Least Sensitive (Most Robust) Fuzzy "Exclusive Or" Operations

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***Abstract*—In natural language, "or" sometimes means "inclusive or" and sometimes means "exclusive or". To adequately describe commonsense and expert knowledge, it is therefore important to have not only t-conorms describing fuzzy "inclusive or" operations, but also fuzzy "inclusive or" operations $f_\oplus(a,b)$. Since the degrees of certainty are only approximately defined, it is reasonable to require that the corresponding operation be the least sensitive to small changes in the inputs. In this paper, we show that the least sensitive fuzzy "exclusive or" operation has the form $f_\oplus(a,b)=\min(\max(a,b),\max(1-a,1-b))$.**

I. Introduction

"Exclusive or" operation: a reminder. In commonsense and expert reasoning, people use logical connectives like "and" and "or". Depending on the context, commonsense "or" can mean:

- either "inclusive or" – when "A or B" means that it is also possible to have both A and B,
- or "exclusive or" – when "A or B" means that one of the statements holds but not both.

For example, for a dollar, a vending machine can produce either a coke or a diet coke, but not both. This is an example of "exclusive or".

In the traditional 2-valued logic, with two possible truth values 0 (false) and 1 (true), the "exclusive or" operation $\oplus$ is defined as follows: $0\oplus 0=1\oplus 1=0$ and $0\oplus 1=1\oplus 0=1$.

In mathematics and computer science, "inclusive or" is the one most frequently used as a basic operation. However, the "exclusive or" operation is also actively used:

- As we have mentioned, it is used to formalize commonsense reasoning.
- It is used in computer design, since it corresponds to the bit-by-bit addition of binary numbers (the carry is the "and").
- It is also actively used in quantum computing algorithms; see, e.g., [12].

Need for fuzzy logic. One of the main objectives of fuzzy logic is to formalize commonsense and expert reasoning. Fuzzy logic takes into account that experts are usually not absolutely confident in their statements.

Specifically, in the traditional 2-valued logic, for each statement, we have two options:

- either this statement is true, in which case we are absolutely confident in this statement,
- or this statement is false, in which case we are absolutely confident that this statement is false.

In fuzzy logic, we allow intermediate degrees of belief. Specifically, to each statement S, we associate a "degree" $d(S)$ – a number from the interval $[0,1]$ that describes how confident we are in this statement:

- if we are absolutely confident in the statement S, we assign degree $d(S)=1$;
- if we are absolutely confident that the statement S is false, we assign degree $d(S)=0$;
- in all other cases, we assign degrees between 0 and 1: $d(S)\in(0,1)$.

Need for fuzzy operations. Based on the expert statements, we need to make conclusions. Conclusions are often based on several statements; in this case, a conclusion is equivalent to a logical combination of expert statement. For example, if we need both A and B to deduce C, then our degree of confidence in C is equal to our degree of confidence $d(A\,\&\,B)$ that both A and B are true. Alternatively, if either A or B imply C, then our degree of confidence in C is equal to our degree of confidence $d(A\vee B)$ that either A or B are true.

Ideally, we should elicit, from the experts, not only their degrees of confidence in the original statements, but also their degree of confidence in all possible logical combinations of these statements. However, in practice, there are exponentially many such combinations, so it is not possible to elicit the degree of confidence in all of them. Since we cannot extract these degrees of confidence from the experts, we therefore need to estimate these degrees based on our degrees of confidence in the individual statements.

For example, when the logical combination is $A\,\&\,B$, we face the following problem:

- we know the degrees of confidence $a=d(A)$ and $b=d(B)$ corresponding to the original statements A and B;
- based on these two values, we must find a reasonable estimate $f_\&(a,b)$ for $d(A\,\&\,B)$.

This estimate is usually called an *"and"-operation* or a *t-norm*.

