6-2011

# Product of Partially Ordered Sets (Posets), with Potential Applications to Uncertainty Logic and Space-Time Geometry

Francisco Zapata
*The University of Texas at El Paso*

Olga Kosheleva
*The University of Texas at El Paso*, olgak@utep.edu

Karen Villaverde
*New Mexico State University - Main Campus*, kvillave@cs.nmsu.edu



Comments:
Technical Report: UTEP-CS-11-28
Published in *International Journal of Innovative Management, Information & Production (IJIMIP)*, 2011, Vol. 2, No. 4, pp. 10-28.

# Product of Partially Ordered Sets (Posets), with Potential Applications to Uncertainty Logic and Space-Time Geometry

Francisco Zapata[1], Olga Kosheleva[1], and Karen Villaverde[2]
[1]University of Texas at El Paso
El Paso, TX 79968, USA
[2]New Mexico State University
Las Cruces, NM 88003, USA
fazg74@gmail.com, olgak@utep.edu, kvillave@cs.nmsu.edu

**Abstract**

One of the main objectives of science and engineering is to help people select the most beneficial decisions. To make these decisions,

- we must know people's preferences,
- we must have the information about different events – possible consequences of different decisions, and
- since information is never absolutely accurate and precise, we must also have information about the degree of certainty.

All these types of information naturally lead to partial orders:

- For preferences, $a \prec b$ means that $b$ is preferable to $a$. This relation is used in decision theory.
- For events, $a \prec b$ means that $a$ can influence $b$. This causality relation is used in space-time physics.
- For uncertain statements, $a \prec b$ means that $a$ is less certain than $b$. This relation is used in logics describing uncertainty such as fuzzy logic.

In many practical situations, we are analyzing a complex system that consists of several subsystems. Each subsystem can be described as a separate ordered space. To get a description of the system as a whole, we must therefore combine these ordered spaces into a single space that describes the whole system.

In this paper, we consider the general problem of how to combine two ordered spaces $A_1$ and $A_2$ into one. We also analyze which properties of orders are preserved under the resulting products.

# 1 Formulation of the Problem.

**Partially ordered sets (posets) in space-time geometry.** Starting from general relativity, space-time models are usually formulated in terms of appropriate physical fields, e.g., a metric field; see, e.g., [5]. These fields assume that the space-time is smooth. However, there are important situations of non-smoothness:

- *singularities* like the Big Bang or a black hole, and
- *quantum fluctuations.*

According to modern physics (see, e.g., [5]), a proper description of the corresponding non-smooth space-time models means that we no longer have a metric field, we only have a *causality* relation $\preceq$ between events – a partial order; see, e.g., [1, 4, 8, 9].

*Comment.* We will use the standard notation $a \prec b$ meaning that $a \preceq b$ and $a \neq b$, i.e., equivalently, that $a \preceq b$ and $b \not\preceq a$.

**Products of space-time posets.** Sometimes, we need to consider *pairs* of events – e.g., in situations like quantum entanglement, situations of importance to quantum computing [7]. How to extend partial orders on posets $A_1$ and $A_2$ to a partial order on the set $A_1 \times A_2$ of all such pairs?

**Posets in uncertainty logic: need for products.** A similar partial order $\preceq$ is useful in describing degrees of expert's certainty, where $a \preceq a'$ means that $a$ corresponds to less certainty than $a'$; see, e.g., [2, 6].

Sometimes, two (or more) experts evaluate a statement $S$. Then, our certainty in $S$ is described by a *pair* $(a_1, a_2)$, where $a_i \in A_i$ is the $i$-th expert's degree of certainty. When our certainty in $S$ is described by a *pair* $(a_1, a_2) \in A_1 \times A_2$, we must define a *partial order* on the set $A_1 \times A_2$ of all pairs.

**What we do in this paper.** In this paper, we consider the general problem of how to combine two ordered spaces $A_1$ and $A_2$ into one. We also analyze what properties of orders are preserved under the resulting products.

*Comment.* Some of our results were presented at the 14th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics SCAN2010, Lyon, France, September 2730, 2010 [10].

## 2 Products of Partially Ordered Sets: What Is Known

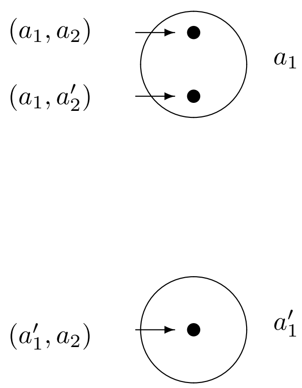
**Known examples of product operations.** At present, two product operations are known [1, 8]:

- *Cartesian product*: $(a_1, a_2) \preceq (a'_1, a'_2) \Leftrightarrow (a_1 \preceq_1 a'_1 \,\&\, a_2 \prec_2 a'_2)$, and
- *lexicographic product*

$$(a_1, a_2) \preceq (a'_1, a'_2) \Leftrightarrow ((a_1 \preceq a'_1 \,\&\, a_1 \neq a'_1) \vee (a_1 = a'_1 \,\&\, a_2 \preceq a'_2)).$$

**Physical meaning of lexicographic order.** For space-time models, a possible meaning of a lexicographic product $A_1 \times A_2$ is that $A_1$ is *macroscopic* space-time, and $A_2$ is *microscopic* space-time. When $a'_1$ macroscopically precedes $a_1$, i.e., when $a'_1 \prec_1 a_1$, then, of course, the microscopic events should not matter – and we should have $(a'_1, a'_2) \preceq (a_1, a_2)$.

On the other hand, when $a'_1 = a_1$, i.e., when, from the macroscopic viewpoint, the two events $a'_1$ and $a_1$ are indistinguishable, we need to go to the microscopic level to see which of these two events causally influences another one, i.e., $(a_1, a'_2) \preceq (a_1, a_2) \Leftrightarrow a'_2 \preceq_2 a_2$.




**Logical meaning of Cartesian product.** The Cartesian product means that our confidence in $S$ is higher than in $S'$ if and only if it is higher for both experts. In other words, the Cartesian product corresponds to a *maximally cautious* approach.

**Logical meaning of lexicographic product.** In contrast, a lexicographic product means that we have *absolute confidence* in the first expert, and we only use the opinion of the 2nd expert when, to the 1st expert, the degrees of certainty are equivalent.

# 3 Products of Partially Ordered Sets: Towards a General Description

**A natural question.** A natural question is: what other operations are possible?

**What we prove in this section.** In this section, we prove that every non-degenerate product operation satisfying the above properties coincides with one of these two products.

**Reasonable assumptions on the product.** It is reasonable to assume that the validity of the relation $(a_1, a_2) \preceq (a_1', a_2')$ depends only on whether $a_1 \preceq_1 a_1'$, $a_1' \preceq_1 a_1$, $a_2 \preceq_2 a_2'$, and/or $a_2' \preceq_2 a_2$.

