5-2011

Uniqueness of Reconstruction for Yager's t-Norm Combination of Probabilistic and Possibilistic Knowledge

Nitaya Buntao

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

Comments:

Technical Report: UTEP-CS-11-18

Published in *International Journal of Intelligent Systems*, 2012, Vol. 27, No. 1, pp. 16-22.

Uniqueness of Reconstruction for Yager's t-Norm Combination of Probabilistic and Possibilistic Knowledge

Nitaya Buntao[1] and Vladik Kreinovich[2]
[1]Department of Applied Statistics
King Mongkut's University of Technology North Bangkok
1518 Piboonsongkhram Road, Bangsue
Bangkok 10800 Thailand, taltanot@hotmail.com
[2]Computer Science Department, University of Texas at El Paso
El Paso, TX 79968, USA, vladik@utep.edu

Abstract

Often, about the same real-life system, we have both measurement-related probabilistic information expressed by a probability measure $P(S)$ and expert-related possibilistic information expressed by a possibility measure $M(S)$. To get the most adequate idea about the system, we must combine these two pieces of information. For this combination, R. Yager – borrowing an idea from fuzzy logic – proposed to use a t-norm $f_{\&}(a,b)$ such as the product $f_{\&}(a,b) = a \cdot b$, i.e., to consider a set function $f(S) = f_{\&}(P(S), M(S))$. A natural question is: can we uniquely reconstruct the two parts of knowledge from this function $f(S)$? In our previous paper, we showed that such a unique reconstruction is possible for the product t-norm; in this paper, we extend this result to a general class of t-norms.

1 Formulation of the Problem

Need to combine probabilistic and possibilistic knowledge. In many practical situations, we have both *probabilistic* information about some objects – e.g., information coming from measurements with known probability of measurement errors – and *possibilistic* information – describing expert knowledge. In the probabilistic case, for every set S, we have a probability $P(S) \in [0,1]$ that the actual (unknown) state s of the object belongs to the set S. In the possibilistic case, for each set S, we know the possibility $M(S) \in [0,1]$ that s belongs to S.

It is often desirable to combine these two numbers $P(S)$ and $M(S)$ into a single value $f(S)$.

Yager's approach: the use of t-norms. [5, 6] We need to combine two degrees from the interval $[0,1]$. The desired combination must satisfy some reasonable properties; for example:

- if it is not possible for the state s to be in the set S, i.e., if $M(S) = 0$, then the resulting degree $f(S)$ must also reflect this impossibility, i.e., we should have $f(S) = 0$;
- if the probability $P(S)$ of s being in the set S is equal to 0, i.e., if $P(S) = 0$, then we should also have $f(S) = 0$,
- etc.

Different procedures of combining such degrees have been actively analyzed in fuzzy logic; see, e.g., [2, 3]. In particular, procedures that satisfy the above properties (and several other similar properties) are known as *t-norms* (or and-operations) $f_\&(a,b)$. It is therefore reasonable to combine $P(S)$ and $M(S)$ by using a t-norm, i.e., to consider the set function $f(S) = f_\&(P(S), M(S))$.

One of the simplest (and most widely used) t-norms is the algebraic product $f_\&(a,b) = a \cdot b$. In this case, we get a combination with a set function $f(S) = P(S) \cdot M(S)$.

Uniqueness: a natural question. A natural question is: once we have the combined measure $f(S) = f_\&(P(S), M(S))$, can we reconstruct both $P(S)$ and $M(S)$?

Continuous case. We will consider a continuous case, in which the set X of all possible states is either an n-dimensional space $\mathbb{R}^n$ or its open subset, and we restrict ourselves to open subsets $S \subseteq X$. We assume that a probability measure $P(S)$ is described by a continuous probability density function $\rho(x) \geq 0$ for which $P(S) = \int_S \rho(x)\,dx$ and $\int_X \rho(x)\,dx = 1$. Similarly, we assume that a possibility measure is described by a continuous possibility function $\mu(x) \geq 0$ for which $M(S) = \sup\limits_{x \in S} \mu(x)$ and $\sup\limits_{x \in X} \mu(x) = 1$. We will also assume that a t-norm $f_\&(a,b)$ is continuous.