The t-norm must satisfy reasonable conditions. For example, if about each of the two statements A and B we know whether

it is true or false, i.e., if $a = d(A)$ and $b = d(B)$ both belong to the set $\{0, 1\}$, then we can easily find out whether $A \,\&\, B$ is true or not. The value $f_\&(a, b)$ should coincide with this result, i.e., in other words, the t-norm should be the extension of the corresponding crisp operation: $f_\&(1, 1) = 1$ and $f_\&(0, 0) = f_\&(0, 1) = f_\&(1, 0) = 0$.

Another requirements is that, since intuitively, $A \,\&\, B$ and $B \,\&\, A$ mean the same thing, the corresponding degrees $f_\&(a, b)$ and $f_\&(b, a)$ should coincide: $f_\&(a, b) = f_\&(b, a)$. In mathematical terms, this means that the operation $f_\&(a, b)$ must be commutative.

Similarly, since intuitively, $A \,\&\, (B \,\&\, C)$ and $(A \,\&\, B) \,\&\, C$ mean the same thing, the corresponding degrees $f_\&(a, f_\&(b, c))$ and $f_\&(f_\&(a, b), c)$ should coincide $f_\&(a, f_\&(b, c)) = f_\&(f_\&(a, b), c)$. In mathematical terms, this means that the operation $f_\&(a, b)$ must be associative.

To estimate the degree $d(A \vee B)$ for the usual (inclusive) "or", we similarly need a commutative and associative operation $f_\vee(a, b)$ for which $f_\vee(0, 0) = 0$ and $f_\vee(0, 1) = f_\vee(1, 0) = f_\vee(1, 1) = 1$. Such operations are called *"or"-operations* or *t-conorms*.

Need for fuzzy "exclusive or" operations. Since "exclusive or" is also used in commonsense reasoning, there is a practical need for a fuzzy versions of this operation, i.e., an operation $f_\oplus(a, b)$ that uses the degrees $a = d(A)$ and $b = d(B)$ of two statements A and B to estimate our degree of confidence in the statement $A \oplus B$.

If about each of the two statements A and B we know whether it is true or false, i.e., if $d(a)$ and $d(B)$ both belong to the set $\{0, 1\}$, then we can easily find out whether $A \oplus B$ is true or not. The value $f_\oplus(a, b)$ should coincide with this result, i.e., in other words, the fuzzy "exclusive or" operation should be the extension of the corresponding crisp operation:

$$f_\oplus(0, 0) = f_\oplus(1, 1) = 0; \quad f_\oplus(0, 1) = f_\oplus(1, 0) = 1. \tag{1}$$

Several fuzzy "exclusive or" operations have been proposed. Several fuzzy "exclusive or" operations have been proposed; see, e.g., [1]. These fuzzy versions are actively used in machine learning; see, e.g., [3], [7], [8], [14]. In particular, some of these papers (especially [8]) use a natural extension of fuzzy "exclusive or" from a binary to a k-ary operation.

Need for sensitivity. Fuzzy logic operations deal with experts' degrees of certainty in their statements. These degrees are not precisely defined, the same expert can assign, say, 0.7 and 0.8 to the same degrees of belief. It is therefore reasonable to require that the result of the fuzzy operation does not change much if we slightly change the inputs. A reasonable way to formalize this requirement is to require that the operation $f(a, b)$ satisfies the following property:

$$|f(a, b) - f(a', b')| \le k \cdot \max(|a - a'|, |b - b'|), \tag{2}$$

for some real number k. Operations that satisfy this property are called *k-sensitive* or *k-robust.*

Need for the least sensitivity. In general, there are different sensitive operations which satisfy the inequality (2) for different values k. The smaller the value k, the less sensitive the corresponding operation.

It is therefore reasonable to look for the *least sensitive* (*most robust*) operations, i.e., for the operations that are k-sensitive for the smallest possible value k.

For t-norms and t-conorms, the least sensitivity requirement leads to reasonable operations. It is known that there is only one least sensitive t-norm ("and"-operation) $f_\&(a, b) = \min(a, b)$, and only one least sensitive t-conorm ("or"-operation) $f_\vee(a, b) = \max(a, b)$; see, e.g., [9], [10], [13].

