

5-2011

When Is Busemann Product a Lattice? A Relation Between Metric Spaces and Corresponding Space-Time Models

Hans-Peter Künzi
University of Cape Town

Francisco Zapata
The University of Texas at El Paso

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

Follow this and additional works at: https://scholarworks.utep.edu/cs_techrep



Part of the [Computer Engineering Commons](#)

Comments:

Technical Report: UTEP-CS-11-24

Recommended Citation

Künzi, Hans-Peter; Zapata, Francisco; and Kreinovich, Vladik, "When Is Busemann Product a Lattice? A Relation Between Metric Spaces and Corresponding Space-Time Models" (2011). *Departmental Technical Reports (CS)*. 614.

https://scholarworks.utep.edu/cs_techrep/614

This Article is brought to you for free and open access by the Computer Science at ScholarWorks@UTEP. It has been accepted for inclusion in Departmental Technical Reports (CS) by an authorized administrator of ScholarWorks@UTEP. For more information, please contact lweber@utep.edu.

When Is Busemann Product a Lattice? A Relation Between Metric Spaces and Corresponding Space-Time Models

Hans-Peter Künzi¹, Francisco Zapata², and Vladik Kreinovich²

¹Department of Mathematics and Applied Mathematics
University of Cape Town
Rondebosch 7701, South Africa
Hans-Peter.Kunzi@uct.ac.za

²Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
fazg74@gmail.com, vladik@utep.edu

Abstract

The causality relation of special relativity is based on the assumption that the speed of all physical processes is limited by the speed of light. As a result, an event (t, x) occurring at moment t at location x can influence an event (y, s) if and only if $s \geq t + \frac{d(x, y)}{c}$. We can simplify this formula if we use units of time and distance in which $c = 1$ (e.g., by using a light second as a unit of distance). In this case, the above causality relation takes the form $s \geq t + d(x, y)$. Since the actual space can be non-Euclidean, H. Busemann generalized this ordering relation to the case when points x, y , etc. are taken from an arbitrary metric space X . A natural question is: when is the resulting ordered space – called a Busemann product – a lattice? In this paper, we provide a necessary and sufficient condition for it being a lattice: it is a lattice if and only if X is a real tree, i.e., a metric space in which every two points are connected by exactly one arc, and this arc is geodesic (i.e., metrically isomorphic to an interval on a real line).

1 Formulation of the Problem

Special relativity: brief reminder. To uniquely describe an event, we need to describe the moment of time t at which it occurs and its spatial location x . In other words, an event can be characterized by a pair (t, x) , where $t \in \mathbb{R}$ is

a real number and x is an element of the metric space X describing the proper physical space.

Such pairs form a *space-time* $\mathbb{R} \times X$. How can we describe the causality relation \leq on this space-time, i.e., the relation $a \leq b$ meaning that an event a can causally influence the event b ?

According to the special relativity theory, the speed of all processes is limited by the speed of light c . So, an event (t, x) can influence an event (s, y) if $t \leq s$ and if it is possible for a signal from x to reach y in time $s - t$. During this time, the signal can cover at most the distance $c \cdot (s - t)$, so this condition can be expressed as $(t, x) \leq (s, y) \Leftrightarrow d(x, y) \cdot c \cdot (s - t)$.

This condition can be simplified even further if, instead of using different units for measuring space and time, we use the same units for both, i.e., if we use, as a unit of distance, the distance c that the light covers in one second. In these new units, the numerical value of the speed of light is 1, so the causality relation takes the following simplified form:

$$(s, y) \geq (t, x) \Leftrightarrow s - t \geq d(x, y). \quad (1)$$

Busemann product. In the special relativity theory, the proper space X is a usual Euclidean space. Starting with general relativity, however, physicists realized that the actual space-time is curved. Thus, it is reasonable to consider space-time models $\mathbb{R} \times X$ with non-Euclidean metric spaces X and causality relation (1). Such models were first considered by H. Busemann [2] and are thus called *Busemann products* of the real line \mathbb{R} and the metric space X (see also [3, 4]).