It is also reasonable to assume that if $a_1 \preceq_1 a_1'$ and $a_2 \preceq_2 a_2'$ then

$$(a_1, a_2) \preceq (a_1', a_2').$$

**Definition 1.**

- *By a* product operation*, we mean a Boolean function* $P : \{T, F\}^4 \to \{T, F\}$.
- *For every two partially ordered sets* $A_1$ *and* $A_2$*, we define the following relation on* $A_1 \times A_2$*:*

$$(a_1, a_2) \preceq (a_1', a_2') \stackrel{\rm def}{=} P(a_1 \preceq_1 a_1', a_1' \preceq_1 a_1, a_2 \preceq_2 a_2', a_2' \preceq_2 a_2).$$

- *We say that a product operation is* consistent *if* $\preceq$ *is always a partial order, and*

$$(a_1 \preceq_1 a_1' \,\&\, a_2 \preceq_2 a_2') \Rightarrow (a_1, a_2) \preceq (a_1', a_2').$$

**Theorem 1.** *Every consistent product operation is the Cartesian or the lexicographic product.*

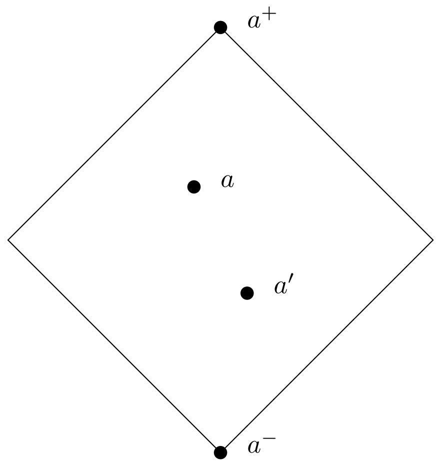
*Comment.* For reader's convenience, the proofs of all the results are placed in a special Proofs section.

# 4 Auxiliary Results: General Idea and First Example

**General idea.** For each property of an ordered set $A$, we analyze which properties need to be satisfied for $A_1$ and $A_2$ so that the corresponding property is satisfies in $A_1 \times A_2$.

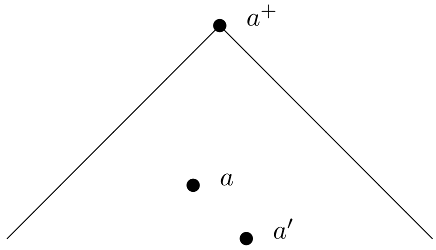
**First example: connectedness property.** As a first example, let us consider the following

- *Connectedness property (CP):* for every two points $a, a' \in A$, there exists an interval that contains $a$ and $a'$: $\forall a\, \forall a'\, \exists a^-\, \exists a^+\, (a^- \preceq a, a' \preceq a^+)$.

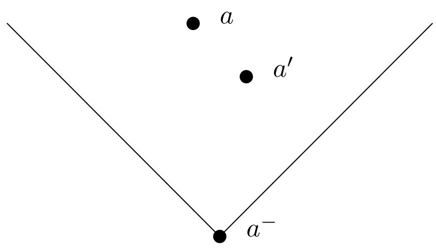



**Connectedness property: analysis.** One can easily see that a partially ordered set $A$ satisfies the connectedness property if and only if it satisfies the following two properties:

- $A$ is *upward-directed*: $\forall a\, \forall a'\, \exists a^+\, (a, a' \preceq a^+)$;




- $A$ is *downward-directed*: $\forall a\, \forall a'\, \exists a^-\, (a^- \preceq a, a')$.




So, to check when the product satisfies the connectivity property, it is sufficient to check when the product is upward- and downward-directed.

**Results for Cartesian product.** For both $\preceq$- and $\prec$-Cartesian products, we get the following results.

**Proposition 1.** *A Cartesian product $A_1 \times A_2$ is upward-directed if and only if both $A_1$ and $A_2$ are upward-directed.*

**Proposition 2.** *A Cartesian product $A_1 \times A_2$ is downward-directed if and only if both $A_1$ and $A_2$ are downward-directed.*

**Results for lexicographic product.** For the lexicographic product, we get the following results:

**Definition 2.**

- *An element $\overline{a} \in A$ is called* maximal *if there are no elements $a$ with $\overline{a} \prec a$.*
- *An element $\underline{a} \in A$ is called* minimal *if there are no elements $a$ with $a \prec \underline{a}$.*

**Proposition 3.** *A lexicographic product $A_1 \times A_2$ is upward-directed $\Leftrightarrow$ the following two conditions hold:*

- *the set $A_1$ is upward-directed, and*
- *if $A_1$ has a maximal element $\overline{a}_1$, then $A_2$ is upward-directed.*

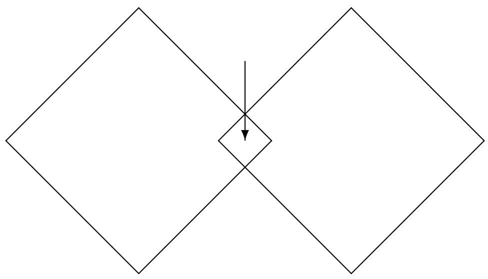
**Proposition 4.** *A lexicographic product $A_1 \times A_2$ is downward-directed $\Leftrightarrow$ the following two conditions hold:*

- *the set $A_1$ is downward-directed, and*
- *if $A_1$ has a minimal element $\underline{a}_1$, then $A_2$ is downward-directed.*

# 5 Auxiliary Result: Second Example

**Second example: intersection property.** As a second example, let us consider the following

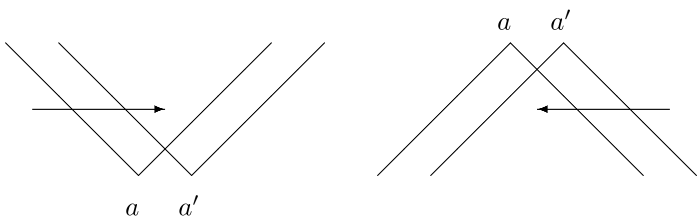
- *Intersection property:* the intersection of every two intervals is an interval.



*Comment.* This property is satisfied for intervals on the real line.

**Intersection property: analysis.** Similarly to the connectivity property, the intersection property can also be reduced to two properties:

- the intersection of every two future cones $Q_a^+ \stackrel{\text{def}}{=} \{b : a \preceq b\}$ is a future cone;
- the intersection of every two past cones $Q_a^- \stackrel{\text{def}}{=} \{b : b \preceq a\}$ is a past cone.



If both properties hold, then a non-empty intersection of every two intervals $[a,b] = Q_a^+ \cap Q_b^-$ is an interval.

**Definition 3.**

- *An ordered set for which the intersection $Q_a^+ \cap Q_{a'}^+$ of every two future cones $Q_a^+$ and $Q_{a'}^+$ is a future cone is called an* upper semi-lattice.
- *For every two elements $a, a'$, the element $a''$ for which $Q_a^+ \cap Q_{a'}^+ = Q_{a''}^+$ is called a* join *of $a$ and $a'$ and is denoted by $a \vee a'$.*
- *An ordered set for which the intersection $Q_a^- \cap Q_{a'}^-$ of every two past cones $Q_a^-$ and $Q_{a'}^-$ is a past cone is called a* lower semi-lattice.
- *For every two elements $a, a'$, the element $a''$ for which $Q_a^- \cap Q_{a'}^- = Q_{a''}^-$ is called a* meet *of $a$ and $a'$ and is denoted by $a \wedge a'$.*

**What we plan to do.** We plan to analyze when the Cartesian and lexicographic products are upper and lower semi-lattices.