What we do in this paper. In this paper, we describe the least sensitive fuzzy "exclusive or" operation.

II. Main Result

Definition 1. *A function $f : [0, 1] \times [0, 1] \to [0, 1]$ is called a* fuzzy "exclusive or" *operation if it satisfies the following conditions: $f(0, 0) = f(1, 1) = 0$ and $f(0, 1) = f(1, 0) = 1$.*

Comment. We could also require other conditions, e.g., commutativity and associativity. However, our main objective is to select a single operation which is the least sensitive. The weaker the condition, the larger the class of operations that satisfy these conditions, and thus, the stronger the result that our operation is the least sensitive in this class.

Thus, to make our result as strong as possible, we selected the weakest possible condition – and thus, the largest possible class of "exclusive or" operations.

Definition 2. *Let $k > 0$ be a real number. We say that a function $f : [0, 1] \times [0, 1] \to [0, 1]$ is* k-sensitive *if for all a, b, a', and b', the following condition is satisfied:*

$$|f(a, b) - f(a', b')| \le k \cdot \max(|a - a'|, |b - b'|).$$

Definition 3. *Let F be a class of functions from $[0, 1] \times [0, 1]$ to $[0, 1]$. We say that a function $f \in F$ is* the least sensitive *in the class F if for some real number k,*

- *this function f is k-sensitive, and*
- *for every $k' < k$, no function $f' \in F$ is k'-sensitive.*

Theorem. *In the class of all fuzzy "exclusive or" operations, the following function is the least sensitive:*

$$f_\oplus(a, b) = \min(\max(a, b), \max(1 - a, 1 - b)). \tag{3}$$

Comments.

- The operation (3) can be understood as follows. In the crisp (two-valued) logic, "exclusive or" $\oplus$ can be described in terms of the "inclusive or" operation $\vee$ as

$$a \oplus b \Leftrightarrow (a \vee b) \,\&\, \neg(a \,\&\, b).$$

If we:

- replace $\vee$ with the least sensitive "or"-operation $f_\vee(a,b) = \max(a,b)$,
- replace & with the least sensitive "and"-operation $f_\&(a,b) = \min(a,b)$, and
- replace $\neg$ with the least sensitive negation operation $f_\neg(a) = 1 - a$,

then we get the expression (3) given in the Theorem.

- The above operation is associative and has a value a_0 (equal to 0.5) which satisfies the property $a \oplus a_0 = a$ for all a. Thus, from the mathematical viewpoint, this operation is an example of a *nullnorm*; see, e.g., [2].

III. Proof of the Main Result

We will prove that the Theorem is true for $k = 1$.

1°. First, let us prove that the operation (3) indeed satisfies the condition (2) with $k = 1$. In other words, let us prove that for every $\varepsilon > 0$, if $|a - a'| \le \varepsilon$ and $|b - b'| \le \varepsilon$, then $|f_\oplus(a,b) - f_\oplus(a',b')| \le \varepsilon$.

1.1°. It is known (see, e.g., [9], [10], [13]) that the functions $\min(a,b)$, $\max(a,b)$, and $1-a$ satisfy the condition (2) with $k = 1$. In particular, this means that if $|a - a'| \le \varepsilon$ and $|b - b'| \le \varepsilon$, then we have

$$|\max(a,b) - \max(a',b')| \le \varepsilon \tag{4}$$

and also

$$|(1-a) - (1-a')| \le \varepsilon \text{ and } |(1-b) - (1-b')| \le \varepsilon. \tag{5}$$

1.2°. From (5), by using the property (2) for the max operation, we conclude that

$$|\max(1-a, 1-b) - \max(1-a', 1-b')| \le \varepsilon. \tag{6}$$

1.3°. Now, from (4) and (6), by using the property (2) for the min operation, we conclude that

$$|\min(\max(a,b), \max(1-a,1-b)) - \min(\max(a',b'), \max(1-a',1-b'))| \le \varepsilon. \tag{7}$$

The statement is proven.