A natural question: when is the Busemann product $\mathbb{R} \times X$ a lattice? From the viewpoint of ordered space, a natural question is: when is the Busemann product a lattice?

In the simplest case of a 1-D Euclidean space (and thus, 2-D space-time) it is a lattice. Indeed, in this case, $d(x, y) = |x - y|$ and since $|z| = \max(z, -z)$, the relation $s - t \geq d(x, y) = |x - y| = \max(x - y, y - x)$ is equivalent to $s - t \geq x - y$ and $s - t \geq y - x$. By moving terms t and x related to the event (t, x) to one side of each of these inequalities, and terms s and y related to the event (s, y) to another side, we get an equivalent form: $s + y \geq t + x$ and $s - y \geq t - x$. So, if instead of the original coordinates t and x , we use new coordinates $u = t + x$ and $v = t - x$, the ordering relation between two events (u, v) and (u', v') takes the form

$$(u', v') \geq (u, v) \Leftrightarrow ((u' \geq u) \& (v' \geq v)).$$

One can easily check that for this relation, every two elements (u, v) and (u', v') have the greatest lower bound (*meet*) $(u, v) \wedge (u', v') = (\min(u, u'), \min(v, v'))$ and least upper bound (*join*) $(u, v) \vee (u', v') = (\max(u, u'), \max(v, v'))$ – i.e., that it is indeed a lattice.

On the other hand, for the 3-D Euclidean space, the Busemann product – i.e., the causality relation of the Special Relativity theory – is *not* a lattice.

Indeed, for a lattice, the intersection of two *future cones* $a^+ = \{b : b \geq a\}$ and $(a')^+ = \{b : b \geq a'\}$ is also a future cone: namely, the future cone of the join $a \vee a'$. For special relativity, the future cone is, from the geometric viewpoint, an actual cone

$$\begin{aligned} (s, y_1, y_2, y_3) \geq (t, x_1, x_2, x_3) &\Leftrightarrow (s - t) \geq d(x, y) \Leftrightarrow \\ (s \geq t \ \&\ (s - t)^2 \geq d^2(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2) &\Leftrightarrow \\ (s \geq t \ \&\ (s - t)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2 \geq 0). \end{aligned}$$

It is also easy to see that the intersection of two geometric cones is, in general, *not* a cone – and thus, this ordered space is not a lattice.

It is therefore reasonable to ask: when is a Busemann product a lattice? The definition of a lattice means that for every two elements, we have a meet and a join. If for every two elements, we have a meet, this is called a *lower semi-lattice*; if for every two elements, we have a join, this is called an *upper semi-lattice*. A lattice is thus an ordered space which is at the same time a lower and an upper semi-lattice. We can therefore also ask: when is a Busemann product a lower semi-lattice? an upper semi-lattice? In this paper, we provide a necessary and sufficient condition for the Busemann product to be a lattice, a lower semi-lattice, and/or an upper semi-lattice.

2 Main Result

The answer comes in terms of *real trees* (*R-trees*), i.e., metric spaces in which every two points x and y are connected by exactly one *arc* – a homeomorphic embedding of an interval into this space, and this arc is *geodesic*, i.e., is formed by points x_α , $\alpha \in [0, d(x, y)]$ for which $d(x_\alpha, x_\beta) = |\alpha - \beta|$; see, e.g., [1].

An example of a real tree is a *hedgehog set* – a collection of several intervals with a common starting point O , in which the distance on each interval is Euclidean, between two points x and y on different intervals is defined as $d(x, y) = d(x, O) + d(O, y)$.

