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Towards a General Description of Translation-Invariant and Translation-Covariant Linear Transformations: A Natural Justification of Fourier Transforms and Fuzzy Transforms

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Abstract: In many practical situations, we are interested in the dependencies that do not change with time, i.e., that do not change when we change the origin of the time axis. The corresponding translation-invariant transformations are easy to describe: they correspond to convolutions, or, equivalently, to fuzzy transforms.
It turns out that if we relax the invariance condition and require only that the transformation be translation-*convariant* (i.e., that it appropriately changes under translation), we get exactly two classes of transformations: Fourier transforms and fuzzy transforms. This result explain why both transforms have been successfully used in data processing.

1 From General to Linear Transformations: Motivations and Main Formulas

General transformations: brief reminder. For most real-life systems, their behavior and their state depend on the influence of other systems. For example, the state of a controlled system depends on what control we apply: the position and velocity of a car is determined by how much acceleration, breaking, and turning we applied; the state of a growing plant depend on how much water, minerals, and warmth it received at different moments of time, etc.

In systems terms, what we apply to the system is called an *input*, and the result of this application is called an *output*. In this section, we will denote the input by $x(s)$ and the output by $Y(t)$. In terms of these notations, each value $Y(t)$ of the output is determined by the values $x(s)$ of the input at different moments of time s.

In systems theory, the mapping that transforms the function $x(s)$ describing the input into a function $Y(t)$ that describes the output is called an *input-output transformation*, or simply *transformation*.

Comment about noise. In real life, the output is not uniquely determined by the input: due to inevitable noise, for the same input, the output may be somewhat different.

In this paper, we only consider the average output – and we do not provide a detailed analysis of the noise-induced random component of the output (i.e., of the random deviations between the actual output and the average output corresponding to the given input).

Inputs and outputs beyond control examples. The same input-output relation is applicable not only to controlled systems, but to other systems as well.

For example, due to inevitable inertia, a measuring instrument does not reproduce the input signal $x(s)$ exactly, it produces a somewhat distorted output signal $Y(t)$ – which is, however, uniquely determined by the input signal $x(s)$ (provided that we ignore the effects of the random noise).

Applications beyond dependence on time. Similar input-output relations hold for systems in which both the input and the output are of more general type than simply functions of time. For example, for an image processing system, the input is the input image, i.e., a function $I_{\rm in}(x_1, x_2)$ describing how the brightness depends on the spatial coordinates x_1 and x_2, and the output is the output image $I_{\rm out}(x_1, x_2)$.

Simplest case: 1-D systems. In general, we need several variables to describe the state of a system at a given moment of time. For example, at any given moment of time t, the state of a car can be described by its two spatial coordinates, two components of the velocity vector, and an angle describing the car's orientation.

Similarly, we usually need several variables to describe the control input. For example, to describe the control applied to a car, we need to describe two parameters: linear acceleration and the rotational acceleration (corresponding to turns).

The general behavior of such systems can be very complex and difficult to analyze. In this paper, we start our analysis with the simplest possible case, when we need only one variable to describe the input, and we need only one variable to describe the output.

For such 1-D systems, the input $x(s)$ at any given moment of time s is characterized by a single number, and the output $Y(t)$ at any given moment of time is also characterized by a single number.

From general systems to linear systems. For general systems, each value $Y(t)$ of the output is determined by the input values $x(s)$ at different moments of time s. In many practical situations, the input is relatively small. As a result, we can expand the dependence of $Y(t)$ on $x(s)$ in Taylor series and only keep linear terms in this expansion. How can we describe the general form of such a linear dependence?

In the situations when we have only finitely many moments of time $S_1, \ldots, S_n$ and thus, only finitely many input variables $x(S_1), \ldots, x(S_n)$, the general linear dependence can be described as

$$Y(t) = c(t) + c(t, S_1) \cdot x(S_1) + \ldots + c(t, S_n) \cdot x(S_n), \quad (1)$$

for appropriate coefficients $c(t)$ and $c(t, S_i)$.

In practice, we have a potentially infinite number of different moments of time s and thus, the potentially infinite number of input variables $x(s)$. To properly take into account the effect of all these variables, it is reasonable to consider more

and more dense values S_i which cover a larger and larger interval. When the values S_i get closer and closer to each other, the sum (1) tends to the corresponding integral.