2°. Let us now assume that $f(a,b)$ is an "exclusive or" operation that satisfies the condition (2) with $k = 1$. Let us prove that then $f(a,b)$ coincides with the function (3).

2.1°. Let us first prove that $f(0.5, 0.5) = 0.5$.

By the definition of the exclusive or operation, we have $f(0,0) = 0$ and $f(0,1) = 1$. Due to the property (2), we have

$$|f(0,0) - f(0.5,0.5)| \le \max(|0-0.5|, |0-0.5|) = 0.5 \tag{8}$$

thus,

$$f(0.5,0.5) \le f(0,0) + 0.5 = 0 + 0.5 = 0.5. \tag{9}$$

Similarly, due to the property (2), we have

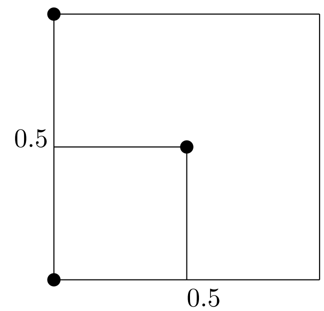
$$|f(0,1) - f(0.5,0.5)| \le \max(|0-0.5|, |1-0.5|) = 0.5 \tag{10}$$

thus,

$$f(0.5,0.5) \ge f(0,1) - 0.5 = 1 - 0.5 = 0.5. \tag{11}$$

From (9) and (11), we conclude that $f(0.5,0.5) = 0.5$.

This proof uses the value $f(0,0)$, $f(0.5,0.5)$, and $f(0,1)$ of the function $f(a,b)$ at three points $(0,0)$, $(0.5,0.5)$, and $(0,1)$. These three points are marked on the following figure:



2.2°. Let us now prove that $f(a,a) = a$ for $a \le 0.5$.

Due to the property (2), we have

$$|f(0,0) - f(a,a)| \le \max(|0-a|, |0-a|) = a \tag{12}$$

thus,

$$f(a,a) \le f(0,0) + a = 0 + a = a. \tag{13}$$

Similarly, due to the property (2), we have

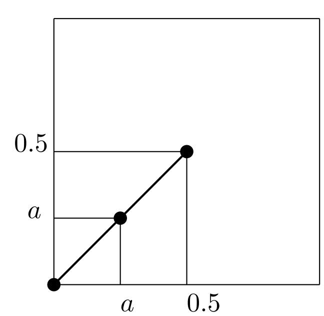
$$|f(0.5,0.5) - f(a,a)| \le \max(|0.5-a|, |0.5-a|) = 0.5 - a \tag{14}$$

thus,

$$f(a,a) \ge f(0.5,0.5) - (0.5-a) = 0.5 - (0.5-a) = a. \tag{15}$$

From (13) and (15), we conclude that $f(a,a) = a$.

This proof uses the value $f(0,0)$, $f(a,a)$, and $f(0.5,0.5)$ of the function $f(a,b)$ at three points $(0,0)$, (a,a), and $(0.5,0.5)$. These three points are marked on the following figure:



2.3°. Let us now prove that $f(1-a, 1-a) = a$ for $a \le 0.5$.

Due to the property (2), we have

$$|f(1,1) - f(1-a, 1-a)| \le \max(|a|, |a|) = a \qquad (16)$$

thus,

$$f(1-a, 1-a) \le f(1,1) + a = 0 + a = a. \qquad (17)$$

Similarly, due to the property (2), we have

$$|f(0.5, 0.5) - f(1-a, 1-a)| \le \max(|0.5-a|, |0.5-a|) =$$

$$0.5 - a \qquad (18)$$

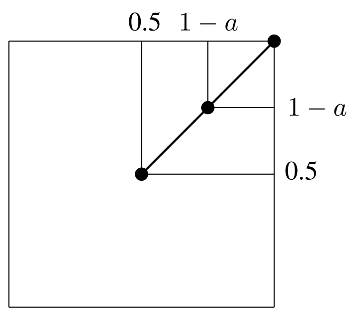
thus,

$$f(1-a, 1-a) \ge f(0.5, 0.5) - (0.5 - a) =$$

$$0.5 - (0.5 - a) = a. \qquad (19)$$

From (17) and (19), we conclude that $f(1-a, 1-a) = a$.