Definition 1. *Let X be a metric space with distance d . A set $\mathbb{R} \times X$ with an ordering relation $(s, y) \geq (t, x) \Leftrightarrow s - t \geq d(x, y)$ is called a Busemann product.*

Theorem. *For each metrics space X , the following conditions are equivalent to each other:*

- *the Busemann product $\mathbb{R} \times X$ is a lattice;*
- *the Busemann product $\mathbb{R} \times X$ is a lower semi-lattice;*
- *the Busemann product $\mathbb{R} \times X$ is an upper semi-lattice;*
- *the space X is a real tree.*

Proof.

1°. In this proof, we will use the following equivalent characterization of real trees: a metric space X is a real tree if and only if the following two conditions are satisfied:

- every two points x and y can be connected by a geodesic arc, and
- for every point x_α on the geodesic arc connecting x and y , and for every other point z , either x_α lies on a geodesic arc connecting x and z , or x_α lies on a geodesic arc connecting y and z .

These conditions are intuitively clear: when we go from x to z in a tree, we may follow the path from x to z for a while, but there is a branching point at which the paths deviate.

- If this branching point is after x_α , then x_α is on the geodesic path from x to y .
- If the branching happens before x_α , then the geodesic path from z to y should go pass x_α – otherwise, the paths from x to y , from y to z , and from x to z would form a loop, which cannot happen in a tree.

2°. First, let us prove that if $\mathbb{R} \times X$ is a lower semi-lattice, then X is a real tree. For the upper semi-lattice, the proof is similar.

3°. Let us first prove that for every two points $x, y \in X$, and for every $\alpha \in (0, d(x, y))$, there exists a point x_α for which $d(x, x_\alpha) = \alpha$ and $d(x_\alpha, y) = d(x, y) - \alpha$.

Indeed, let us consider the following four points: (α, x) , $(d(x, y) - \alpha, y)$, $(-\alpha, x)$, and $(\alpha - d(x, y), y)$. By using the definition of the Busemann product order, we can easily check that each of the first two points follows each of the second two points:

$$\begin{aligned} (\alpha, x) &\geq (-\alpha, x), & (\alpha, x) &\geq (\alpha - d(x, y), y), \\ (d(x, y) - \alpha, y) &\geq (-\alpha, x), & (d(x, y) - \alpha, y) &\geq (\alpha - d(x, y), y). \end{aligned}$$

Since the Busemann product $\mathbb{R} \times X$ is a lower semi-lattice, the first two points (α, x) and $(d(x, y) - \alpha, y)$ have a meet, i.e., a point $(s, z) \stackrel{\text{def}}{=} (\alpha, x) \wedge (d(x, y) - \alpha, y)$ which precedes both of them and which follows both of the points $(-\alpha, x)$ and $(\alpha - d(x, y), y)$:

$$\begin{aligned} (\alpha, x) &\geq (s, z), & (d(x, y) - \alpha, y) &\geq (s, z), \\ (s, z) &\geq (-\alpha, x), & (s, z) &\geq (\alpha - d(x, y), y). \end{aligned}$$

These orders mean that the following four inequalities are satisfied:

$$\begin{aligned} \alpha - s &\geq d(x, z); & d(x, y) - \alpha - s &\geq d(y, z); \\ s + \alpha &\geq d(x, z); & s + d(x, y) - \alpha &\geq d(y, z). \end{aligned}$$

Adding the first and the third inequalities and dividing the sum by two, we conclude that $\alpha \geq d(x, z)$. Similarly, by adding the second and the fourth inequalities and dividing the sum by two, we conclude that $d(x, y) - \alpha \geq d(y, z)$.

We cannot have strict inequality in any of these two inequalities, because if, e.g., $\alpha > d(x, z)$, then by adding it to $d(x, y) - \alpha \geq d(y, z)$, we could conclude that $d(x, y) > d(x, z) + d(y, z)$ – which contradicts to the triangle inequality. Thus, we must have equality, i.e., we must have $d(x, z) = \alpha$ and $d(y, z) = d(x, y) - \alpha$. The statement is proven: the point z is our desired point x_α .