So, a general transformation linear 1-D transformation can be written as follows: $Y(t) = c(t) + \int c(t,s) \cdot x(s)\,ds$, for appropriate functions $c(t)$ and $c(t,s)$.

When we do not apply any input, i.e., when $x(s) = 0$ for all s, then we get $Y(t) = c(t)$. Thus, if we identify input $x(t)$ with a control action, the function $c(t)$ describes the state of the un-controlled system, for which $x(t) = 0$. We are interested in predicting the *deviations* $y(t) \stackrel{\text{def}}{=} Y(t) - c(t)$ between the actual state and the un-controlled state. For this deviation, the dependence on $x(s)$ takes an even simpler form

$$y(t) = \int c(t,s) \cdot x(s)\,ds. \qquad (2)$$

This is the dependence that we will consider in this paper. To describe this dependence, it is sufficient to consider a single function $c(t,s)$. This function is usually called a *kernel* of the transformation (2).

Definition 1. *By a* linear transformation*, we mean a mapping of the type (2).*

Comment. In this paper, we will mainly consider the case when the function $c(t,s)$ is continuous, smooth (differentiable), etc. However, in some practical cases, this function is not continuous and not smooth.

As an example of such a situation, let us consider an ideal non-distorting input-output system in which, for every input x, the output $y(t)$ is identical to the input $x(t)$ at this same moment of time. In such a system, the value $y(t)$ depends only on the value $x(t)$ at this moment of time t but not on the values $x(s)$ for $s \neq t$. Thus, we must have $c(t,s) = 0$ for all $s \neq t$.

If the function $c(t,s)$ was a continuous function of its variables t and s, then we would be able to take $s_n = t + \dfrac{1}{n}$ and in the limit $n \to \infty$, when $s_n \to t$, get $c(t,t) = \lim c(s_n,t) = 0$. Since we already know that $c(t,s) = 0$ for $s \neq t$, we would thus conclude that $c(t,s) = 0$ for all s and t – and so, that the transformation (2) transforms every input signal $x(s)$ into an identical 0: $y(t) = 0$. This contradicts to the above assumption that $y(t) = x(t)$. So, the function c is not continuous.

For this ideal non-distorting transformation, not only the function $c(t,s)$ is not continuous, it is, strictly speaking, not a function at all – rather a limit of functions. Such useful limits are known as *generalized function* or *distributions*.

2 TRANSLATION-INVARIANT TRANSFORMATIONS: A GENERAL DESCRIPTION

Translation: motivations and reminder. For many real-life systems, the same input repeated after some time should lead to the exact same output. This is not always true: e.g., a system can start running out of battery power, or the material from which the system is built can start showing fatigue. However, in many cases, the above property is indeed true. How can we describe this property in precise terms?

First, we need to describe what it means that we apply the same input after a certain time t_0. Suppose that the original input was described by the function $x(s)$. We call the new input the same if it has the exact same form – but in the new time coordinate s_1, in which the starting point is t_0 time units after the original one.

If we change the original starting point (which corresponded to $s = 0$) with a new starting point which is t_0 units of time later, then the new time s_1 is equal to $s_1 = s - t_0$. Thus, in terms of the original time coordinate, the new input $x(s_1)$ has the form $x(s - t_0)$.

Informally, we simply shift all the moments of time by t_0. Because of this meaning, the operation transforming s into $s - t_0$ is called a *shift*, or a *translation*.

Translation-invariance: definition. The property that we are trying to formalize is as follows: We start with the input $x(s)$, and we produce the output $y(t)$. Then, we select some time shift t_0 and take the input $x(s_1) = x(s - t_0)$ which looks exactly the same as the original input $x(s)$ – except that is now described in new translated coordinates $s_1 = s - t_0$; we expect that in these new coordinates, the output $y_{t_0}(t)$ also take the exact same form as before, i.e., we expect the output to be equal to $y_{t_0}(t) = y(t_1) = y(t - t_0)$.

Thus, we require that the relation between input and output does not change ("is invariant") when we apply a time shift (translation). Such invariance is called *translation-invariance*.