What is known and what we do in this paper. In [1], we showed that reconstruction is unique for the case when the t-norm is the algebraic product. In this paper, we extend this result to a general class of t-norms.

2 First Result: Reconstructing $P(S)$ from $f(S) = f_\&(P(S), M(S))$

Reminder. In this paper, we consider situations in which the universal set X is an open subset of an n-dimensional space $\mathbb{R}^n$, a probability measure is defined by a continuous probability density function, and a possibility measure is defined by a continuous possibility function.

Theorem 1. *Let $f_\&(a,b)$ be a continuous t-norm, let $P(S)$ and $P'(S)$ be probability measures on the same set X, and let $M(S)$ and $M'(S)$ be possibility measures on X. If for every open set $S \subseteq X$, we have $f_\&(P(S),M(S)) = f_\&(P'(S),M'(S))$, then $P(S) = P'(S)$ for all sets S.*

Comment. In other words, if we know the combined measure

$$f(S) = f_\&(P(S),M(S)),$$

then we can uniquely reconstruct the probability measure.

Proof.

1°. For every point $x_0 \in X$ and for every positive real number δ, let $B_\delta(x_0) \stackrel{\text{def}}{=} \{x : d(x,x_0) < \delta\}$ denote an open ball with a center in x and radius δ. In this proof, we will consider sets of the type $S \cup B_\delta(x_0)$ in the limit $\delta \to 0$.

We want to know the limit of

$$f(S \cup B_\delta(x_0)) = f_\&(P(S \cup B_\delta(x_0)), M(S \cup B_\delta(x_0)))$$

when $\delta \to 0$. Since the t-norm $f_\&(a,b)$ is continuous, it is sufficient to find the limits of $P(S \cup B_\delta(x_0))$ and $M(S \cup B_\delta(x_0))$; then, the limit of $f(S \cup B_\delta(x_0))$ is simply equal to the result of applying the t-norm $f_\&(a,b)$ to the limits of $P(S \cup B_\delta(x_0))$ and $M(S \cup B_\delta(x_0))$.

2°. Let us start with computing the limit of $P(S \cup B_\delta(x_0))$. A probability measure is monotonic and additive, so we have

$$P(S) \le P(S \cup B_\delta(x_0)) \le P(S) + P(B_\delta(x_0)).$$

Let us show that $P(B_\delta(x_0)) \to 0$ as $\delta \to 0$; this will imply that

$$P(S \cup B_\delta(x_0)) \to P(S).$$

Indeed, since the probability density function $\rho(x)$ is continuous, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x,x_0) \le \delta$ implies that $|\rho(x) - \rho(x_0)| \le \varepsilon$. Let us pick any $\varepsilon_0 > 0$ (e.g., $\varepsilon_0 = 1$). Then, there exists a $\delta_0 > 0$ for which $d(x,x_0) \le \delta_0$ implies that $|\rho(x) - \rho(x_0)| \le \varepsilon_0$.

In this case, for every $\delta \le \delta_0$, if $x \in B_\delta(x_0)$, then $d(x,x_0) < \delta \le \delta_0$ hence $\rho(x) \le \rho(x_0) + \varepsilon_0$. Thus,

$$0 \le P(B_\delta(x_0)) = \int_{B_\delta(x_0)} \rho(x)\,dx \le (\rho(x_0) + \varepsilon_0) \cdot V(B_\delta(x_0)).$$

When $\delta \to 0$, the sum $\rho(x_0) + \varepsilon_0$ is a constant and the volume $V(B_\delta(x_0)) \sim \delta^n$ tends to 0, so indeed $P(B_\delta(x_0)) \to 0$ and $P(S \cup B_\delta(x_0)) \to P(S)$.