In this case, from $d(x, z) = \alpha$ and $\alpha - s \geq d(x, z)$, we conclude that $s \leq 0$. Similarly, from $d(x, z) = \alpha$ and $\alpha + s \geq d(x, z)$, we conclude that $s \geq 0$. Since $s \leq 0$ and $s \geq 0$, we have $s = 0$. Thus, $(\alpha, x) \wedge (d(x, y) - \alpha, y) = (0, z)$.

4°. Let us now prove that for every two points $x, y \in X$, and for every $\alpha \in (0, d(x, y))$, there exists only one point x_α for which $d(x, x_\alpha) = \alpha$ and $d(x_\alpha, y) = d(x, y) - \alpha$.

Indeed, we have already shown that one such point exists – the point x_α for which $(\alpha, x) \wedge (d(x, y) - \alpha, y) = (0, x_\alpha)$. Let us assume that for some point $x'_\alpha \neq x_\alpha$, we have $d(x, x'_\alpha) = \alpha$ and $d(x'_\alpha, y) = d(x, y) - \alpha$. Then, by definition of the Busemann product order, we have $(\alpha, x) \geq (0, x'_\alpha)$ and $(d(x, y) - \alpha, y) \geq (0, x'_\alpha)$. By the definition of a meet, we then conclude that $(0, x_\alpha) \geq (0, x'_\alpha)$. By definition of the Busemann product order, this means that $0 \geq d(x_\alpha, x'_\alpha)$, i.e., that $d(x_\alpha, x'_\alpha) = 0$ and $x_\alpha = x'_\alpha$. Uniqueness is proven.

5°. Now, we can conclude that every two points $x, y \in X$ are connected by a geodesic arc.

We have already shown that for every α , there exists a unique point x_α for which $d(x, x_\alpha) = \alpha$ and $d(x_\alpha, y) = d(x, y) - \alpha$. We want to prove that these points x_α form a geodesic arc, i.e., that for every $\alpha < \beta$, we have $d(x_\alpha, x_\beta) = \beta - \alpha$. Indeed, due to Part 1 of this proof, if we take points x_α and y with $d(x_\alpha, y) = d(x, y) - \alpha$, then there exists a point x'_β for which $d(x_\alpha, x'_\beta) = \beta - \alpha$ and

$$d(x'_\beta, y) = d(x_\alpha, y) - (\beta - \alpha) = (d(x, y) - \alpha) - (\beta - \alpha) = d(x, y) - \beta.$$

Due to the triangle inequality,

$$d(x, x'_\beta) \leq d(x, x_\alpha) + d(x_\alpha, x'_\beta) \leq \alpha + (\beta - \alpha) = \beta,$$

so $d(x, x'_\beta) \leq \beta$. We cannot have $d(x, x'_\beta) < \beta$, since then we would have

$$d(x, y) \leq d(x, x'_\beta) + d(x'_\beta, y) < \beta + (d(x, y) - \beta) < d(x, y),$$

i.e., $d(x, y) < d(x, y)$, a contradiction. Thus, we have $d(x, x'_\beta) = \beta$ and $d(x'_\beta, y) = d(x, y) - \beta$. Due to Part 2 of our proof, this means that $x'_\beta = x_\beta$. Thus, $d(x_\alpha, x'_\beta) = \beta - \alpha$ implies that $d(x_\alpha, x_\beta) = \beta - \alpha$. The statement is proven.

6°. Let us now prove that for every $x, y \in X$, for every $\alpha \in (0, d(x, y))$, and for every point $z \in X$, we have either $d(x, z) = d(x, x_\alpha) + d(x_\alpha, z)$ or $d(y, z) =$

$d(y, x_\alpha) + d(x_\alpha, z)$. In other words, x_α either lies on a geodesic arc connecting x and z or on a geodesic arc connecting y and z . This would mean that X is a real tree.