Definition 2. *We say that a linear transformation (2) from functions to functions is* translation-invariant *if for every real number t_0, whenever the transformation transforms a function $x(s)$ into a function $y(t)$, it also transforms a translated function $x(s - t_0)$ into the similarly translated function $y(t - t_0)$.*

Translation-invariant transformations have been described in signal processing:

Proposition 1. *A linear transformation (2) is translation-invariant if and only the corresponding kernel $c(t,s)$ has the form $A(t - s)$ for some function $A(t)$.*

For such functions $c(t,s)$, the linear transformation (2) takes the form $y(t) = \int A(t - s) \cdot x(s)\,ds$. This transformation is called a *convolution* of functions $A(t)$ and $x(s)$. It also naturally appears in fuzzy logic techniques – and is therefore called *fuzzy transform*, or *F-transform*, for short; see, e.g., [3, 4].

3 FROM TRANSLATION-INVARIANCE TO TRANSLATION-COVARIANCE

Fourier transforms: reminder. One of the main tools of signal processing is Fourier transform

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \int \exp(-\mathrm{i} \cdot \omega \cdot s) \cdot x(s)\,ds,$$

where $\mathrm{i} \stackrel{\text{def}}{=} \sqrt{-1}$.

Comment. In addition to Fourier transform, signal processing also uses *Laplace transform* $\int \exp(-p \cdot s) \cdot x(s)\,dt$. Laplace transform is, in effect, Fourier transform corresponding to imaginary values $\omega = \mathrm{i} \cdot p$.

Fourier transform of a translated signal: reminder. One of the reasons why Fourier transform is so useful is that it behaves nicely under translation. Specifically, if instead of the original signal $x(s)$, we consider a translated signal $x_{t_0}(s) =$

$x(s-t_0)$, then the Fourier transform $X_{t_0}(\omega)$ of this translated signal takes the form

$$X_{t_0}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \int \exp(-\mathrm{i} \cdot \omega \cdot s) \cdot x_{t_0}(s)\, ds =$$

$$\frac{1}{\sqrt{2\pi}} \cdot \int \exp(-\mathrm{i} \cdot \omega \cdot s) \cdot x(s - t_0)\, dt. \quad (3)$$

Let us introduce the new variable $s_1 = s - t_0$. In terms of this new variable, $s = s_1 + t_0$, $ds = ds_1$, so (3) takes the form

$$X_{t_0}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \int \exp(-\mathrm{i} \cdot \omega \cdot (s_1 + t_0)) \cdot x(s_1)\, ds_1.$$

Here,

$$\exp(-\mathrm{i} \cdot \omega \cdot (s_1 + t_0)) = \exp(-\mathrm{i} \cdot \omega \cdot s_1) \cdot \exp(-\mathrm{i} \cdot \omega \cdot t_0). \quad (4)$$

The factor $\exp(-\mathrm{i} \cdot \omega \cdot t_0)$ does not depend on t_1 and can therefore be placed outside the integral. Thus, we get

$$X_{t_0}(\omega) = \exp(-\mathrm{i} \cdot \omega \cdot t_0) \cdot \frac{1}{\sqrt{2\pi}} \cdot \int \exp(-\mathrm{i} \cdot \omega \cdot s_1) \cdot x(s_1)\, dt_1.$$

The corresponding integral is simply $X(\omega)$, so we get

$$X_{t_0}(\omega) = \exp(-\mathrm{i} \cdot \omega \cdot t_0) \cdot X(\omega). \quad (5)$$

In other words, once we know all the values $X(\omega)$ of the Fourier transform of the original signal $x(s)$, we can easily find all the values of the Fourier transform $X_{t_0}(\omega)$ of the translated signal $x_{t_0}(s) = x(s - t_0)$: it is sufficient to multiply the corresponding components $X(\omega)$ by the corresponding factors $\exp(-\mathrm{i} \cdot \omega \cdot t_0)$.

Comment about notations. Traditionally, for the Fourier transforms, the variable is denoted by ω. However, since we want to consider Fourier transform as an example of a general transformation (2) in which the transformation result is denoted by $y(t)$, we will use the same general notation for the Fourier transform as well. In this notation, the formula (5) takes the form

$$y_{t_0}(t) = \exp(-\mathrm{i} \cdot t \cdot t_0) \cdot y(t). \quad (6)$$

Towards the notion of translation-covariance. We now have two examples in which, once we know the transformation $y(t)$ of the original signal $x(s)$, we can easily find the transformation $y_{t_0}(t)$ of the translated signal $x_{t_0}(s) = x(s - t_0)$:

For convolution (fuzzy transformation), we can find each value $y_{t_0}(t)$ as the value of the original transformation $y(t)$ at a translated moment of time: $y_{t_0}(t) = y(t - t_0)$.