This proof uses the value $f(0.5, 0.5)$, $f(1-a, 1-a)$, and $f(1,1)$ of the function $f(a,b)$ at the following three points: $(0.5, 0.5)$, $(1-a, 1-a)$, and $(1,1)$. These three points are marked on the following figure:



2.4°. Let us now prove that $f(a, 1-a) = 1-a$ for $a \le 0.5$.

Due to the property (2), we have

$$|f(0,1) - f(a, 1-a)| \le \max(|a|, |a|) = a \qquad (20)$$

thus,

$$f(a, 1-a) \ge f(0,1) - a = 1 - a. \qquad (21)$$

Similarly, due to the property (2), we have

$$|f(0.5, 0.5) - f(a, 1-a)| \le \max(|0.5-a|, |0.5-a|) =$$

$$0.5 - a \qquad (22)$$

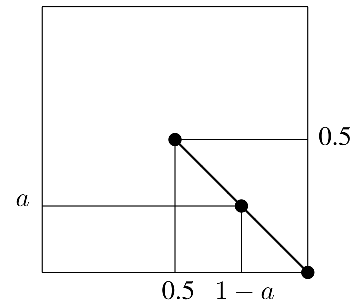
thus,

$$f(a, 1-a) \le f(0.5, 0.5) + (0.5 - a) =$$

$$0.5 + (0.5 - a) = 1 - a. \qquad (23)$$

From (21) and (23), we conclude that $f(a, 1-a) = 1-a$.

2.5°. Similarly, by considering the three values $f(0.5, 0.5)$, $f(1-a, a)$, and $f(1, 0)$, we can prove that $f(1-a, a) = 1-a$ for $a \le 0.5$.



2.6°. From Parts 2.2, 2.3, 2.4, and 2.5 of this proof, we can now conclude that the formula (6) holds when $b = a$ and when $b = 1 - a$.

Indeed, if $a = b$, then we have either $a \le 0.5$ or $a \ge 0.5$. In the first case, we have the expression $f(a,a)$ described in Part 2.2. In the second case, $a = 1 - a'$, where $a' \stackrel{\text{def}}{=} 1 - a$. In this case, $f(a,b) = f(a,a) = f(1-a', 1-a')$ for $a' \le 0.5$, i.e., we have the expression described in Part 2.3.

If $b = 1-a$ and $a \le 0.5$, then we have the case $f(a, 1-a)$ described in Part 2.4. If $b = 1-a$ and $a \ge 0.5$, then $a = 1-a'$ for $a' \le 0.5$, and $b = 1 - a = a'$. In this case, $f(a,b) = f(1-a', a')$, the case described in Part 2.5.

2.7°. Let us now prove that the formula (6) holds for arbitrary a and b.

In principle, we can have four cases depending on whether $b \le a$ or $b \ge a$ and on whether $b \le 1-a$ or $b \ge 1-a$. Without losing generality, let us consider the case when $b \le a$ and $b \le 1-a$; the other three cases can be proven in a similar way.

For this case, by adding two inequalities $b \le a$ and $b \le 1-a$, we get $2b \le 1$ hence $b \le 0.5$. Thus, from Parts 2.2 and 2.3 of this proof, we conclude that $f(b,b) = b$ and $f(1-b, b) = 1-b$.

