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Uniqueness of Reconstruction for Yager’s t-Norm
Combination of Probabilistic and Possibilistic Knowledge

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Abstract

Often, about the same real-life system, we have both measurement-related probabilistic information expressed by a probability measure \(P(S)\) and expert-related possibilistic information expressed by a possibility measure \(M(S)\). To get the most adequate idea about the system, we must combine these two pieces of information. For this combination, R. Yager – borrowing an idea from fuzzy logic – proposed to use a t-norm \(f_t(a,b)\) such as the product \(f_t(a,b) = a \cdot b\), i.e., to consider a set function \(f(S) = f_t(P(S), M(S))\). A natural question is: can we uniquely reconstruct the two parts of knowledge from this function \(f(S)\)? In our previous paper, we showed that such a unique reconstruction is possible for the product t-norm; in this paper, we extend this result to a general class of t-norms.

1 Formulation of the Problem

Need to combine probabilistic and possibilistic knowledge. In many practical situations, we have both probabilistic information about some objects – e.g., information coming from measurements with known probability of measurement errors – and possibilistic information – describing expert knowledge. In the probabilistic case, for every set \(S\), we have a probability \(P(S) \in [0,1]\) that the actual (unknown) state \(s\) of the object belongs to the set \(S\). In the possibilistic case, for each set \(S\), we know the possibility \(M(S) \in [0,1]\) that \(s\) belongs to \(S\).

It is often desirable to combine these two numbers \(P(S)\) and \(M(S)\) into a single value \(f(S)\).
Yager’s approach: the use of t-norms. [5, 6] We need to combine two degrees from the interval [0, 1]. The desired combination must satisfy some reasonable properties; for example:

- if it is not possible for the state \( s \) to be in the set \( S \), i.e., if \( M(S) = 0 \),
  then the resulting degree \( f(S) \) must also reflect this impossibility, i.e., we should have \( f(S) = 0 \);

- if the probability \( P(S) \) of \( s \) being in the set \( S \) is equal to 0, i.e., if \( P(S) = 0 \),
  then we should also have \( f(S) = 0 \);

- etc.

Different procedures of combining such degrees have been actively analyzed in fuzzy logic; see, e.g., [2, 3]. In particular, procedures that satisfy the above properties (and several other similar properties) are known as t-norms (or and-operations) \( f(a; b) \). It is therefore reasonable to combine \( P(S) \) and \( M(S) \) by using a t-norm, i.e., to consider the set function \( f(S) = f_k(P(S), M(S)) \).

One of the simplest (and most widely used) t-norms is the algebraic product \( f_k(a, b) = a \cdot b \). In this case, we get a combination with a set function \( f(S) = P(S) \cdot M(S) \).

Uniqueness: a natural question. A natural question is: once we have the combined measure \( f(S) = f_k(P(S), M(S)) \), can we reconstruct both \( P(S) \) and \( M(S) \)?

Continuous case. We will consider a continuous case, in which the set \( X \) of all possible states is either an \( n \)-dimensional space \( \mathbb{R}^n \) or its open subset, and we restrict ourselves to open subsets \( S \subseteq X \). We assume that a probability measure \( P(S) \) is described by a continuous probability density function \( \rho(x) \geq 0 \) for which \( P(S) = \int_S \rho(x) \, dx \) and \( \int_X \rho(x) \, dx = 1 \). Similarly, we assume that a possibility measure is described by a continuous possibility function \( \mu(x) \geq 0 \) for which \( M(S) = \sup_{x \in S} \mu(x) \) and \( \sup_{x \in X} \mu(x) = 1 \). We will also assume that a t-norm \( f_k(a, b) \) is continuous.

What is known and what we do in this paper. In [1], we showed that reconstruction is unique for the case when the t-norm is the algebraic product. In this paper, we extend this result to a general class of t-norms.

2 First Result: Reconstructing \( P(S) \) from \( f(S) = f_k(P(S), M(S)) \)

Reminder. In this paper, we consider situations in which the universal set \( X \) is an open subset of an \( n \)-dimensional space \( \mathbb{R}^n \), a probability measure is defined by a continuous probability density function, and a possibility measure is defined by a continuous possibility function.
Theorem 1. Let $f_k(a, b)$ be a continuous t-norm, let $P(S)$ and $P'(S)$ be probability measures on the same set $X$, and let $M(S)$ and $M'(S)$ be possibility measures on $X$. If for every open set $S \subseteq X$, we have $f_k(P(S), M(S)) = f_k(P'(S), M'(S))$, then $P(S) = P'(S)$ for all sets $S$.