For the Fourier transform, we can find each value $y_{t_0}(t)$ by multiplying the corresponding value $y(t)$ of the original transformation by an appropriate coefficient: $y_{t_0}(t) = \exp(-\mathrm{i} \cdot t \cdot t_0) \cdot y(t)$.

It is reasonable to consider a *general* type of such "easiness", where, to find each value $y_{t_0}(t)$ of the new transformation, it is sufficient to take a single value of the original transformation $y(v(t, t_0))$ at some point $v(t, t_0)$ – and if needed, multiply it by an appropriate factor $f(t, t_0)$ depending on t and on t_0.

Thus, we arrive at the following definition.

Definition 3. *We say that a linear transformation (2) is* translation-covariant *if there exist functions $v(t, t_0)$ and $f(t, t_0)$ such that for every real number t_0, whenever the transformation transforms a function $x(s)$ into a function $y(t)$, it should also transform a translated function $x_{t_0}(s) = x(s - t_0)$ into a function $y_{t_0}(t) = f(t, t_0) \cdot y(v(t, t_0))$.*

Examples. For the fuzzy transform, we have $f(t, t_0) = 1$ and $v(t, t_0) = t - t_0$. For the Fourier transform, we have $f(t, t_0) = \exp(-\mathrm{i} \cdot t \cdot t_0)$ and $v(t, t_0) = t$.

Comment. The terminology comes from physics, specifically from relativity theory, where [1]: Some physical quantities do not change their numerical values if we change the coordinate system; such properties are called *invariant*. Some quantities do change their numerical values when we change a coordinate system – but these values can be easily computed based on the values of this quantity in the original coordinates; such quantities are called *covariant*.

For example, the length of a 3-dimensional vector is *invariant* with respect to rotations, while the coordinates of this vector are *covariant*.

Our objective. The main objective of this paper is to describe all possible translation-covariant transformations.

Challenge. The description of all possible translation-covariant transformations is not a trivial task since, in principle, we can combine Fourier and fuzzy transforms.

For example, we can start with a fuzzy transform $y^{(1)}(t)$ and a Fourier transform $y^{(2)}(t)$, and then define a new translation-covariant transform $y^{(3)}(t)$ as follows:

- $y^{(3)}(t) = y^{(1)}(\tan(t))$ when $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and
- $y^{(3)}(t) = y^{(2)}(t)$ for all other values t.

One can show that this transform is indeed translation-covariant.

What we do in this paper. In this paper, we describe all possible translation-covariant transformations. We will show that, similar to the above example, every such transformation locally coincides either with a fuzzy transform, or with the Fourier transform. This result explains why both transforms have been successfully used in data processing.

4 Translation-Covariant Transformations: Towards a General Description

Equivalent transformations. In our definition of the translation-covariance, we require that each value of the transformation $y_{t_0}(t)$ of a translated input $x_{t_0}(s) = x(s - t_0)$ can be obtained by multiplying one the values $y(v(t, t_0))$ of the transformation $y(t)$ of the original signal $x(s)$ by an appropriate factor $f(t, t_0)$.

Thus, it is reasonable to expect that the transformation retains this property if we simply multiply all its values by some function $m(t)$ and/or reshuffle the values $y(t)$. Let us describe these changes in precise terms.

Definition 4. *We say that a transformation*

$$y^{(1)}(t) = \int c^{(1)}(t, s) \cdot x(s)\, ds$$

is multiplication-equivalent *to the transformation*

$$y(t) = \int c(t,s) \cdot x(s)\,ds. \tag{7}$$

if for some function $m(t) \neq 0$ and for every input $x(s)$, we have $y^{(1)}(t) = m(t) \cdot y(t)$.