**Proposition 5.** *A Cartesian product $A_1 \times A_2$ is an upper semi-lattice if and only if both $A_1$ and $A_2$ are are upper semi-lattices.*

**Proposition 6.** *A Cartesian product $A_1 \times A_2$ is a lower semi-lattice if and only if both $A_1$ and $A_2$ are are upper semi-lattices.*

To describe when a lexicographic product is an upper semi-lattice, we need two introduce the following auxiliary notions:

**Definition 4.**

- *We say that an ordered set $A$ is* linearly (totally) ordered *if for every two elements $a, a' \in A$, we have either $a \preceq a'$ or $a' \preceq a$.*
- *We say that an ordered set is a* conditional upper semi-lattice *if for all $a$ and $a'$ for which the future cones $Q_a^+$ and $Q_{a'}^+$ intersect, this intersection is also a future cone.*
- *We say that an ordered set is a* conditional lower semi-lattice *if for all $a$ and $a'$ for which the past cones $Q_a^-$ and $Q_{a'}^-$ intersect, this intersection is also a past cone.*
- *We say that an element $a^+$ is the* next *element to $a$ if for every $a'$, the condition $a \prec a'$ is equivalent to $a^+ \preceq a'$.*
- *We say that an ordered set is* sequential up *if every element has a next one.*

- *We say that an element $a^-$ is the* previous *element to $a$ if for every $a'$, the condition $a' \prec a$ is equivalent to $a' \preceq a^-$.*

- *We say that an ordered set is* sequential down *if every element has a previous one.*

**Proposition 7.** *The lexicographic product $A_1 \times A_2$ is an upper semi-lattice if and only $A_1$ is an upper semi-lattice and one of the following conditions holds:*

- *$A_1$ is linearly ordered and $A_2$ is an upper semi-lattice;*

- *$A_2$ is an upper semi-lattice that has the smallest element;*

- *$A_1$ is sequential up, $A_2$ is a conditional upper semi-lattice, and $A_2$ has the smallest element.*

**Proposition 8.** *The lexicographic product $A_1 \times A_2$ is a lower semi-lattice if and only $A_1$ is a lower semi-lattice and one of the following conditions holds:*

- *$A_1$ is linearly ordered and $A_2$ is a lower semi-lattice;*

- *$A_2$ is a lower semi-lattice that has the largest element;*

- *$A_1$ is sequential down, $A_2$ is a conditional lower semi-lattice, and $A_2$ has the largest element.*

# 6 Proofs

**Proof of Theorem 1.**

$1^\circ$. According to the definition, whether $(a_1, a_2) \preceq (a'_1, a_2)$ depends on the two relation: the relation between $a_1$ and $a'_1$ and on the relation between $a_2$ and $a'_2$. For each pair $a_i$ and $a'_i$, we have four possible relations:

- the relation $a_i \prec_i a'_i$; we will denote this case by $+$;

- the relation $a'_i \prec_i a_i$; we will denote this case by $-$;

- the relation $a_i = a'_i$; we will denote this relation by $=$; and

- the relation $a_i \not\preceq_i a'_i$ and $a'_i \not\preceq_i a_i$; we will denote this relation by $\|$.

The case when we have relation $R_1$ for $a_1$ and $a'_1$ and relation $R_2$ for $a_2$ and $a'_2$ will be denoted by $R_1R_2$. So, we have 16 possible pairs of relations: $++$, $+-$, $+=$, $+\|$, $-+$, $--$, etc. To describe the product, it is sufficient to describe which of these 16 pairs correspond to $(a_1, a_2) \preceq (a'_1, a_2)$.

Due to the consistency requirement, pairs $++$, $+=$, $=+$, and $==$ always result in $\preceq$, so it is sufficient to classify the remaining 12 pairs. If only these four pairs result in $\preceq$, then we have the Cartesian product. So, to prove our theorem, it is sufficient to prove that if at least one other pair leads to $\preceq$, then

we get a lexicographic product. To prove this, let us consider the remaining 12 pairs one by one.

$2^\circ$. Let us first consider pairs that contain $-$.

$2.1^\circ$. Let us prove that the pair $--$ cannot lead to $\preceq$. Indeed, when both $A_1$ and $A_2$ are real lines $\mathbb{R}$ with the usual order, due to the fact that $++$ leads to $\preceq$, we get $(0,0) \preceq (1,1)$, while due to the fact that $--$ leads to $\preceq$, we get $(1,1) \preceq (0,0)$. Hence, we have $(0,0) \preceq (1,1)$ and $(1,1) \preceq (0,0)$ but $(0,0) \neq (1,1)$ – a contradiction to antisymmetry.

$2.2^\circ$. Similarly, the pair $-=$ cannot lead to $\preceq$ because otherwise, for the same example $A_1 = A_2 = \mathbb{R}$, we would get $(0,0) \preceq (1,0)$ and $(1,0) \preceq (0,0)$ but $(0,0) \neq (1,0)$ – also a contradiction to antisymmetry.

$2.3^\circ$. Let us now consider the pair $-\,\|$.

To prove that it cannot lead to $\preceq$, we consider $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$ with Cartesian order. In this case,

$$(0,0) \parallel_2 (1,-2)$$

and $(1,-2) \parallel_2 (-1,-1)$. Thus, if $-\,\|$ leads to $\preceq$, we have $(0,(0,0)) \preceq (-1,(1,-2))$ and $(-1,(1,-2)) \preceq (-2,(-1,-1))$. Thus, due to transitivity of $\preceq$, we get $(0,(0,0)) \preceq (-2,(-1,-1))$. On the other hand, due to consistency, from $-2 \preceq_1 0$ and $(-1,-1) \preceq_2 (0,0)$, we conclude that $(-2,(-1,-1)) \preceq (0,(0,0))$ – a contradiction with antisymmetry.

$2.4^\circ$. Similarly, pairs $=-$ and $\|\,-$ cannot lead to $\preceq$. Thus, the only pais containing $-$ that can potentially lead to $\preceq$ are pairs containing a $+$.

$3^\circ$. Let us prove a similar property for pairs containing $\|$. We already know that pairs $\|\,-$ and $-\,\|$ cannot lead to $\preceq$, so it is sufficient to consider pairs $\|=$, $=\|$, and $\|\|$.

$3.1^\circ$. To prove that the pair $=\|$ cannot lead to $\preceq$, let us consider the same case $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$. In this case, due to

$$(0,0) \parallel_2 (1,-2)$$

and $(1,-2) \parallel_2 (-1,-1)$, if $=\|$ leads to $\preceq$, we have $(0,(0,0)) \preceq (0,(1,-2))$ and $(0,(1,-2)) \preceq (0,(-1,-1))$. Thus, due to transitivity of $\preceq$, we get

$$(0,(0,0)) \preceq (0,(-1,-1)).$$

On the other hand, due to consistency, from $0 \preceq_1 0$ and $(-1,-1) \preceq_2 (0,0)$, we conclude that $(0,(-1,-1)) \preceq (0,(0,0))$ – a contradiction with antisymmetry.

$3.2^\circ$. Similarly, it is possible to prove that the pair $\|=$ cannot lead to $\preceq$.

$3.3^\circ$. To prove that the pair $\|\|$ cannot lead to $\preceq$, let us consider the case when $A_1 = A_2 = \mathbb{R} \times \mathbb{R}$. In this case, due to

$$(0,0) \parallel_i (1,-2)$$

and $(1,-2) \parallel_i (-1,-1)$, if $\parallel\parallel$ leads to $\preceq$, we have $((0,0),(0,0)) \preceq ((1,-2),(1,-2))$ and $((1,-2),(1,-2)) \preceq ((-1,-1),(-1,-1))$. Thus, due to transitivity of $\preceq$, we get $((0,0),(0,0)) \preceq ((-1,-1),(-1,-1))$. On the other hand, due to consistency, from $(-1,-1) \preceq_i (0,0)$, we conclude that $((-1,-1),(-1,-1)) \preceq ((0,0),(0,0))$ – a contradiction with antisymmetry.