3°. Let us now compute the limit of $M(S \cup B_\delta(x_0))$ when $\delta \to 0$. From the definition of a possibility measure, it follows that $M(A \cup B) = \max(M(A), M(B))$

for all A and B; in particular, $M(S\cup B_\delta(x_0))=\max(M(S),M(B_\delta(x_0)))$. Since $\max(a,b)$ is a continuous function, it is sufficient to find the limit of $M(B_\delta(x_0))$.

The possibility function $\mu(x)$ is also assumed to be continuous, so for every $\varepsilon>0$, there exists a $\delta>0$ such that $d(x,x_0)\le\delta$ implies that $|\mu(x)-\mu(x_0)|\le\varepsilon$, i.e., for all $x\in B_\delta(x_0)$, we have

$$\mu(x_0)-\varepsilon\le\mu(x)\le\mu(x_0)+\varepsilon.$$

Since all the values $\mu(x)$ are between $\mu(x_0)-\varepsilon$ and $\mu(x_0)+\varepsilon$, the largest of these values $M(B_\delta(x_0))=\sup_{B_\delta(x_0)}\mu(x)$ also lies within the same interval:

$$\mu(x_0)-\varepsilon\le M(B_\delta(x_0))\le\mu(x_0)+\varepsilon.$$

Thus, for every $\varepsilon>0$ there exists a δ for which $|M(B_\delta(x_0))-\mu(x)|\le\varepsilon$. By definition of the limit, this means that $M(B_\delta(x_0))\to\mu(x)$. So, due to the continuity of the maximum function,

$$M(S\cup B_\delta(x_0))=\max(M(S),M(B_\delta(x_0)))\to\max(M(S),\mu(x)).$$

4°. Since the t-norm $f_\&(a,b)$ is continuous and we know the limits for

$$P(S\cup B_\delta(x_0)) \text{ and } M(S\cup B_\delta(x_0)),$$

we conclude that

$$f(S\cup B_\delta(x_0))=f_\&(P(S\cup B_\delta(x_0)),M(S\cup B_\delta(x_0)))\to$$

$$f_\&(P(S),\max(M(S),\mu(x))),$$

i.e., that

$$\lim_{\delta\to0}f(S\cup B_\delta(x_0))=f_\&(P(S),\max(M(S),\mu(x_0))).$$

5°. We now want to find the largest value of $f_\&(P(S),\max(M(S),\mu(x)))$, i.e.,

$$\sup_{x_0\in X}f_\&(P(S),\max(M(S),\mu(x_0))).$$

Since the t-norm is monotonic, it is sufficient to find the largest possible value of $\max(M(S),\mu(x_0))$:

$$\sup_{x_0\in X}f_\&(P(S),\max(M(S),\mu(x_0)))=f_\&\left(P(S),\sup_{x_0\in X}\max(S,\mu(x_0))\right).$$

By definition of a possibility measure,

$$M(X)=\sup_{x_0\in X}\mu(x_0)=1.$$

Since $\mu(x_0) \le \max(S, \mu(x_0)) \le 1$, we can thus conclude that $\sup\limits_{x_0 \in X} \max(S, \mu(x_0)) = 1$ and thus, $\sup\limits_{x_0 \in X} f_\&(P(S), \max(M(S), \mu(x_0))) = f_\&(P(S), 1)$. By definition of a t-norm, $f_\&(a, 1) = a$, hence $f_\&(P(S), 1) = P(S)$ and thus, for every set S,

$$\sup_{x_0 \in X} f_\&(P(S), \max(M(S), \mu(x_0))) = P(S).$$

We already know how to describe $f_\&(P(S), \max(M(S), \mu(x_0)))$ in terms of the combined function $f(S)$: $f_\&(P(S), \max(M(S), \mu(x_0))) = \lim\limits_{\delta \to 0} f(S \cup B_\delta(x_0))$; thus,

$$P(S) = \sup_{x_0 \in X} \lim_{\delta \to 0} f(S \cup B_\delta(x_0)).$$

This formula describes the probability measure in terms of the combined measure. So, the probability measure can indeed be uniquely reconstructed form the combined measure. The theorem is proven.