Comment. It is easy to check that the formula $y^{(1)}(t) = m(t) \cdot y(t)$ indeed defines a linear transformation of type (2): Indeed, from (7), it follows that

$$y^{(1)}(t) = m(t) \cdot \int c(t,s) \cdot x(s)\,ds.$$

Since the factor $m(t)$ does not depend on s, it can be placed inside the integral:

$$y^{(1)}(t) = \int m(t) \cdot c(t,s) \cdot x(s)\,ds.$$

So, the new transformation has the form (2) with the new function $c^{(1)}(t,s) = m(t) \cdot c(t,s)$.

Proposition 2. *If a transformation is translation-covariant, then every multiplication-equivalent transformation is also translation-covariant.*

Definition 5. *We say that a transformation*

$$y^{(1)}(t) = \int c^{(1)}(t,s) \cdot x(s)\,ds$$

is permutation-equivalent *to the transformation*

$$y(t) = \int c(t,s) \cdot x(s)\,ds. \tag{8}$$

if for some one-to-one function $p(t)$ and for every input $x(s)$, we have $y^{(1)}(t) = y(p(t))$.

Comment. It is easy to check that the formula $y^{(1)}(t) = y(p(t))$ indeed defines a linear transformation of type (2): Indeed, from (8), it follows that

$$y^{(1)}(t) = \int c(p(t),s) \cdot x(s)\,ds. \tag{9}$$

So, the new transformation has the form (2) with the new function $c^{(1)}(t,s) = c(p(t),s)$.

Proposition 3. *If a transformation is translation-covariant, then every permutation-equivalent transformation is also translation-covariant.*

The general case can be described as follows:

Definition 6. *We say that a transformation*

$$y^{(1)}(t) = \int c^{(1)}(t,s) \cdot x(s)\,ds$$

is equivalent *to the transformation*

$$y(t) = \int c(t,s) \cdot x(s)\,ds$$

if for some function $m(t) \neq 0$ and for some one-to-one function $p(t)$, for every input $x(s)$, we have $y^{(1)}(t) = m(t) \cdot y(p(t))$.

For the general case, a similar result holds:

Proposition 4. *If a transformation is translation-covariant, then every equivalent transformation is also translation-covariant.*

From equivalence to reduction. Translation-covariant transformations are not necessarily equivalent to Fourier transform. For example, if we transform the original function $x(s)$ into a single value of the Fourier transform, then we also get a translation-covariant transformation – but this new transformation is not equivalent to the original Fourier transform, since it has lost most of the information about the original Fourier transform.

To describe such situation, we will supplement the notion of equivalence with a notion of *reduction:*

Definition 7. *Let t_g be a real number, and let*

$$y^{(1)}(t) = \int c^{(1)}(t,s) \cdot x(s)\,ds \tag{10}$$

be a linear transformation. We say that the t_g-th component $y^{(1)}(t_g)$ of the transformation (10) can be reduced *to the transformation $y(t) = \int c(t,s) \cdot x(s)\,ds$ if there exist values m and p for which, for every input $x(s)$, we have $y^{(1)}(t_g) = m \cdot y(p)$.*

Definition 8. *Let t_g be a real number. We say that a transformation $y^{(1)}(t) = \int c^{(1)}(t,s) \cdot x(s)\,ds$ can be t_g*-locally reduced *to the transformation $y(t) = \int c(t,s) \cdot x(s)\,ds$ if there exists an open interval $I = (t_-, t_+)$ (finite or infinite) containing t_g and functions $m(t)$ and $p(t)$ defined on this interval for which, for every input $x(s)$ and for all $t \in I$, we have $y^{(1)}(t) = m(t) \cdot y(p(t))$.*

Comment. One can easily check that two transformations are equivalent if the first can be reduced to the second one (with $I = R$) and the second one can be reduced to the first one. In this sense, reduction is a local one-sided analogue of equivalence.

Smooth transformations. In this paper, we will consider transformations in which the function $c(t,s)$ is smooth (= twice continuously differentiable) and the corresponding functions $v(t,t_0)$ and $f(t,t_0)$ are also smooth.

Both fuzzy transforms with a smooth function $A(t)$ and the Fourier transform are smooth in this sense.

Comments. A similar result holds for some non-smooth functions as well, if we consider generalized functions – since for some non-smooth functions, we can describe their "derivatives" as generalized functions.