Here, $b \le 1-a$ implies that $a \le 1-b$, hence $b \le a \le 1-b$. Due to the property (2), we have

$$|f(a,b) - f(b,b)| \le \max(|a-b|, |b-b|) = a - b. \qquad (24)$$

For every real number z, we have $z \le |z|$. In particular, when $z = x - y$ for some real numbers x and y, we have $x - y \le |x-y|$. Hence, we get $x \le y + |x-y|$. In particular, for $x = f(a,b)$ and $y = f(b,b)$, we have

$$f(a,b) \le f(b,b) + |f(a,b) - f(b,b)|. \qquad (25)$$

Due to (24), we conclude that

$$f(a,b) \le f(b,b) + |f(a,b) - f(b,b)| \le f(b,b) + (a-b). \quad (26)$$

We have already mentioned that $f(b,b) = b$, hence

$$f(a,b) \le f(b,b) + (a-b) = b + (a-b) = a. \qquad (27)$$

Similarly, due to the property (2), we have

$$|f(a,b) - f(1-b, b)| \le \max(|a - (1-b)|, |b-b|) =$$

$$(1-b) - a, \qquad (28)$$

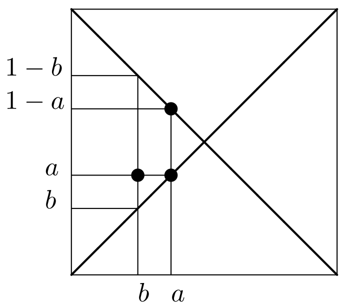
thus,

$$f(a,b) \geq f(1-b,b) - ((1-b)-a) =$$

$$(1-b) - ((1-b)-a) = a. \quad (29)$$

From (27) and (29), we conclude that $f(a,b) = a$.

This proof uses the value $f(a,b)$, $f(b,b)$, and $f(1-b,b)$ of the function f at three points (a,b), (b,b), and $(1-b,b)$. These three points are marked on the following figure:



Similarly:

- for $b \leq a$ and $b \geq 1-a$, i.e., when $1-a \leq b \leq a$, by considering the points $(a, 1-a)$ and (a,a), we conclude that $f(a,b) = 1-a$;
- for $b \geq a$ and $b \leq 1-a$, i.e., when $a \leq b \leq 1-a$, by considering the points (a,a) and $(a, 1-a)$, we conclude that $f(a,b) = b$;
- for $b \geq a$ and $b \geq 1-a$, i.e., when $1-b \leq a \leq b$, by considering the points $(1-b,b)$ and (b,b), we conclude that $f(a,b) = 1-b$.

In other words, we prove that the formula (6) holds for all a and b. The theorem is proven.

IV. Fuzzy "Exclusive Or" Operations Which Are The Least Sensitive On Average

Average sensitivity: reminder. As we have mentioned earlier, the fuzzy degrees are given with some uncertainty. In other words, different experts – and even the same expert at different times – would assign somewhat different numerical values to the same degree of certainty. In the main part of the paper, we have showed how to select fuzzy operations $c = f(a,b)$ in such a way that "in the worst case", the change in a and b would lead to the smallest possible change in the value $c = f(a,b)$.

Another reasonable possibility is to select fuzzy operations $c = f(a,b)$ in such a way that "on average", the change in a and b would lead to the smallest possible change in the value $c = f(a,b)$.

For each pair of values a and b, it is reasonable to assume that the differences Δa and Δb between the different numerical values corresponding to the same degree of certainty are independent random variables with 0 mean and small variance σ^2. Since the differences Δa and Δb are small, we can expand the difference $\Delta c = f(a+\Delta a, b+\Delta b) - f(a,b)$ in Taylor series with respect to Δa and Δb and keep only linear terms in this expansion:

$$\Delta c \approx \frac{\partial f}{\partial a} \cdot \Delta a + \frac{\partial f}{\partial b} \cdot \Delta b. \quad (30)$$

Since the variance is independent with 0 mean, the mean of Δc is also 0, and variance of Δc is equal to

$$\sigma^2(a,b) = \left(\left(\frac{\partial f}{\partial a} \right)^2 + \left(\frac{\partial f}{\partial b} \right)^2 \right) \cdot \sigma^2. \quad (31)$$

This is the variance for given a and b. To get the average variance, it is reasonable to average this value over all possible values of a and b, i.e., to consider the value

$$I \cdot \sigma^2,$$

where

$$I \stackrel{\text{def}}{=} \int_{a=0}^{a=1} \int_{b=0}^{b=1} \left(\left(\frac{\partial f}{\partial a} \right)^2 + \left(\frac{\partial f}{\partial b} \right)^2 \right) da\, db. \quad (32)$$

Thus, the average sensitivity is the smallest if, among all possible functions $f(a,b)$ satisfying the given constraints, we select a function for which the integral I takes the smallest possible value.