Comment. In other words, if we know the combined measure

$$f(S) = f_k(P(S), M(S)),$$

then we can uniquely reconstruct the probability measure.

Proof.

1°. For every point $x_0 \in X$ and for every positive real number $\delta$, let $B_\delta(x_0) \overset{\text{def}}{=} \{x : d(x, x_0) < \delta\}$ denote an open ball with a center in $x$ and radius $\delta$. In this proof, we will consider sets of the type $S \cup B_\delta(x_0)$ in the limit $\delta \to 0$.

We want to know the limit of

$$f(S \cup B_\delta(x_0)) = f_k(P(S \cup B_\delta(x_0)), M(S \cup B_\delta(x_0)))$$

when $\delta \to 0$. Since the t-norm $f_k(a, b)$ is continuous, it is sufficient to find the limits of $P(S \cup B_\delta(x_0))$ and $M(S \cup B_\delta(x_0))$; then, the limit of $f(S \cup B_\delta(x_0))$ is simply equal to the result of applying the t-norm $f_k(a, b)$ to the limits of $P(S \cup B_\delta(x_0))$ and $M(S \cup B_\delta(x_0))$.

2°. Let us start with computing the limit of $P(S \cup B_\delta(x_0))$. A probability measure is monotonic and additive, so we have

$$P(S) \leq P(S \cup B_\delta(x_0)) \leq P(S) + P(B_\delta(x_0)).$$

Let us show that $P(B_\delta(x_0)) \to 0$ as $\delta \to 0$; this will imply that

$$P(S \cup B_\delta(x_0)) \to P(S).$$

Indeed, since the probability density function $\rho(x)$ is continuous, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x, x_0) \leq \delta$ implies that $|\rho(x) - \rho(x_0)| \leq \varepsilon$. Let us pick any $\delta_0 > 0$ (e.g., $\delta_0 = 1$). Then, there exists a $\delta_0 > 0$ for which $d(x, x_0) \leq \delta_0$ implies that $|\rho(x) - \rho(x_0)| \leq \varepsilon_0$.

In this case, for every $\delta \leq \delta_0$, if $x \in B_\delta(x_0)$, then $d(x, x_0) < \delta \leq \delta_0$ hence $\rho(x) \leq \rho(x_0) + \varepsilon_0$. Thus,

$$0 \leq P(B_\delta(x_0)) = \int_{B_\delta(x_0)} \rho(x) \, dx \leq (\rho(x_0) + \varepsilon_0) \cdot V(B_\delta(x_0)).$$

When $\delta \to 0$, the sum $\rho(x_0) + \varepsilon_0$ is a constant and the volume $V(B_\delta(x_0)) \sim \delta^n$ tends to 0, so indeed $P(B_\delta(x_0)) \to 0$ and $P(S \cup B_\delta(x_0)) \to P(S)$.

3°. Let us now compute the limit of $M(S \cup B_\delta(x_0))$ when $\delta \to 0$. From the definition of a possibility measure, it follows that $M(A \cup B) = \max(M(A), M(B))$
for all $A$ and $B$; in particular, $M(S \cup B_{\delta}(x_0)) = \max(M(S), M(B_{\delta}(x_0)))$. Since $\max(a, b)$ is a continuous function, it is sufficient to find the limit of $M(B_{\delta}(x_0))$.

The possibility function $\mu(x)$ is also assumed to be continuous, so for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x, x_0) \leq \delta$ implies that $|\mu(x) - \mu(x_0)| \leq \varepsilon$, i.e., for all $x \in B_{\delta}(x_0)$, we have

$$\mu(x_0) - \varepsilon \leq \mu(x) \leq \mu(x_0) + \varepsilon.$$  

Since all the values $\mu(x)$ are between $\mu(x_0) - \varepsilon$ and $\mu(x_0) + \varepsilon$, the largest of these values $M(B_{\delta}(x_0)) = \sup_{B_{\delta}(x_0)} \mu(x)$ also lies within the same interval:

$$\mu(x_0) - \varepsilon \leq M(B_{\delta}(x_0)) \leq \mu(x_0) + \varepsilon.$$  

Thus, for every $\varepsilon > 0$ there exists a $\delta$ for which $|M(B_{\delta}(x_0)) - \mu(x)| \leq \varepsilon$. By definition of the limit, this means that $M(B_{\delta}(x_0)) \to \mu(x)$. So, due to the continuity of the maximum function,

$$M(S \cup B_{\delta}(x_0)) = \max(M(S), M(B_{\delta}(x_0))) \to \max(M(S), \mu(x)).$$