4°. Thus, due to Part 2 and 3 of this proof, the only additional pairs that can, in principle, lead to $\preceq$ are pairs containing $+$, i.e., pairs $+-$, $+\parallel$, $-+$, and $-\parallel$.

5°. Let us prove that the pair $+-$ leads to $\preceq$ if and only if the pair $+\parallel$ leads to $\preceq$.

5.1°. Let us first prove that if the pair $+-$ leads to $\preceq$, then the pair $+\parallel$ also leads to $\preceq$.

Indeed, let us consider the case when $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$. If $+-$ leads to $\preceq$, then $0 \prec_1 1$ and $(-1,-1) \prec_2 (0,0)$ imply $(0,(0,0)) \preceq (1,(-1,-1))$. Due to consistency, $1 \preceq_1 1$ and $(-1,-1) \preceq_2 (-1,1)$ lead to $(1,(-1,-1)) \preceq (1,(-1,1))$. Due to transitivity of $\preceq$, we get $(0,(0,0)) \preceq (1,(-1,1))$. In this case, $\preceq$ holds for a pair for which $0 \prec_1 1$ and $(0,0) \parallel_2 (-1,1)$, i.e., for a pair of type $+\parallel$. By our definition of an order on the product, this means that $\preceq$ must hold for all pairs of this type, i.e., that the pair $+\parallel$ indeed leads to $\preceq$.

5.2°. Let us now prove that if the pair $+\parallel$ leads to $\preceq$, then the pair $+-$ also leads to $\preceq$.

Let us consider the same case $A_1 = \mathbb{R}$ and $A_2 = \mathbb{R} \times \mathbb{R}$. If $+\parallel$ leads to $\preceq$, then $0 \prec_1 1$ and and $(1,-2) \parallel_2 (-1,-1)$ imply $(0,(0,0)) \preceq (1,(1,-2))$, and $1 \prec_1 2$ and $(0,0) \parallel_2 (1,-2)$ imply and $(1,(1,-2)) \preceq (2,(-1,-1))$. Due to transitivity of $\preceq$, we get $(0,(0,0)) \preceq (2,(-1,-1))$. In this case, $\preceq$ holds for a pair for which $0 \prec_1 2$ and $(-1,-1) \prec_2 (0,0)$, i.e., for a pair of type $+-$. By our definition of an order on the product, this means that $\preceq$ must hold for all pairs of this type, i.e., that the pair $+-$ indeed leads to $\preceq$.

6°. Similarly, we can prove that the pair $-+$ leads to $\preceq$ if and only if the pair $\parallel +$ leads to $\preceq$. Thus, adding $+-$ is equivalent to adding $+\parallel$, and adding $-+$ is equivalent to adding $\parallel +$.

If we add $+-$ (and hence $+\parallel$), we get the lexicographic product $A_1 \times A_2$. If we add $-+$ (and hence $\parallel +$), we get the lexicographic product $A_2 \times A_1$. Thus, to complete the proof, it is sufficient to show that we cannot simultaneously add $+-$ and $-+$.

7°. Let us prove that $+-$ and $-+$ cannot simultaneously lead to $\preceq$.

We will prove this by contradiction. Let us assume that adding both $+-$ and $-+$ always leads to a consistent partial order. In this case, let us take $A_1 = A_2 = \mathbb{R}$. Since $+-$ leads to $\preceq$, the conditions $0 \prec_1 1$ and $-2 \prec_2 0$ lead to $(0,0) \preceq (1,-2)$. Similarly, since $-+$ leads to $\preceq$, from $-1 \prec_1 1$ and $-2 \prec_2 -1$, we conclude that $(1,-2) \preceq (-1,-1)$. By transitivity of $\preceq$, we can now conclude

that $(0,0) \preceq (-1,-1)$. However, due to consistency, $(-1,-1) \preceq (0,0)$ – a contradiction to anti-symmetry.

The statement is proven, and so is the theorem.

**Proof of Proposition 1.**

1°. Let us assume that the $\preceq$-Cartesian product $A_1 \times A_2$ is upward-directed. We want to prove that $A_1$ is upward-directed. (For $A_2$, the proof of similar.)

In other words, we want to prove that for every $a_1, a'_1 \in A_1$, there exists an element $a_1^+ \in A_1$ for which $a_1, a'_1 \preceq_1 a_1^+$. Let us take any $a_1, a'_1 \in A_1$, and any $a_2 \in A_2$. Then, since the product $A_1 \times A_2$ is upward-directed, there exists an element $a^+ = (a_1^+, a_2^+) \in A_1 \times A_2$ for which

$$(a_1, a_2) \preceq a^+ = (a_1^+, a_2^+) \text{ and } (a'_1, a_2) \preceq a^+ = (a_1^+, a_2^+).$$

By definition of an order on $A_1 \times A_2$, we thus conclude that $a_1 \preceq_1 a_1^+$ and $a'_1 \preceq_1 a_1^+$. Thus, $A_1$ is indeed upward-directed.

2°. Let us now assume that both $A_i$ are upward-directed. We want to prove that $A_1 \times A_2$ is upward-directed, i.e., that for any two elements $a = (a_1, a_2) \in A_1 \times A_2$ and $a' = (a'_1, a'_2) \in A_1 \times A_2$, there exists an element $a^+$ for which $a \preceq a^+$ and $a' \preceq a^+$.

Indeed, since the set $A_1$ is upward-directed, there exists an element $a_1^+$ for which $a_1 \preceq_1 a_1^+$ and $a'_1 \preceq_1 a_1^+$. Similarly, since the set $A_2$ is upward-directed, there exists an element $a_2^+$ for which $a_2 \preceq_2 a_2^+$ and $a'_2 \preceq_2 a_2^+$. By definition the the order on the Cartesian product, we can now conclude that $(a_1, a_2) \preceq (a_1^+, a_2^+)$ and $(a'_1, a'_2) \preceq (a_1^+, a_2^+)$. Thus, the set $A_1 \times A_2$ is upward-directed.

**Proof of Proposition 2** is similar to the proof of Proposition 1.

**Proof of Proposition 3.**

1°. Let us assume that $A_1 \times A_2$ is upward-directed.

1.1°. Let us prove that $A_1$ is upward-directed, i.e., that for every $a_1 \in A_1$ and $a'_1 \in A_1$, there exists an element $a_1^+$ for which $a_1 \preceq_1 a_1^+$ and $a'_1 \preceq_1 a_1^+$.

Indeed, since the product $A_1 \times A_2$ is upward-directed, for any $a_2 \in A_2$, there exists an element $a^+ = (a_1^+, a_2^+)$ for which $(a_1, a_2) \preceq a^+)$ and $(a'_1, a_2) \preceq a^+$. By definition of the lexicographic product, this implies that $a_1 \preceq_1 a_1^+$ and $a'_1 \preceq_1 a_1^+$. Thus, the set $A_1$ is upward-directed.

1.2°. Let us now assume that, in addition to $A_1 \times A_2$ being upward-directed, the set $A_1$ has a maximal element $\overline{a}_1$. Let us prove that under these assumptions, the set $A_2$ is also upward-directed, i.e., for every $a_2, a'_2 \in A_2$, there exists an element $a_2^+$ for which $a_2 \preceq_2 a_2^+$ and $a'_2 \preceq_2 a_2^+$.