3 Second Result: For Strictly Archimedean t-Norms, We Can Also Reconstruct $M(S)$ from $f(S) = f_\&(P(S), M(S))$

Discussion. In the previous section, we showed that we can uniquely reconstruct the probability measure $P(S)$ from the combined measure $f(S) = f_\&(P(S), M(S))$. Let us show that for strictly Archimedean t-norms, we can also reconstruct the possibility measure $M(S)$.

When $\rho(x) = 0$ for all points x from some region S, this means that the probability $P(S) = 0$ of this region is 0, so points x from this region are not possible. We can therefore exclude these points from our universal set X, and assume that $\rho(x) > 0$ for all $x \in X$. Such probability measures will be called *strictly positive.*

Theorem 2. *Let $f_\&(a, b)$ be a strictly Archimedean continuous t-norm, let $P(S)$ and $P'(S)$ be strictly positive probability measures on the same set X, and let $M(S)$ and $M'(S)$ be possibility measures on X. If for every open set $S \subseteq X$, we have $f_\&(P(S), M(S)) = f_\&(P'(S), M'(S))$, then $P(S) = P'(S)$ and $M(S) = M'(S)$ for all sets S.*

Reminder. A strictly Archimedean t-norm (see, e.g., [2, 3]) can be described, e.g., by the property that when $a > 0$ and $b > 0$, the function $f_\&(a, b)$ is strictly increasing, i.e., if $0 < a < a'$ and $0 < b < b'$, then $f_\&(a, b) < f_\&(a', b)$ and $f_\&(a, b) < f_\&(a, b')$. The algebraic product t-norm $f_\&(a, b) = a \cdot b$ is a classical example of a strictly Archimedean t-norm.

Comment. The restriction to strictly Archimedean t-norms is not very restrictive, since, as shown in [4], an arbitrary t-norm with an arbitrary accuracy can be approximated by a strictly Archimedean one. Thus, for any given accuracy, strict Archimedean t-norms are sufficient for representing experts' "and" operations.

Proof. According to Theorem 1, the fact that

$$f_{\&}(P(S), M(S)) = f_{\&}(P'(S), M'(S))$$

for all open sets S implies that $P(S) = P'(S)$ for all such sets. Thus, for every open set S, we have $f_{\&}(P(S), M(S)) = f_{\&}(P(S), M'(S))$. For strictly positive probability measures, with continuous positive density function $\rho(x) > 0$, the probability $P(S) = \int_S \rho(x)\,dx$ is always positive $P(S) > 0$.

Thus, we cannot have $M(S) < M'(S)$, because then, due to the above strict monotonicity property of strictly Archimedean t-norms, we would have $f_{\&}(P(S), M(S)) < f_{\&}(P(S), M'(S))$. Similarly, we cannot have $M'(S) < M(S)$, because then, due to the above property of strictly Archimedean t-norms, we would have $f_{\&}(P(S), M'(S)) < f_{\&}(P(S), M(S))$. Since we cannot have $M(S) < M'(S)$ and we cannot have $M'(S) < M(S)$, the only remaining possibility is $M(S) = M'(S)$. The theorem is proven.

4 Auxiliary Result: For t-Norms Which Are Not Strictly Archimedean, We Cannot Always Reconstruct $M(S)$ from $f(S) = f_{\&}(P(S), M(S))$

Let us show that the requirement that the t-norm be strictly Archimedean is necessary. Specifically, we will show that even for the simplest possible non-strictly-Archimedean t-norm $f_{\&}(a, b) = \min(a, b)$, we sometimes cannot uniquely reconstruct $M(S)$ from $f(S) = f_{\&}(P(S), M(S))$. Specifically, we will show an example of a strictly positive probability measure $P(S)$ and two different possibility measures $M(S) \neq M'(s)$ for which $\min(P(S), M(S)) = \min(P(S), M'(S))$ for all open sets S.

