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A Simple Physics-Motivated Equivalent Reformulation of P=NP that Makes This Equality (Slighty) More Plausible

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Abstract

In our opinion, one of the reasons why the problem P$\stackrel{?}{=}$NP is so difficult is that while there are good intuitive arguments in favor of P$\neq$NP, there is a lack of intuitive arguments in favor of P=NP. In this paper, we provide such an argument – based on the fact that in physics, many dependencies are scale-invariant, their expression does not change if we simply change the unit in which we measure the corresponding input quantity (e.g., replace meters by centimeters). It is reasonable to imagine similar behavior for time complexity $t_A(n)$ of algorithms A: that the form of this dependence does not change if we change change the unit in which we measure the input length (e.g., from bits to bytes). One can then easily prove that the existence of such scale-invariant algorithms for solving, e.g., propositional satisfiability is equivalent to P=NP. This equivalent reformulation of the formula P=NP is, in our opinion, much more intuitively reasonable than the original formula – at least to those who are familiar with the importance of scale-invariance in physics.

Need for a better intuitive understanding of the P=NP option. In history of mathematics, solutions to many long-standing problems came when the consequences of the corresponding statements being true or false became clearer. For example, mathematicians have tried, for many centuries, to deduce the V-th Postulate – that for every point P outside a line ℓ, there is no more than one line ℓ' going through P and parallel to ℓ – from other postulates of geometry. The independence proof appeared only after the results of Gauss, Bolyai, and Lobachevsky made geometry without this postulate more intuitively clear; see, e.g., [2].

For this viewpoint, maybe one of the difficulties in solving the P$\stackrel{?}{=}$NP prob-

lem is that while there are good intuitive arguments in favor of P≠NP, there is a definite lack of intuitively convincing arguments in favor of P=NP.

Example of intuitive arguments in favor of P≠NP. Example of arguments in favor of P≠NP are numerous, many of them boil down to the following: if P=NP, we will have feasible algorithms for solving classes of problems which are now considered highly creative – and for which, therefore, such algorithms are intuitively unlikely.

One example of a highly creative activity area is mathematics, where one of main objectives is, given a statement S, to prove either this statement or its negation $\neg$. We are usually interested in proofs which can be checked by human researchers, and are, thus, of reasonable size. In the usual formal systems of mathematics, the correctness of a formal proof can be checked in polynomial time. So, the problem of finding a reasonable-size proof of a given statement S (or of its negation) belongs to the class NP. If P was equal to NP, then we would be able to have a polynomial-time algorithm for proving theorems – a conclusion which most mathematicians consider unlikely.

Similarly, in theoretical physics, one of the main challenged is to find formulas that describe the observed data. The size of such a formula cannot exceed the amount of data, so the size is feasible. Once a formula is proposed, checking whether all the data is consistent with this formula is easy; thus, the problem of searching for such a formula is in the class NP. So, if P was equal to NP, we would have a feasible algorithm for the activity which is now considered one of the most creative ones – judged, e.g., by the fact that Nobel Prizes in Physics get a lot of publicity and bring a lot of prestige.

What we do in this paper. In this paper, we propose a physics-motivated argument in favor P=NP.

Physical motivations: the idea of scale invariance. The value of a physical quantity can be measured by using different units. For example, length can be measured in meters, in centimeters, in inches, etc. When we replace the original unit by a new unit which is λ times larger, all numerical values x change, from x to $x' = \dfrac{x}{\lambda}$, so that $x = \lambda \cdot x'$; this transformation is known as *re-scaling*.

For many physical processes, there is no preferred value of a physical quantity; see, e.g., [3]. For such processes, it is reasonable to require that the corresponding dependence have the same form no matter what measuring unit we use. For example, the dependence of the pendulum's period T on its length L has the form $T = f(L) = 2\pi \cdot \sqrt{\dfrac{L}{g}} = c \cdot \sqrt{L}$ for an appropriate constant c. If we change the unit of length, so that $L = \lambda \cdot L'$, we get a *similar* dependence $T = f(\lambda \cdot L') = c \cdot \sqrt{\lambda \cdot L'} = c \cdot \sqrt{\lambda} \cdot \sqrt{L'}$. If we now accordingly re-scale time, to new units which are $\sqrt{\lambda}$ times larger, then we get the exact *same* dependence

in the new units $T' = c \cdot \sqrt{L'}$. Since we get the same formula for all measuring unit, physicists say that the pendulum formula is *scale-invariant*.

In general, a dependence $y = f(x)$ is called scale-invariant if each re-scaling of x can be compensated by an appropriate re-scaling of y, i.e., if for every λ, there is a value $C(\lambda)$ for which $f(\lambda \cdot x) = C(\lambda) \cdot f(x)$ for all x and λ. For continuous functions, this functional equation leads to the power law $f(x) = c \cdot x^\alpha$; see, e.g., [1].