Average sensitivity: known results. [11], [13]

- For negation operations, this approach selects the standard function
$$f_\neg(a) = 1-a.$$
- For "and"-operations (t-norms), this approach selects $f_\&(a,b) = a \cdot b$.
- For "or"-operations (t-conorms), this approach selects $f_\vee(a,b) = a + b - a \cdot b$.

New result: formulation. We consider *"exclusive or"* operations, i.e., functions $f(a,b)$ from $[0,1] \times [0,1]$ to $[0,1]$ for which $f(0,b) = b$, $f(a,0) = a$, $f(1,b) = 1-b$, and $f(a,1) = 1-a$.

Our main result is that among all such operations, the operation which is the least sensitive on average has the form

$$f_\oplus(a,b) = a + b - 2 \cdot a \cdot b. \quad (33)$$

Comment. This operation can be explained as follows:

- First, we represent the classical (2-valued) "exclusive or" operation $a \oplus b$ as $(a \vee b) \& (\neg a \vee \neg b)$.
- Then, to get a fuzzy analogue of this operation, we replace $p \vee q$ with $p + q - p \cdot q$, $\neg p$ with $1-p$, and $p \,\&\, q$ with $\max(p+q-1, 0)$.

Indeed, in this case,

$$a \vee b = a + b - a \cdot b;$$

$$\neg a \vee \neg b = (1-a) \vee (1-b) =$$

$$(1-a) + (1-b) - (1-a) \cdot (1-b) =$$

$$1 - a + 1 - b - (1 - a - b + a \cdot b) =$$

$$1-a+1-b-1+a+b-a\cdot b=1-a\cdot b,$$

and thus,

$$(a\vee b)+(\neg a\vee\neg b)-1=$$
$$a+b-a\cdot b+1-a\cdot b-1=$$
$$a+b-2\cdot a\cdot b.$$

For values $a,b\in[0,1]$, we have $a^2\le a$ and $b^2\le b$, hence

$$(a\vee b)+(\neg a\vee\neg b)-1=$$
$$a+b-2\cdot a\cdot b\ge a^2+b^2-2\cdot a\cdot b=$$
$$(a-b)^2\ge 0,$$

therefore, indeed

$$(a\vee b)\&(\neg a\vee\neg b)=$$
$$\max((a\vee b)+(\neg a\vee\neg b)-1,0)=$$
$$(a\vee b)+(\neg a\vee\neg b)-1.$$

This replacement operation sounds arbitrary, but the resulting "exclusive or" operation is uniquely determined by the sensitivity requirement.

V. Proof of the Auxiliary Result

It is known similarly to the fact that the minimum of a function is always attained at a point where its derivative is 0, the minimum of a functional is always attained at a function where its *variational derivative* is equal to 0 (see, e.g., [5]; see also [11], [13]):

$$\frac{\delta L}{\delta f}=\frac{\partial L}{\partial f}-\sum_i\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial f_i}\right)=0,$$

where $f_{,i}\stackrel{\rm def}{=}\dfrac{\partial f}{\partial x_i}$.

Applying this *variational equation* to the functional $I=\int L\,da\,db$, with $L=\left(\dfrac{\partial f}{\partial a}\right)^2+\left(\dfrac{\partial f}{\partial b}\right)^2$, we conclude that

$$-\frac{\partial}{\partial a}\left(2\cdot\frac{\partial f}{\partial a}\right)-\frac{\partial}{\partial b}\left(2\cdot\frac{\partial f}{\partial b}\right)=0,$$

i.e., we arrive at the equation

$$\nabla^2 f=0,\tag{34}$$

where $\nabla\stackrel{\rm def}{=}\left(\dfrac{\partial f}{\partial a},\dfrac{\partial f}{\partial b}\right)$ and

$$\nabla^2 f=\frac{\partial^2 f}{\partial a^2}+\frac{\partial^2 f}{\partial b^2}.$$