Indeed, we know that $(\alpha, x) \wedge (d(x, y) - \alpha, y) = (0, x_\alpha)$. Let us find s for which $(\alpha, x) \geq (-s, z)$ and $(d(x, y) - \alpha, y) \geq (-s, z)$. The first desired relation means that $\alpha + s \geq d(x, z)$, i.e., that

$$s \geq d(x, z) - \alpha = d(x, z) - d(x, x_\alpha).$$

The second relation means that $d(x, y) - \alpha + s \geq d(y, z)$, i.e., that

$$s \geq d(y, z) - (d(x, y) - \alpha) = d(y, z) - d(y, x_\alpha).$$

So, if we take

$$s = \max(d(x, z) - d(x, x_\alpha), d(y, z) - d(y, x_\alpha)),$$

both inequalities will be satisfied and thus, we will have $(\alpha, x) \geq (-s, z)$ and $(d(x, y) - \alpha, y) \geq (-s, z)$.

By definition of the meet, this means that $(0, x_\alpha) \geq (-s, z)$, i.e., that $s \geq d(x_\alpha, z)$. The value s is defined as the largest of the two expressions, so it is equal to one of them.

If s is equal to the first expression $s = d(x, z) - d(x, x_\alpha)$, then the above inequality $s \geq d(x_\alpha, z)$ takes the form $d(x, z) - d(x, x_\alpha) \geq d(x_\alpha, z)$, i.e., equivalently, $d(x, z) \geq d(x, x_\alpha) + d(x_\alpha, z)$. Since by the triangle inequality, we have $d(x, z) \leq d(x, x_\alpha) + d(x_\alpha, z)$, we thus conclude that $d(x, z) = d(x, x_\alpha) + d(x_\alpha, z)$.

If s is equal to the second expression $s = d(y, z) - d(y, x_\alpha)$, then the above inequality $s \geq d(x_\alpha, z)$ takes the form $d(y, z) - d(y, x_\alpha) \geq d(x_\alpha, z)$, i.e., equivalently, $d(y, z) \geq d(y, x_\alpha) + d(x_\alpha, z)$. Since by the triangle inequality, we have $d(y, z) \leq d(y, x_\alpha) + d(x_\alpha, z)$, we thus conclude that $d(y, z) = d(y, x_\alpha) + d(x_\alpha, z)$.

The statement is proven.

7°. To complete our proof, we need to show that if X is a real tree, then the Busemann product is a lattice.

Let us assume that the metric space X is a real tree, and let us consider two points (t, x) and (s, x) in the Busemann product $\mathbb{R} \times X$. Let us show that the meet of these points exists (for the join, the proof is similar).

7.1°. If $t - s \geq d(x, y)$, then $(t, x) \geq (s, y)$, so the smallest point (s, y) is the desired meet.

7.2°. If $s - t \geq d(x, y)$, then $(s, y) \geq (t, x)$, so the smallest point (t, x) is the desired meet.

7.3°. Let us now consider the remaining case when $d(x, y) > |t - s|$. In this case, $-d(x, y) \leq t - s \leq d(x, y)$ hence $0 \leq t - s + d(x, y) \leq 2d(x, y)$ and thus, $0 \leq \alpha \leq d(x, y)$, where we denoted $\alpha \stackrel{\text{def}}{=} \frac{t - s + d(x, y)}{2}$. We will prove that in

this case, the desired meet is the element (t_0, x_α) , where $t_0 \stackrel{\text{def}}{=} \frac{t + s - d(x, y)}{2}$ and x_α is a point on the geodesic arc connecting x and y for which $d(x, x_\alpha) = \alpha$.

Note that indeed $(t, x) \geq (t_0, x_\alpha)$ and $(s, y) \geq (t_0, x_\alpha)$.