Scale-invariance is ubiquitous in physics: e.g., it helps explain most fundamental equations of physics, such as Einstein's equations of General Relativity, Schrödinger'e equations of quantum mechanics, Maxwell's equations, etc. [4]. It is also useful in explaining many semi-empirical computer-related formulas; see, e.g., [**?**].

Maybe some algorithms are scale-invariant. One of the main concepts underlying P and NP is the concept of computational complexity $t_A(n)$ of an algorithm A, which is defined as the largest running time of this algorithm on all inputs of length $\le n$. Similar to physics, in principle, we can use different units to measure the input length: we can use bits, bytes, Kilobytes, Megabytes, etc. It is therefore reasonable to conjecture that for some algorithms, the dependence $t_A(n)$ is scale-invariant – i.e., that its form does not change if we simply change a unit for measuring input length.

It should be mentioned that for discrete variables n, scale-invariance cannot be defined in exactly the same way, since the fractional length n/λ does not always make sense. Thus, we require scale-invariance only *asymptotically*, when $n \to \infty$.

Definition.

- *We say that functions $f(n)$ and $g(n)$ are* asymptotically equivalent *(and denote it by $f(n) \sim g(n)$) if $f(n)/g(n) \to 1$ when $n \to \infty$.*
- *We say that a function $f(n)$ from natural numbers to natural numbers is* asymptotically scale-invariant *if for every integer k, there exists an integer $C(k)$ for which $f(k \cdot n) \sim C(k) \cdot f(n)$.*
- *We say that an algorithm A is* scale-invariant *if its computational complexity function $t_A(n)$ is scale-invariant.*

Now, we are ready to present the promised equivalent reformulation of P=NP, a reformulation that – in view of the ubiquity of scale invariance in physics – provides *some* intuitive argument in favor of this equality.

Proposition. P=NP *if and only if there exists a scale-invariant algorithm for solving propositional satisfiability SAT.*

Proof. If P=NP, then there exists a polynomial-time algorithm A for solving SAT, i.e., an algorithm for which $t_A(n) \le C \cdot n^\alpha$ for some C and α. We can modify this algorithm as follows: first, we run A, then wait until the moment $C \cdot n^\alpha$. Thus modified algorithm A' also solves SAT, and its running time $t_{A'}(n) = C \cdot n^\alpha$ is clearly scale-invariant.

Vice versa, let us assume that A is scale-invariant algorithm for solving SAT. For $k = 2$, this means that for some number $C(2)$, the ratio $\dfrac{t_A(2n)}{C(2) \cdot t_A(n)}$ tends to 1 as $n \to \infty$. By definition of the limit, that there exists an N such that for all $n \ge N$, we have $\dfrac{t_A(2n)}{C(2) \cdot t_A(n)} \le 2$, i.e., $t_A(2n) \le 2 \cdot C(2) \cdot t_A(n)$. By induction, for values $n = 2^k \cdot N$, we can now prove that $t_A(2^k \cdot N) \le (2 \cdot C(2))^k \cdot t_A(N)$.

For every $n \ge N$, the smallest k for which $2^k \cdot N \ne n$ can be found as $k = \lceil \log_2(n/N) \rceil \le \log_2(n/N) + 1$. By definition, the function $t_A(n)$ is non-decreasing, hence $t_A(n) \le t_A(2^k \cdot N)$ and thus, $t_A(n) \le (2 \cdot C(2))^k \cdot t_A(N)$. Due to the above inequality for k, we get

$$t_A(n) \le (2 \cdot C(2))^{\log_2(n/N)+1} \cdot t_A(N) = (2 \cdot C(2))^{\log_2(n/N)} \cdot 2 \cdot C(2) \cdot f(N).$$

Here,

$$(2 \cdot C(2))^{\log_2(n/N)} = \left(2^{\log_2(2 \cdot C(2))}\right)^{\log_2(n/N)} = 2^{\log_2(2 \cdot C(2)) \cdot \log_2(n/N)} =$$

$$\left(2^{\log_2(n/N)}\right)^{\log_2(2 \cdot C(2))} = \left(\frac{n}{N}\right)^\alpha,$$

where $\alpha \stackrel{\text{def}}{=} \log_2(2 \cdot C(2))$, so $t_A(n) \le \left(\dfrac{n}{N}\right)^\alpha \cdot 2 \cdot C(2) \cdot f(N)$. Thus, the SAT-solving algorithm A is indeed polynomial time, and hence, P=NP. The proposition is proven.

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