Indeed, since the product $A_1 \times A_2$ is upward-related, there exists an element $a^+ = (a_1^+, a_2^+) \in A_1 \times A_2$ for which $(\overline{a}_1, a_2) \preceq a^+$ and $(\overline{a}_1, a'_2) \preceq a^+$. By

definition of the lexicographic product, this implies that $\overline{a}_1 \preceq_1 a_1^+$, i.e., that either $\overline{a}_1 \prec_1 a_1^+$ or $\overline{a}_1 = a_1^+$. Since the element $\overline{a}_1$ is maximal, we cannot have $\overline{a}_1 \prec_1 a_1^+$, so we have $a_1^+ = \overline{a}_1$. In this case, $(\overline{a}_1, a_2) \preceq a^+ = (\overline{a}_1, a_2^+)$, so by definition of the lexicographic order, we get $a_2 \preceq_2 a_2^+$. Similarly, we get $a_2' \preceq_2 a_2^+$, so the set $A_2$ is indeed upward-directed.

$2^\circ$. Let us now assume that $A_1$ is upward-directed, and that if $A_1$ has a maximal element, then $A_2$ is upward-directed. Let us prove that $A_1 \times A_2$ is upward-directed.

Indeed, let us take any two elements $a = (a_1, a_2)$ and $a' = (a_1', a_2')$ from $A_1 \times A_2$, and let us show that there exists an element $a^+$ for which $a \preceq a^+$ and $a' \preceq a^+$.

Since $A_1$ is upward-directed, there exists an element $a_1^+ \in A_1$ for which $a_1 \preceq_1 a_1^+$ and $a_1' \preceq_1 a_1^+$.

Since $a_1 \preceq_1 a_1^+$, we have either $a_1 \prec_1 a_1^+$ or $a_1 = a_1^+$. Similarly, since $a_1' \preceq_1 a_1^+$, we have either $a_1' \prec_1 a_1^+$ or $a_1' = a_1^+$. Thus, by considering both $a_1$ and $a_1'$, we have four possible situations:

- situation when $a_1 \prec_1 a_1^+$ and $a_1' \prec_1 a_1^+$;
- situation when $a_1 \prec_1 a_1^+$ and $a_1' = a_1^+$;
- situation when $a_1 = a_1^+$ and $a_1' \prec_1 a_1^+$; and
- situation when $a_1 = a_1^+$ and $a_1' = a_1^+$,

Let us consider these four situations one by one.

When $a_1 \prec_1 a_1^+$ and $a_1' \prec_1 a_1^+$, then, by the definition of lexicographic order, we have $(a_1, a_2) \preceq (a_1^+, a_2')$ and $(a_1', a_2) \preceq (a_1^+, a_2')$. So, in this case, we can take $(a_1^+, a_2')$ as the desired element $a^+$.

When $a_1 \prec_1 a_1^+$ and $a_1' = a_1^+$, then, by the definition of lexicographic order, we have $(a_1, a_2) \preceq (a_1^+, a_2')$. We also have $(a_1', a_2) = (a_1^+, a_2')$ hence $(a_1', a_2) \preceq (a_1^+, a_2')$. So, in this case, we can also take $(a_1^+, a_2')$ as the desired element $a^+$.

When $a_1 = a_1^+$ and $a_1' \prec_1 a_1^+$, then, by the definition of lexicographic order, we have $(a_1', a_2') \preceq (a_1^+, a_2)$. We also have $(a_1, a_2) = (a_1^+, a_2)$ hence $(a_1, a_2) \preceq (a_1^+, a_2)$. So, in this case, we can also take $(a_1^+, a_2)$ as the desired element $a^+$.

Finally, let us consider the situation when $a_1 = a_1^+ = a_1'$. In this situation, there are two possible sub-situations: when this element $a_1$ is maximal and when it is not maximal.

If the element $a_1$ is not maximal, then there exists a value $a_1^+$ for which $a_1 \prec_1 a_1^+$. In this case, by definition of the lexicographic product, we have $(a_1, a_2) \preceq (a_1^+, a_2)$ and $(a_1', a_2) \preceq (a_1^+, a_2)$. So, we can take $(a_1^+, a_2)$ as the desired element $a^+$.

If the element $a_1$ is maximal, then $A_2$ is upward-directed. Thus, there exists an element $a_2^+ \in A_2$ for which $a_2 \preceq_2 a_2^+$ and $a_2' \preceq_2 a_2^+$. For this element, we have $(a_1, a_2) \preceq (a_1, a_2^+)$ and $(a_1, a_2') \preceq (a_1, a_2^+)$. So, we can take $(a_1, a_2^+)$ as the desired element $a^+$.

In all four situations, there exists an element $a^+$ for which $a \preceq a^+$ and $a' \preceq a^+$. The statement is proven, and so is the proposition.

**Proof of Proposition 4** is similar to the proof of Proposition 3.

**Proof of Proposition 5.**

1°. Let us assume that the Cartesian product $A_1 \times A_2$ is an upper semi-lattice. Let us prove that in this case, $A_1$ is also an upper semi-lattice (for $A_2$, the proof is the same).

We need to prove that for every two elements $a_1, a'_1 \in A_1$, there exists an element $a_1^+$ for which $Q^+_{a_1} \cap Q^+_{a'_1} = Q^+_{a_1^+}$, i.e., for which, for every element $b_1$, we have

$$(a_1 \preceq_1 b_1 \,\&\, a'_1 \preceq_1 b_1) \Leftrightarrow a_1^+ \preceq_1 b_1. \tag{1}$$

To prove it, let us take an arbitrary element $a_2 \in A_2$ and consider two elements $a = (a_1, a_2)$ and $a' = (a'_1, a_2)$. Since the set $A_1 \times A_2$ is an upper semi-lattice, there exists an element $a^+ = (a_1^+, a_2^+)$ which is a join of the elements $a$ and $a'$, i.e., for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preceq (b_1, b_2) \,\&\, (a'_1, a_2) \preceq (b_1, b_2)) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \tag{2}$$

Let us prove that the first component $a_1^+$ of this join $a^+$ is the desired join of $a_1$ and $a'_1$, i.e., that for this first component, the condition (1) holds.

$\Leftarrow$ If $a_1^+ \preceq_1 b_1$, then by the definition of the Cartesian product, we have $(a_1^+, a_2^+) \le (b_1, a_2^+)$. Applying the $\Leftarrow$ part of the condition (2) with $b_2 = a_2^+$, we conclude that $(a_1, a_2) \preceq (b_1, a_2^+)$ and $(a'_1, a_2) \preceq (b_1, a_2^+)$. By the definition of the Cartesian product, the first condition implies that $a_1 \preceq_1 b_1$, and the second condition implies that $a'_1 \preceq_1 b_1$.

$\Rightarrow$ Vice versa, let us assume that $a_1 \preceq_1 b_1$ and $a'_1 \preceq_1 b_1$. We need to prove that $a_1^+ \preceq_1 b_1$. In this case, by the definition of the Cartesian product, we have $(a_1, a_2) \le (b_1, a_2)$ and $(a'_1, a_2) \le (b_1, a_2)$. Applying the $\Rightarrow$ part of the condition (2) with $b_2 = a_2$, we conclude that $(a_1^+, a_2^+) \preceq (b_1, a_2)$. By the definition of the Cartesian product, this implies that $a_1^+ \preceq_1 b_1$.