We need to prove that for every q and z , if $(t, x) \geq (q, z)$ and $(s, y) \geq (q, z)$ then $(t_0, x_\alpha) \geq (q, z)$. By the property of a real tree,

- either x_α lies on a geodesic arc connecting x and z ,
- or x_α lies on a geodesic arc connecting y and z .

Without losing generality, let us consider the first case, in which $d(x, z) = d(x, x_\alpha) + d(x_\alpha, z)$. We know that $(t, x) \geq (q, z)$ and $(s, y) \geq (q, z)$, i.e., that $t - q \geq d(x, z)$ and $s - q \geq d(y, z)$. We need to prove that $(t_0, x_\alpha) \geq (q, z)$, i.e., that $t_0 - q \geq d(x_\alpha, z)$. Since we are in the first case, we have $d(x_\alpha, z) = d(x, z) - d(x, x_\alpha) = d(x, z) - \alpha$. By definition of α , this means that

$$d(x_\alpha, z) = d(x, z) - \frac{t - s + d(x, y)}{2}.$$

Substituting this expression for $d(x_\alpha, z)$ and the definition of t_0 into the desired inequality $t_0 - q \geq d(x_\alpha, z)$, we get an equivalent inequality

$$\frac{t + s - d(x, y)}{2} - q \geq d(x, z) - \frac{t - s + d(x, y)}{2} = d(x, z) + \frac{s - t - d(x, y)}{2}.$$

By canceling identical terms $s/2$ and $-d(x, y)/2$ on both sides, and by moving $t/2$ into the left-hand side of this inequality, we get an equivalent inequality $t - q \geq d(x, z)$ which we assumed to be true. The statement is proven, and so is the theorem.

3 Open Questions

Case of quasimetrics. In the main text, we only considered metric spaces X , in which $d(x, y) = d(y, x)$, but a similar construction of a Busemann product order can be described for a *quasimetric*, i.e., a function which is not necessarily symmetric; see, e.g., [4]. It is desirable to extend our results to such quasimetrics.

More general Busemann products. The space $\mathbb{R} \times X$ is not just an ordered space: similarly to the case of special relativity, it can be equipped by a function describing proper time [2]:

$$\tau((t, x), (s, y)) = \sqrt[\alpha]{\max((s - t)^\alpha - d^\alpha(x, y), 0)}.$$

This function – called *kinematic metric* – satisfies the following two conditions:

- if $\tau(a, b) > 0$ then $a \geq b$, and

- the *anti-triangle inequality*: if $a \geq b \geq c$, then $\tau(a, c) \geq \tau(a, b) + \tau(b, c)$.

For each ordered space E with a function τ that satisfies these two conditions, and for each metric space X , we can define a Busemann product as the following ordering relation of $E \times X$:

$$(t, x) \geq (s, y) \Leftrightarrow \tau(t, s) \geq d(x, y).$$

It is desirable to analyze when this order is a lattice.

Acknowledgments

This work was partly supported by a CONACyT scholarship, by the National Science Foundation grants HRD-0734825 and DUE-0926721, and by Grant 1 T36 GM078000-01 from the National Institutes of Health. The author from South Africa also would like to thank the South African National Research Foundation for partial financial support.

References

- [1] M. Bestvina, “R-trees in topology, geometry, and group theory”, In: R. J. Daverman and R. B. Sher (eds.), *Handbook of Geometric Topology*, North-Holland, Amsterdam, 2002, pp. 55–91.
- [2] H. Busemann, *Timelike spaces*, PWN Publishers, Warszawa, 1967.
- [3] V. Kreinovich, “Space-time is ‘square times’ more difficult to approximate than Euclidean space”, *Geombinatorics*, 1996, Vol. 6, No. 1, pp. 19–29.
- [4] H.-P. A. Künzi and V. Kreinovich, “Static Space-Times Naturally Lead to Quasi-Pseudometrics”, *Theoretical Computer Science*, 2008, Vol. 405, No. 1–2, pp. 64–72.