As a universal set X, let us take the interval $[0, 1]$. As $P(S)$, we take the uniform probability measure, with $\rho(x) = 1$ for all $x \in X$. The possibility functions $\mu(x)$ and $\mu'(x)$ defining the possibility measures $M(S)$ and $M'(S)$ are as follows: $\mu(x) = 1$ and $\mu'(x) = \min(0.5 + x, 1)$ for all x.

In this case, for every set S, we have $M(S) = \sup\limits_{x \in S} \mu(x) = 1$. In particular, this means that $M(S) \geq 0.5$ for every set S. Since $\mu'(x) \geq 0.5$ for all x, for every set S, we have $M'(S) = \sup\limits_{x \in S} \mu'(x) \geq 0.5$.

We will prove that $\min(P(S), M(S)) = \min(P(S), M'(S))$ for all open sets S by considering two possible cases: $P(S) \leq 0.5$ and $P(S) > 0.5$

If $P(S) \le 0.5$, then $P(S) \le M(S)$ and $P(S) \le M'(s)$, hence

$$\min(P(S), M(S)) = P(S), \quad \min(P(S), M'(S)) = P(S),$$

and therefore, $\min(P(S), M(S)) = \min(P(S), M'(S))$.

If $P(S) > 0.5$, this means that the set S must contain some points x_0 from the second half $[0.5, 1]$ of the interval $X = [0, 1]$: indeed, otherwise, if $S \subseteq [0, 0.5]$, we would then have $P(S) \le P([0, 0.5]) = 0.5$ but we have $P(S) > 0.5$. For all points $x_0 \in [0.5, 1]$, we have $\mu'(x_0) = 1$. Thus, in this case, we have $M'(S) = \sup\limits_{x \in S} \mu(x) \ge \mu(x_0) = 1$ hence $M'(S) = 1$. We already know that $M(S) = 1$. Thus, $\min(P(S), M(S)) = \min(P(S), 1) = P(S)$, $\min(P(S), M'(S)) = \min(P(S), 1) = P(S)$, and therefore, $\min(P(S), M(S)) = \min(P(S), M'(S))$.

The desired equality have thus been proven for both possible cases. The example has been proven.

Acknowledgments

This work was supported in part by the National Science Foundation grants HRD-0734825 and DUE-0926721 and by Grant 1 T36 GM078000-01 from the National Institutes of Health. The work of N. Buntao was supported by a grant from the Office of the Higher Education Commission, Thailand, under the Strategic Scholarships for Frontier Research Network.

The authors are thankful to Ron Yager, Carol Walker, and Elbert Walker for valuable discussions.

References

[1] N. Buntao and V. Kreinovich, "How to Combine Probabilistic and Possibilistic (Expert) Knowledge: Uniqueness of Reconstruction in Yager's (Product) Approach", *International Journal of Innovative Management, Information & Production (IJIMIP)*, 2011, Vol. 2, No. 1, pp. 1–8.

[2] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Upper Saddle River, New Jersey: Prentice Hall, 1995.

[3] H. T. Nguyen and E. A. Walker, *First Course on Fuzzy Logic*, CRC Press, Boca Raton, Florida, 2006.

[4] H. T. Nguyen and V. Kreinovich, "Archimedean t-norms and t-conorms as universal approximators", *International Journal of Approximate Reasoning*, 1998, Vol. 18, Iss. 3-4, pp. 239–249.

[5] R. R. Yager, "A Measure Based Approach to the Fusion of Uncertain Information", *Proceedings of the 30th Annual Conference of the North American Fuzzy Information Processing Society NAFIPS'2011*, El Paso, Texas, March 18–20, 2011.

[6] R. R. Yager, "A measure based approach to the fusion of possibilistic and probabilistic uncertainty," *Fuzzy Optimization and Decision Making*, to appear.