2°. It is easy to prove that if $A_1$ and $A_2$ are upper semi-lattices, then their Cartesian product $A_1 \times A_2$ is also an upper semi-lattice, with $(a_1, a_2) \vee (a'_1, a'_2) = (a_1 \vee a'_1, a_2 \vee a'_2)$.

The proposition is proven.

**Proof of Proposition 6** is similar to the proof of Proposition 5.

**Proof of Proposition 7.**

1°. Let us assume that $A_1 \times A_2$ is an upper semi-lattice. Let us prove that in this case, $A_1$ is an upper semi-lattices, and one of the three properties described in the formulation of Proposition 7 holds.

1.1°. Let us prove that $A_1$ is an upper semi-lattice, i.e., that for every two elements $a_1, a_1' \in A_1$, there exists an element $a_1^+$ for which $Q^+_{a_1} \cap Q^+_{a_1'} = Q^+_{a_1^+}$, i.e., for which, for every element $b_1$, we have

$$(a_1 \preceq_1 b_1 \,\&\, a_1' \preceq_1 b_1) \Leftrightarrow a_1^+ \preceq_1 b_1. \tag{3}$$

Let us take an arbitrary element $a_2 \in A_2$ and consider two elements $a = (a_1, a_2)$ and $a' = (a_1', a_2)$. Since the set $A_1 \times A_2$ is an upper semi-lattice, there exists an element $a^+ = (a_1^+, a_2^+)$ which is a join of the elements $a$ and $a'$, i.e., for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preceq (b_1, b_2) \,\&\, (a_1', a_2) \preceq (b_1, b_2)) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \tag{4}$$

Let us prove that the first component $a_1^+$ of this join $a^+$ is the desired join of $a_1$ and $a_1'$, i.e., that for this first component, the condition (3) holds.

$\Leftarrow$ If $a_1^+ \preceq_1 b_1$, then, by the definition of the lexicographic product, we have $(a_1^+, a_2^+) \leq (b_1, a_2^+)$. Applying the $\Leftarrow$ part of the condition (4) with $b_2 = a_2^+$, we conclude that $(a_1, a_2) \preceq (b_1, a_2^+)$ and $(a_1', a_2) \preceq (b_1, a_2^+)$. By the definition of the lexicographic product, the first condition implies that $a_1 \preceq_1 b_1$, and the second condition implies that $a_1' \preceq_1 b_1$.

$\Rightarrow$ Vice versa, let us assume that $a_1 \preceq_1 b_1$ and $a_1' \preceq_1 b_1$. We need to prove that $a_1^+ \preceq_1 b_1$. To prove this statement, we will consider four possible cases:

- case when $a_1 \prec_1 b_1$ and $a_1' \prec_1 b_1$;
- case when $a_1 \prec_1 b_1$ and $a_1' = b_1$;
- case when $a_1 = b_1$ and $a_1' \prec_1 b_1$; and
- case when $a_1 = b_1$ and $a_1' = b_1$.

1) When $a_1 \prec_1 b_1$ and $a_1' \prec_1 b_1$, then, by the definition of the lexicographic order, we have $(a_1, a_2^+) \preceq (b_1, a_2^+)$ and $(a_1', a_2^+) \preceq (b_1, a_2^+)$. Thus, due to the $\Rightarrow$ part of (4), with $b_2 = a_2^+$, we get $(a_1^+, a_2^+) \preceq (b_1, a_2^+)$. So, by the definition of the lexicographic product, we have $a_1^+ \preceq_1 b_1$.

2) When $a_1 \prec_1 b_1$ and $a_1' = b_1$, we have $a_1 \prec_1 a_1'$. Here, $(a_1', a_2) = (a_1', a_2)$ hence $(a_1', a_2) \preceq (a_1', a_2)$, and $a_1 \prec_1 a_1'$ implies that $(a_1, a_2) \prec (a_1', a_2)$. Thus, due to the $\Rightarrow$ part of (4), with $b_1 = a_1'$ and $b_2 = a_2$, we get $(a_1^+, a_2^+) \preceq (a_1', a_2)$. So, by the definition of the lexicographic product, we have $a_1^+ \preceq_1 a_1' = b_1$.

3) When $a_1 = b_1$ and $a_1' \prec_1 b_1$, we have $a_1' \prec_1 a_1$. Here, $(a_1, a_2) = (a_1, a_2)$ hence $(a_1, a_2) \preceq (a_1, a_2)$, and $a_1' \prec_1 a_1$ implies that $(a_1', a_2) \prec (a_1, a_2)$. Thus, due to the $\Rightarrow$ part of (4), with $b_1 = a_1$ and $b_2 = a_2$, we get $(a_1^+, a_2^+) \preceq (a_1, a_2)$. So, by the definition of the lexicographic product, we have $a_1^+ \preceq_1 a_1 = b_1$.

4) Finally, when $a_1 = b_1 = a'_1$, we have $a_1 = a'_1$. In this case, $(a_1, a_2) = (a'_1, a_2)$, so the condition $(a_1, a_2) \preceq (b_1, b_2)$ is simply equivalent to $(a'_1, a_2) \preceq (b_1, b_2)$. In this case, the condition (4) simply means that

$$(a_1, a_2) \preceq (b_1, b_2) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \tag{5}$$

For $(b_1, b_2) = (a_1, a_2)$, the $\Rightarrow$ part of the formula (5) implies that $(a_1^+, a_2^+) \preceq (a_1, a_2)$. For $(b_1, b_2) = (a_1^+, a_2^+)$, the $\Leftarrow$ part the formula (5) implies that $(a_1, a_2) \preceq (a_1^+, a_2^+)$. Thus, $(a_1^+, a_2^+) = (a_1, a_2)$, and hence, $a_1^+ = a_1$ $(= a'_1)$. Since $a_1^+ = a_1 = a'_1$, the condition (3) is clearly satisfied.

In all four cases, we have the desired proof, so the statement is proven. So, $A_1$ is indeed an upper semi-lattice.

1.2°. Let us prove that if the set $A_1$ is not linearly ordered, then the set $A_2$ has the smallest element.

Indeed, the fact that the set $A_1$ is not linearly ordered means that there exist values $a_1$ and $a'_1$ for which $a_1 \not\preceq_1 a'_1$ and $a'_1 \not\preceq_1 a_1$.

Let us take an arbitrary $a_2 \in A_2$. Since $A_1 \times A_2$ is an upper semi-lattice, for $a = (a_1, a_2)$ and $a' = (a'_1, a_2)$, there exists an element $a^+ = (a_1^+, a_2^+)$ for which $Q_a^+ \cap Q_{a'}^+ = Q_{a^+}^+$, i.e., for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preceq (b_1, b_2) \,\&\, (a'_1, a_2) \preceq (b_1, b_2)) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \tag{6}$$

Let us prove that $a_2^+$ is the smallest element of the set $A_2$.

Indeed, the formula (6) implies, for $(b_1, b_2) = (a_1^+, a_2^+)$, we have $(a_1^+, a_2^+) = (b_1, b_2)$ and hence, $(a_1^+, a_2^+) \preceq (b_1, b_2)$. Thus, due to the $\Leftarrow$ part of the formula (6), we conclude that $(a_1, a_2) \preceq (a_1^+, a_2^+)$ and $(a'_1, a_2) \preceq (a_1^+, a_2^+)$. By definition of the lexicographic order, we conclude that $a_1 \preceq_1 a_1^+$ and $a'_1 \preceq_1 a_1^+$.