Definition 9. *In this paper, by a* smooth function*, we mean a twice continuously differentiable function.*

Definition 10. *A linear transformation (2) is called* smooth *if the corresponding function $c(t,s)$ is smooth.*

Definition 11. *We say that a smooth linear transformation (2) is* smoothly translation-covariant *if there exist smooth functions $v(t,t_0)$ and $f(t,t_0)$ such that for every real number t_0, whenever the transformation transforms a function $x(s)$ into a function $y(t)$, it should also transform a translated function $x_{t_0}(s) = x(s - t_0)$ into a function $y_{t_0}(t) = f(t,t_0) \cdot y(v(t,t_0))$.*

Comment. In the following section, we will show that for translation-covariant transformations, it is sufficient to require that the functions $c(t,s)$ and $v(t,t_0)$ are smooth. In this case, the smoothness of the factor function $f(t,t_0)$ follows.

5 MAIN RESULT AND ITS PROOF

Now, we are ready for formulate our main result.

Theorem 1. *Let (2) be a smoothly translation-covariant linear transformation. Then, for every value t_g,*

- *either $y(t_g)$ can be reduced to the Fourier transform,*
- *or the transformation (2) can be t_g-locally reduced to a fuzzy transform.*

Proof: towards a functional equation. translation-covariance means that for every function $x(s)$, once we know its transformation

$$y(t) = \int c(t,s) \cdot x(s)\, ds \tag{11}$$

the transformation

$$y_{t_0}(t) = \int c(t,s) \cdot x(s - t_0)\, ds \tag{12}$$

of the translated input $x_{t_0}(t) = x_{t_0}(t - t_0)$ is related to the original transformation by the formula

$$y_{t_0}(t) = f(t,t_0) \cdot y(v(t,t_0)). \tag{13}$$

Substituting expression (11) and (12) into the formula (13), we conclude that

$$\int c(t,s) \cdot x(s - t_0)\, ds = f(t,t_0) \cdot \int c(v(t,t_0),s) \cdot x(s)\, ds$$

for all possible inputs $x(s)$.

Introducing a new variable $s_1 = s - t_0$ (for which $s = s_1 + t_0$ and $ds = ds_1$) into the left-hand side of this formula, we conclude that

$$\int c(t,s_1+t_0) \cdot x(s_1)\, ds_1 = f(t,t_0) \cdot \int c(v(t,t_0),s) \cdot x(s)\, ds.$$

For convenience, it is useful to rename the variable in the first integral from s_1 back to s. Then, we get

$$\int c(t,s+t_0) \cdot x(s)\, ds = f(t,t_0) \cdot \int c(v(t,t_0),s) \cdot x(s)\, ds.$$

This is true for all inputs $x(s)$. For linear functions of finitely many variables, the two linear functions coincide if and only if all their coefficients coincide. For linear transformations, the same result is true: the coefficients at each value $x(s)$ in both sides must be the same:

$$c(t,s+t_0) = f(t,t_0) \cdot c(v(t,t_0),s) \tag{14}$$

for all possible real numbers t, s_1, and t_0.

From a functional equation to a differential equation. Functional equations are, in general, difficult to solve. Thus, to solve the equation (14), we will reduce it to an easier-to-solve differential equation.

This reduction when all the functions involved in this equation are smooth (differentiable). We assumed that the kernel $c(t,s)$ is smooth, and that the function $v(t,t_0)$ is smooth. Therefore, the only smoothness that we need to prove is that the function $f(t,t_0)$ is smooth as well.

Auxiliary result: the function $f(t,t_0)$ is also smooth. From the equation (14), we conclude that

$$f(t,t_0) = \frac{c(v(t,t_0),s)}{c(t,s+t_0)}. \tag{15}$$

We assumed that the kernel $c(t,s)$ is smooth, and that the function $v(t,t_0)$ is smooth. Thus, from the formula (15), we can conclude that the function $f(t,t_0)$ is also smooth.

From a functional equation to a differential equation (cont-d). Since all three functions $c(t,s)$, $f(t,t_0)$, and $v(t,t_0)$ are smooth, both the left-hand side and the right-hand side of the formula (14) are smooth. Therefore, we can differentiate both sides of this formula by t_0 and take $t_0 = 0$.