4°. Since the t-norm $f_{\&}(a, b)$ is continuous and we know the limits for $P(S \cup B_{\delta}(x_0))$ and $M(S \cup B_{\delta}(x_0))$,

we conclude that

$$f(S \cup B_{\delta}(x_0)) = f_{\&}(P(S \cup B_{\delta}(x_0)), M(S \cup B_{\delta}(x_0))) \to f_{\&}(P(S), \max(M(S), \mu(x))),$$

i.e.,

$$\lim_{\delta \to 0} f(S \cup B_{\delta}(x_0)) = f_{\&}(P(S), \max(M(S), \mu(x))).$$

5°. We now want to find the largest value of $f_{\&}(P(S), \max(M(S), \mu(x)))$, i.e.,

$$\sup_{x_0 \in X} f_{\&}(P(S), \max(M(S), \mu(x))).$$

Since the t-norm is monotonic, it is sufficient to find the largest possible value of $\max(M(S), \mu(x))$:

$$\sup_{x_0 \in X} f_{\&}(P(S), \max(M(S), \mu(x))) = f_{\&}\left(P(S), \sup_{x_0 \in X} \max(M(S), \mu(x))\right).$$

By definition of a possibility measure,

$$M(X) = \sup_{x_0 \in X} \mu(x_0) = 1.$$
Since \( \mu(x_0) \leq \max(S, \mu(x_0)) \leq 1 \), we can thus conclude that \( \sup_{x_0 \in X} \max(S, \mu(x_0)) = 1 \) and thus, \( \sup_{x_0 \in X} f_{k}(P(S), \max(M(S), \mu(x_0))) = f_{k}(P(S), 1) \). By definition of a t-norm, \( f_{k}(a, 1) = a \), hence \( f_{k}(P(S), 1) = P(S) \) and thus, for every set \( S \),
\[
\sup_{x_0 \in X} f_{k}(P(S), \max(M(S), \mu(x_0))) = P(S).
\]
We already know how to describe \( f_{k}(P(S), \max(M(S), \mu(x_0))) \) in terms of the combined function \( f(S) \): \( f_{k}(P(S), \max(M(S), \mu(x_0))) = \lim_{\delta \to 0} f(S \cup B_{\delta}(x_0)) \); thus,
\[
P(S) = \sup_{x_0 \in X} \lim_{\delta \to 0} f(S \cup B_{\delta}(x_0)).
\]
This formula describes the probability measure in terms of the combined measure. So, the probability measure can indeed be uniquely reconstructed form the combined measure. The theorem is proven.

3 Second Result: For Strictly Archimedean t-Norms, We Can Also Reconstruct \( M(S) \) from \( f(S) = f_{k}(P(S), M(S)) \)

Discussion. In the previous section, we showed that we can uniquely reconstruct the probability measure \( P(S) \) from the combined measure \( f(S) = f_{k}(P(S), M(S)) \). Let us show that for strictly Archimedean t-norms, we can also reconstruct the possibility measure \( M(S) \).

When \( \rho(x) = 0 \) for all points \( x \) from some region \( S \), this means that the probability \( P(S) = 0 \) of this region is 0, so points \( x \) from this region are not possible. We can therefore exclude these points from our universal set \( X \), and assume that \( \rho(x) > 0 \) for all \( x \in X \). Such probability measures will be called strictly positive.

Theorem 2. Let \( f_{k}(a, b) \) be a strictly Archimedean continuous t-norm, let \( P(S) \) and \( P'(S) \) be strictly positive probability measures on the same set \( X \), and let \( M(S) \) and \( M'(S) \) be possibility measures on \( X \). If for every open set \( S \subseteq X \), we have \( f_{k}(P(S), M(S)) = f_{k}(P'(S), M'(S)) \), then \( P(S) = P'(S) \) and \( M(S) = M'(S) \) for all sets \( S \).

Reminder. A strictly Archimedean t-norm (see, e.g., [2, 3]) can be described, e.g., by the property that when \( a > 0 \) and \( b > 0 \), the function \( f_{k}(a, b) \) is strictly increasing, i.e., if \( 0 < a < a' \) and \( 0 < b < b' \), then \( f_{k}(a, b) < f_{k}(a', b) \) and \( f_{k}(a, b) < f_{k}(a, b') \). The algebraic product t-norm \( f_{k}(a, b) = a \cdot b \) is a classical example of a strictly Archimedean t-norm.
Comment. The restriction to strictly Archimedean t-norms is not very restrictive, since, as shown in [4], an arbitrary t-norm with an arbitrary accuracy can be approximated by a strictly Archimedean one. Thus, for any given accuracy, strict Archimedean t-norms are sufficient for representing experts’ “and” operations.