We cannot have $a_1 = a_1^+$, since then we would have $a'_1 \preceq_1 a_1$ – which contradicts to our choice of $a_1$ and $a'_1$. Thus, $a_1 \preceq_1 a_1^+$ implies that $a_1 \prec_1 a_1^+$.

Similarly, we cannot have $a'_1 = a_1^+$, since then we would have $a_1 \preceq_1 a'_1$ – which also contradicts to our choice of $a_1$ and $a'_1$. Thus, $a'_1 \preceq_1 a_1^+$ implies that $a'_1 \prec_1 a_1^+$.

Let $a'_2$ be an arbitrary element of the set $A_2$. Since $a_1 \prec_1 a_1^+$ and $a'_1 \prec_1 a_1^+$, we have $(a_1, a_2) \prec (a_1^+, a'_2)$ and $(a'_1, a_2) \prec (a_1^+, a'_2)$. Thus, by applying the $\Rightarrow$ part of the formula (6) to $b_1 = a_1^+$ and $b_2 = a'_2$, we conclude that $(a_1^+, a_2^+) \preceq (a_1^+, a'_2)$. By the definition of the lexicographic order, this means that $a_2^+ \preceq_2 a'_2$. Thus, $a_2^+$ precedes all the elements of $A_2$, so it is indeed the smallest element of $A_2$.

1.3°. Let us now prove that $A_2$ is a conditional semi-lattice, i.e., that for every $a_2 \in A_2$ and $a'_2 \in A_2$ for which $a_2 \preceq_2 a''_2$ and $a'_2 \preceq_2 a''_2$ for some $a''_2$, there exists an element $a^+2$ for which $Q_{a_2}^+ \cap Q_{a'_2}^+ = Q_{a_2^+}^+$, i.e., for which, for every $b_1 \in A_2$, we have

$$(a_2 \preceq_2 b_2 \,\&\, a'_2 \preceq_2 b_2) \Leftrightarrow a_2^+ \preceq_2 b_2. \tag{7}$$

Indeed, let us take an arbitrary element $a_1 \in A_1$. Since the set $A_1 \times A_2$ is an upper semi-lattice, for elements $(a_1, a_2)$ and $(a_1', a_2)$, there exists an element $a^+ = (a_1^+, a_2^+)$ for which

$$Q^+_{(a_1,a_2)} \cap Q^+_{(a_1',a_2)} = Q^+_{(a_1^+,a_2^+)}.$$

In other words, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preceq (b_1, b_2) \,\&\, (a_1, a_2') \preceq (b_1, b_2) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \tag{8}$$

Let us show that the second component $a_2^+$ of this element is the desired join of $a_2$ and $a_2'$.

For that, let us first prove that $a_1^+ = a_1$. Indeed, from $(a_1^+, a_2^+) = (a_1^+, a_2^+)$, we conclude that $(a_1^+, a_2^+) \preceq (a_1^+, a_2^+)$. So, from the $\Leftarrow$ part of the formula (8) for $(b_1, b_2) = (a_1^+, a_2^+)$, we conclude that $(a_1, a_2) \preceq (b_1, b_2)$, i.e., that $(a_1, a_2) \preceq (a_1^+, a_2^+)$. By the definition of the lexicographic order, this implies that $a_1 \preceq_1 a_1^+$.

Now, $a_2 \preceq_2 a_2''$ and $a_2' \preceq_2 a_2''$. So, by definition of the lexicographic order, we have $(a_1, a_2) \preceq (a_1, a_2'')$ and $(a_1, a_2') \preceq (a_1, a_2'')$. Thus, due to the $\Rightarrow$ part of the formula (8), with $(b_1, b_2) = (a_1, a_2'')$, we conclude that $(a_1^+, a_2^+) \preceq (a_1, a_2'')$. Thus, by definition of the lexicographic order, we conclude that $a_1^+ \preceq_1 a_1$. From $a_1 \preceq_1 a_1^+$ and $a_1^+ \preceq_1 a_1$, we conclude that $a^+1 = a_1$. Thus, the formula (8) takes the form

$$((a_1, a_2) \preceq (b_1, b_2) \,\&\, (a_1, a_2') \preceq (b_1, b_2)) \Leftrightarrow (a_1, a_2^+) \preceq (b_1, b_2). \tag{9}$$

Now, we are ready to prove that $a_2^+ = a_2 \vee a_2'$, i.e., that the formula (7) holds.

$\Leftarrow$ Let $a_2^+ \preceq_2 b_2$. Then, by definition of the lexicographic order, we have $(a_1, a^+2) \preceq (a_1, b_2)$. Due to the $\Leftarrow$ part of the formula (9), with $b_1 = a_1$, we get $(a_1, a_2) \preceq (b_1, b_2) = (a_1, b_2)$. By definition of the lexicographic order, this implies that $a_2 \preceq_2 b_2$. Similarly, we conclude that $a_2' \preceq_2 b_2$.

$\Rightarrow$ Let $a_2 \preceq_2 b_2$ and $a_2' \preceq_2 b_2$. Then, by definition of the lexicographic order, we have $(a_1, a_2) \preceq (a_1, b_2)$ and $(a_1, a_2') \preceq (a_1, b_2)$. Due to the $\Rightarrow$ part of the formula (9), with $b_1 = a_1$, we get $(a_1, a_2^+) \preceq (b_1, b_2) = (a_1, b_2)$. By definition of the lexicographic order, this implies that $a_2^+ \preceq_2 b_2$.

The formula (7) is proven.

1.4°. Let us now prove that is $A_2$ is a not an upper semi-lattice, then $A_1$ is sequential up and $A_2$ has the smallest element.

We have already proven that $A_2$ is a conditional upper semi-lattice, i.e., that every two elements $a_2$ and $a_2'$ for which there exists an element $a_2''$ with $a_2 \preceq_2 a_2''$ and $a_2' \preceq_2 a_2''$ have a joint $a_1 \vee a_2'$. Since $A_2$ is not an upper semi-lattice, this means that there exist elements $a_2$ and $a_2'$ for which no element from $A_2$ follows both these elements.

Let us prove that $A_2$ has the smallest element and that $A_1$ is sequential up. For that, let us select any element $a_1 \in A_1$. Since $A_1 \times A_2$ is an upper

semi-lattice, for the elements $a = (a_1, a_2)$ and $a' = (a_1, a'_2)$, there exists a join $a^+ = (a_1^+, a_2^+)$, i.e., an element for which, for every $b_1 \in A_1$ and $b_2 \in A_2$, we have

$$((a_1, a_2) \preceq (b_1, b_2) \,\&\, (a_1, a'_2) \preceq (b_1, b_2) \Leftrightarrow (a_1^+, a_2^+) \preceq (b_1, b_2). \qquad (10)$$

Let us show $a_1^+$ is the next element to $a_1$ and that $a^+2$ is the smallest element of the set $A_2$. This will prove that $A_1$ is sequential up and that the set $A_2$ has the smallest element.

1.4.1°. To prove that $a_1^+$ is the next element to $a_1$, we need to prove that for every $a'_1 \in A_1$, we have $a_1 \prec_1 a'_1 \Leftrightarrow a_1^+ \preceq_1 a'_1$.

$\Rightarrow$ Let $a_1 \prec_1 a'_1$. Then, for an arbitrary $b_2 \in A_2$, by definition of the lexicographic order, we have $(a_1, a_2) \preceq (a'_1, b_2)$ and $(a_1, a'_2) \preceq (a'_1, b_2)$. Thus, due to the $\Rightarrow$ part of the formula (10), with $b_1 = a'_1$, we conclude that $(a_1^+, a_2^+) \preceq (a'_1, b_2)$. Thus, by definition of the lexicographic order, we have $a_1^+ \preceq_1 a'_1$.

