thus, we conclude that

$$\left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 = \frac{n_0}{\pi \cdot \rho(x)}, \quad (16)$$

$$\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 = \frac{c_2}{\sqrt{\rho(x)}}, \quad (17)$$

where we denote

$$c_2 \stackrel{\text{def}}{=} \frac{\sqrt{n_0}}{\sqrt{\pi}}. \quad (18)$$

Thus, we get

$$\frac{c_1}{\sqrt{\rho_d(x)}} = d_0 + \frac{c_2}{\sqrt{\rho(x)}}, \quad (19)$$

$$\sqrt{\rho_d(x)} = \frac{c_1}{d_0 + \dfrac{c_2}{\sqrt{\rho(x)}}}, \quad (20)$$

and

$$\rho_d(x) = \frac{c_1^2}{\left(d_0 + \dfrac{c_2}{\sqrt{\rho(x)}} \right)^2}, \quad (21)$$

From (11), we conclude that

$$c_1^2 = \frac{2 \cdot \sqrt{3}}{9}, \quad (22)$$

hence

$$\rho_d(x) = \frac{2 \cdot \sqrt{3}}{9} \cdot \frac{1}{\left(d_0 + \dfrac{c_2}{\sqrt{\rho(x)}} \right)^2}. \quad (23)$$

The value c_2 must be determined from the equation (1).

Thus, we arrive at the following solution:

Solution: the optimal detector location is characterized by the detector density

$$\rho_d(x) = \frac{2\cdot\sqrt{3}}{9}\cdot\frac{1}{\left(d_0+\dfrac{c_2}{\sqrt{\rho(x)}}\right)^2},$$

where the parameter c_2 must be determined from the equation

$$\int \frac{2\cdot\sqrt{3}}{9}\cdot\frac{1}{\left(d_0+\dfrac{c_2}{\sqrt{\rho(x)}}\right)^2}\,dx = N. \tag{24}$$

Case of $d_0 = 0$. As we have mentioned earlier, in some cases, we have $d_0 = 0$. In this case, the formula (23) takes a simplified form

$$\rho_d(x) = C\cdot\rho(x) \tag{25}$$

for some constant C. In this case, the detector density is exactly proportional to the population density.

Substituting the expression (25) into the constraint (1), we conclude that

$$N = C\cdot N_p, \tag{26}$$

where $N_p = \int \rho(x)\,dx$ is the total population. Thus, $C = \dfrac{N}{N_p}$ and the optimal detector placement (25) takes the form

$$\rho_d(x) = \frac{N}{N_p}\cdot\rho(x). \tag{27}$$

Towards more relevant objective functions: fuzzy techniques may help. In our computations, we assumed that the main objective of the adversary is to maximize the number of people affected by the bio-weapon, i.e., to maximize the value $\int_A \rho(x)\,dx$, where A is the region were people become affected before the bio-weapon is detected.

As we have mentioned, the actual adversary's objective function may differ from this simplified objective function. For example, the adversary may take into account that different locations have different publicity potential. In this case, instead of maximizing the total number of affected people, the adversary may want to maximize the weighted value $\int_A \widetilde{\rho}(x)\,dx$, where $\widetilde{\rho}(x) \stackrel{\text{def}}{=} w(x)\cdot\rho(x)$, and the weight $w(x)$ describes the publicity-related importance of the location x.

From the purely mathematical viewpoint, once we have fixed the weight functions $w(x)$, we get the exact same problem as before – with the only difference that we now have "effective population density" $\widetilde{\rho}(x)$ instead of the original density $\rho(x)$. Thus, if we know the exact weight function $w(x)$, then we find the optimal detector density $\rho_d(x)$ by substituting the effective population density $\widetilde{\rho}(x)$ instead of $\rho(x)$ into the above formulas.

The problem is we do not know the exact weights, we only have expert estimates for these weights, estimates that are formulated in terms of words form natural language. To formalize these estimates, we can use fuzzy techniques; see, e.g., [2], [4].

Once we have the fuzzy values of $w(x)$ and, thus, $\widetilde{\rho}(x)$, the above formulas lead to fuzzy recommendations for the desired detector density $\rho_d(x)$.

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