The equation (34) is known as the *Laplace equation*, and it is known (see, e.g., [4]) that a solution to this equation is uniquely determined by the boundary conditions – i.e., in our case, by the values on all four parts of the boundary of the square $[0,1]\times[0,1]$: lines segments $a=0$, $a=1$, $b=0$, and $b=1$. One can easily show that the above function $f(a,b)=a+b-2\cdot a\cdot b$ satisfies the Laplace equation – since both its second partial derivatives are simply 0s. It is also easy to check that for all four sides, this function coincides with our initial conditions:

- when $a=0$, we get $f(a,b)=0+b-2\cdot 0\cdot b=b$;
- when $a=1$, we get $f(a,b)=1+b-2\cdot 1\cdot b=1-b$;
- when $b=0$, we get $f(a,b)=a+0-2\cdot a\cdot 0=a$;
- when $b=1$, we get $f(a,b)=a+1-2\cdot 1\cdot a=1-a$.

Thus, due to the above property of the Laplace equation, the function $f(a,b)=a+b-2\cdot a\cdot b$ is the only solution to this equation with the given initial condition – therefore, it coincides with the desired *the least sensitive on average* "exclusive or" operation (which satisfies the same Laplace equation with the same boundary conditions).

The theorem is proven.

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References

[1] B. C. Bedregal, R. H. S. Reiser, and G. P. Dimuro, "Xor-implications and e-implications: classes of fuzzy implications based on fuzzy xor", *Electronic Notes in Theoretical Computer Science (ENTCS)*, 2009, Vol. 247, pp. 5–18.

[2] G. Beliakov, A. Pradera, and T. Calvo, *Aggregation Functions: A Guide for Practitioners*, Springer-Verlag, Berlin-Heidelberg-New York, 2007.

[3] L. Y. Cai and H. K. Kwan, "Fuzzy Classifications Using Fuzzy Inference Networks", *IEEE Transactions on Systems, Man, and Cybernetics. Part B: Cybernetics*, 1998, Vol. 28, No. 3, pp. 334–347.

[4] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island, 1998.

[5] I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Dover Publ., New York (2000)

[6] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Upper Saddle River, New Jersey: Prentice Hall, 1995.

[7] H. Le Capitaine and C. Frélicot, "A new fuzzy 3-rules pattern classifier with reject options based on aggregation of membership degrees", *Proceedings of the 12th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems IPMU'08*, Malaga, Spain, 2008.

[8] L. Mascarilla, M. Berthier, and C. Frélicot, "A k-order fuzzy OR operator for pattern classification with k-order ambiguity rejection", *Fuzzy Sets and Systems*, 2008, Vol. 159, No. 15, pp. 2011–2029.

[9] H. T. Nguyen and V. Kreinovich, "Towards theoretical foundations of soft computing applications", *International Journal on Uncertainty, Fuzziness, and Knowledge-Based Systems (IJUFKS)*, 1995, Vol. 3, No. 3, pp. 341-373.

[10] H. T. Nguyen, V. Kreinovich, and D. Tolbert, "On robustness of fuzzy logics". *Proceedings of the 1993 IEEE International Conference on Fuzzy Systems FUZZ-IEEE'93*, San Francisco, California, March 1993, Vol. 1, pp. 543–547.

[11] H. T. Nguyen, V. Kreinovich, and D. Tolbert, "A measure of average sensitivity for fuzzy logics", *International Journal on Uncertainty, Fuzziness, and Knowledge-Based Systems*, 1994, Vol. 2, No. 4, pp. 361–375.

[12] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.

[13] H. T. Nguyen and E. A. Walker, *First Course on Fuzzy Logic*, CRC Press, Boca Raton, Florida, 2006.

[14] W. Pedrycz and G. Succi, "fXOR fuzzy logic networks", *Soft Computing*, 2002, Vol. 7, pp. 115–120.