$\Leftarrow$ Let $a_1^+ \preceq_1 a'_1$. Then, for $b_2 = a_2^+$, by definition of the lexicographic order, we have $(a_1^+, a_2^+) \preceq (a'_1, a_2^+)$. Thus, due to the $\Leftarrow$ part of the formula (10), with $b_1 = a'_1$ and $b_2 = a_2^+$, we conclude that $(a_1, a_2) \preceq (a'_1, a_2^+)$ and $(a_1, a_2) \preceq (a'_1, a_2^+)$. By definition of the lexicographic order, we thus have $a_1 \preceq_1 a'_1$, i.e., either $a_1 \prec_1 a'_1$ or $a_1 = a'_1$.

We cannot have $a'_1 = a_1$ because then, by definition of the lexicographic order, we would have $a_2 \prec_2 a_2^+$ and $a'_2 \preceq_2 a_2^+$, which contradicts to our choice of $a_2$ and $a'_2$ as the elements for which there is no elements following both. Thus, we have $a_1 \prec_1 a'_1$.

1.4.2°. To prove that $a_2^+$ is the smallest element of the set $A_2$, we must prove that for every $b_2 \in A_2$, we have $a_2^+ \preceq_2 b_2$.

Indeed, since $a_1 \prec_1 a_1^+$, for every $b_2 \in A_2$, we have $(a_1, a_2) \preceq (a_1^+, b_2)$ and $(a_1, a'_2) \preceq (a_1^+, b_2)$. Thus, due to the $\Rightarrow$ part of the formula (10), with $b_1 = a_1^+$, we conclude that $(a_1^+, a_2^+) \preceq (a_1^+, b_2)$. By definition of the lexicographic order, this implies that $a_2^+ \leq_2 b_2$. The statement is proven.

2°. Let us now prove that in all three cases described in the formulation of Proposition 7, the lexicographic product $A_1 \times A_2$ is an upper semi-lattice, i.e., for every two elements $a = (a_1, a_2)$ and $a' = (a'_1, a'_2)$, there exists a join $(a_1, a_2) \vee (a'_1, a'_2)$.

In this proof, we simply describe the corresponding joins. Proving that the described elements are indeed joins is (reasonably) easy.

2.1°. Let us first consider the case when $A_1$ is linearly ordered and $A_2$ is an upper semi-lattice.

In this case, since the set $A_1$ is linearly ordered, for every two elements $a = (a_1, a_2)$ and $a' = (a'_1, a'_2)$, we have either $a_1 \prec_1 a'_1$, or $a'_1 \prec_1 a_1$, or $a_1 = a'_1$. Let us consider these three cases one by one.

- When $a_1 \prec_1 a'_1$, by definition of the lexicographic product, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence $(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2)$.
- When $a'_1 \prec_1 a_1$, by definition of the lexicographic product, we have $(a'_1, a'_2) \prec (a_1, a_2)$, hence $(a_1, a_2) \vee (a'_1, a'_2) = (a_1, a_2)$.
- Finally, when $a_1 = a'_1$, we have $(a_1, a_2) \vee (a_1, a'_2) = (a_1, a_2 \vee a'_2)$.

2.2°. Let us now consider the case when $A_1$ is an upper semi-lattice, and $A_2$ is an upper semi-lattice that has the smallest element $\underline{a}_2$.

In this case, similarly to the previous case,

- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence
$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$
- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence
$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$
- When $a_1 = a'_1$, we have
$$(a_1, a_2) \vee (a_1, a'_2) = (a_1, a_2 \vee a'_2).$$

When $a_1 \not\preceq_1 a'_1$ and $a'_1 \not\preceq_1 a_1$, then, since $A_1$ is an upper semi-lattice, there exists an element $a_1 \vee a'_1$. It is easy to prove that this element is different from $a_1$ and $a'_1$. We can then take $(a_1, a_2) \vee (a_1, a'_2) = (a_1 \vee a'_1, \underline{a}_2)$.

2.3°. Finally, let us consider the case when $A_1$ is an upper semi-lattice which is sequential up, $A_2$ is a conditional upper semi-lattice, and $A_2$ has the smallest element $\underline{a}_2$.

To describe the join operation, let us consider all possible relations between $a_1$ and $a'_1$.

- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence
$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$
- When $a_1 \prec_1 a'_1$, we have $(a_1, a_2) \prec (a'_1, a'_2)$, hence
$$(a_1, a_2) \vee (a'_1, a'_2) = (a'_1, a'_2).$$
- When $a_1 \not\preceq_1 a'_1$ and $a'_1 \not\preceq_1 a_1$, we have
$$(a_1, a_2) \vee (a_1, a'_2) = (a_1 \vee a'_1, \underline{a}_2).$$

- When $a_1 = a'_1$, and for elements $a_2, a'_2 \in A_2$, these exists an element $a''_2$ that follows both $a_2$ and $a'_2$, then (since $A_2$ is a conditional upper semi-lattice), there exists a join $a_2 \vee a'_2$ of these two elements. So, we can take
$$(a_1, a_2) \vee (a_1, a'_2) = (a_1, a_2 \vee a'_2).$$

- Finally, if $a_1 = a'_1$, and there are no elements $a''_2$ which follow both $a_2$ and $a'_2$, then we take
$$(a_1, a_2) \vee (a_1, a'_2) = (a_1^+, \underline{a}_2),$$
where $a_1^+$ is the next element to $a_1$.

**Proof of Proposition 8** is similar to the proof of Proposition 7.

**Acknowledgments.** The authors are thankful to all the participants of the 14th GAMMIMACS International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics SCAN2010 (Lyon, France, September 2730, 2010), especially to Vladik Kreinovich and Jurgen Wolff von Gudenberg, for valuable discussions.

This work was partially supported by a CONACyT scholarship to F. Zapata.

# References

[1] H. Busemann, *Timelike spaces*, PWN: Warszawa, 1967.

[2] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Upper Saddle River, New Jersey: Prentice Hall, 1995.

[3] V. Kreinovich and O. Kosheleva, "Computational complexity of determining which statements about causality hold in different space-time models", *Theoretical Computer Science*, 2008, Vol. 405, No. 1–2, pp. 50–63.

[4] E. H. Kronheimer and R. Penrose, "On the structure of causal spaces", *Proc. Cambr. Phil. Soc.*, vol. 63, No. 2, pp. 481–501, 1967.

[5] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, New York: W. H. Freeman, 1973.

[6] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman & Hall/CRC, Boca Raton, Florida, 2006.

[7] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.

[8] R. I. Pimenov, *Kinematic spaces: Mathematical Theory of Space-Time*, N.Y.: Consultants Bureau, 1970.

[9] P. G. Vroegindeweij, V. Kreinovich, and O. M. Kosheleva. "From a connected, partially ordered set of events to a field of time intervals", *Foundations of Physics*, 1980, Vol. 10, No. 5/6, pp. 469–484.

[10] F. Zapata, O. Kosheleva, and K. Villaverde, "Product of partially ordered sets (posets) and intervals in such products, with potential applications to uncertainty logic and spacetime geometry", *Abstracts of the 14th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics SCAN2010*, Lyon, France, September 27–30, 2010, pp. 142–144.