Proof. According to Theorem 1, the fact that

\[ f_\kappa(P(S), M(S)) = f_\kappa(P'(S), M'(S)) \]

for all open sets \( S \) implies that \( P(S) = P'(S) \) for all such sets. Thus, for every open set \( S \), we have \( f_\kappa(P(S), M(S)) = f_\kappa(P(S), M'(S)) \). For strictly positive probability measures, with continuous positive density function \( \rho(x) > 0 \), the probability \( P(S) = \int_S \rho(x) \, dx \) is always positive \( P(S) > 0 \).

Thus, we cannot have \( M(S) < M'(S) \), because then, due to the above strict monotonicity property of strictly Archimedean t-norms, we would have \( f_\kappa(P(S), M(S)) < f_\kappa(P(S), M'(S)) \). Similarly, we cannot have \( M'(S) < M(S) \), because then, due to the above property of strictly Archimedean t-norms, we would have \( f_\kappa(P(S), M'(S)) < f_\kappa(P(S), M(S)) \). Since we cannot have \( M(S) < M'(S) \) and we cannot have \( M'(S) < M(S) \), the only remaining possibility is \( M(S) = M'(S) \). The theorem is proven.

4 Auxiliary Result: For t-Norms Which Are Not Strictly Archimedean, We Cannot Always Reconstruct \( M(S) \) from \( f(S) = f_\kappa(P(S), M(S)) \)

Let us show that the requirement that the t-norm be strictly Archimedean is necessary. Specifically, we will show that even for the simplest possible non-strictly-Archimedean t-norm \( f_\kappa(a, b) = \min(a, b) \), we sometimes cannot uniquely reconstruct \( M(S) \) from \( f(S) = f_\kappa(P(S), M(S)) \). Specifically, we will show an example of a strictly positive probability measure \( P(S) \) and two different possibility measures \( M(S) \neq M'(s) \) for which \( \min(P(S), M(S)) = \min(P(S), M'(S)) \) for all open sets \( S \).

As a universal set \( X \), let us take the interval \([0, 1]\). As \( P(S) \), we take the uniform probability measure, with \( \rho(x) = 1 \) for all \( x \in X \). The possibility functions \( \mu(x) \) and \( \mu'(x) \) defining the possibility measures \( M(S) \) and \( M'(S) \) are as follows: \( \mu(x) = 1 \) and \( \mu'(x) = \min(0.5 + x, 1) \) for all \( x \).

In this case, for every set \( S \), we have \( M(S) = \sup_{x \in S} \mu(x) = 1 \). In particular, this means that \( M(S) \geq 0.5 \) for every set \( S \). Since \( \mu'(x) \geq 0.5 \) for all \( x \), for every set \( S \), we have \( M'(S) = \sup_{x \in S} \mu'(x) \geq 0.5 \).

We will prove that \( \min(P(S), M(S)) = \min(P(S), M'(S)) \) for all open sets \( S \) by considering two possible cases: \( P(S) \leq 0.5 \) and \( P(S) > 0.5 \).
If $P(S) \leq 0.5$, then $P(S) \leq M(S)$ and $P(S) \leq M'(S)$, hence
\[
\min(P(S), M(S)) = P(S), \quad \min(P(S), M'(S)) = P(S),
\]
and therefore, $\min(P(S), M(S)) = \min(P(S), M'(S))$.

If $P(S) > 0.5$, this means that the set $S$ must contain some points $x_0$ from the second half $[0.5, 1]$ of the interval $X = [0, 1]$: indeed, otherwise, if $S \subseteq [0, 0.5]$, we would then have $P(S) \leq P([0,0.5]) = 0.5$ but we have $P(S) > 0.5$. For all points $x_0 \in [0.5,1]$, we have $\mu'(x_0) = 1$. Thus, in this case, we have $M'(S) = \sup_{x \in S} \mu(x) \geq \mu(x_0) = 1$ hence $M'(S) = 1$. We already know that $M(S) = 1$. Thus, $\min(P(S), M(S)) = \min(P(S), 1) = P(S)$, $\min(P(S), M'(S)) = \min(P(S), 1) = P(S)$, and therefore, $\min(P(S), M(S)) = \min(P(S), M'(S))$.

The desired equality have thus been proven for both possible cases. The example has been proven.

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**References**