For $t_0 = 0$, we have $f(t,0) = 1$ and $v(t,t_0) = t$. As a result, we get the following formula

$$\frac{\partial c}{\partial s} = F(t) \cdot c(t,s) - \frac{\partial c}{\partial t} \cdot V(t), \tag{16}$$

where we denoted

$$F(t) = \frac{\partial f(t,t_0)}{\partial t_0}_{|t_0=0}; \quad V(t) = -\frac{\partial v(t,t_0)}{\partial t_0}_{|t_0=0}.$$

Two possibilities. In this proof, we will consider two possible situations: $V(t_g) = 0$ and $V(t_g) \neq 0$.

First case. In the first case, when $V(t_g) = 0$, the equation (16) takes the form $\frac{\partial c}{\partial s}(t_g,s) = F(t_g) \cdot c(t,s)$. Thus, the function $c_g(s) \stackrel{\text{def}}{=} c(t_g,s)$ satisfies the equation $\frac{dc_g}{ds} = F(t_g) \cdot c_g(s)$. Moving all the terms depending on c_g into the left-hand side and all the other terms into the right-hand side, we conclude that $\frac{dc_g}{c_g} = F(t_g) \cdot ds$. Integrating both sides of this equation, we get

$$\ln(c_g) = F(t_g) \cdot s + C \tag{17}$$

for some integration constant C. Taking $\exp$ of both side of the equality (17), to get $c_g = \exp(\ln(c_g))$ in the left-hand side, we conclude that

$$c(t_g,s) = c_g(s) = \exp(F(t_g) \cdot s + C) = e^C \cdot \exp(F(t_g) \cdot s).$$

Thus, in this case, the corresponding value $y(t_g)$ can be reduced to the corresponding component of the Fourier transform, with $m = \exp(C)$ and with $p(t_g) = F(t_g)$.

Second case. Let us now consider the second case, when $V(t_g) \neq 0$. Since the function $v(t,t_0)$ is twice continuously differentiable, its partial derivative $V(t)$ is continuously differentiable.

If $V(t) \neq 0$ for all t, this means that the function $V(t)$ has the same sign for all values t. In this case, as the desired interval I, we take the entire real axis R.

If there exists a value $t < t_g$ for which $V(t_g) = 0$, then let us take, as the left endpoint of the interval I, the least upper bound t_- of all the values $t < t_g$ at which $V(t) = 0$. This point is a limit of points at which $V(t) = 0$. Since the function $V(t)$ is continuous, we can thus conclude that $V(t_-) = 0$. (If there is no such $t < t_g$, then we take $t_- = -\infty$.)

Similarly, if there exists a value $t > t_g$ for which $V(t_g) = 0$, then let us take, as the right endpoint of the interval I, the

greatest lower bound t_+ of all the values $t > t_g$ at which $V(t) = 0$. This point is a limit of points at which $V(t) = 0$. Since the function $V(t)$ is continuous, we can thus conclude that $V(t_+) = 0$. (If there is no such $t < t_g$, then we take $t_- = +\infty$.)

On the resulting interval $I = (t_i, t_+)$, the function $V(t)$ has the same sign. On this interval, we can simplify the formula (16) if we introduce a new coordinate $t_1 = T(t)$ for which $dt_1 = \dfrac{dt}{V(t)}$. This can be done if we take $T(t) = \int \dfrac{dt}{V(t)}$. Since the function $V(t)$ has the same sign, the function $T(t)$ is strictly monotonic: either strictly increasing or strictly decreasing. Thus, on the interval I, we can define an inverse function $T^{-1}(t)$.

If there are values $t_- > -\infty$ and/or $t_+ < +\infty$ at which $V(t_\pm) = 0$, then the integral $T(t)$ is not defined beyond these values. Indeed, since the function $V(t)$ is differentiable, we have

$$V(t_\pm + \Delta t) = V(t_\pm) + V'(t_\pm) \cdot \Delta t + o(\Delta t) =$$

$$V'(t_\pm) \cdot \Delta t + o(\Delta t).$$

Thus, in the vicinity of the point $t_\pm$, the corresponding integral

$$\int \frac{dt}{V(t)} = \int \frac{d(\Delta t)}{V(t_\pm + \Delta t)} \sim \frac{1}{V'(t_\pm)} \cdot \int \frac{d(\Delta t)}{\Delta t} =$$

$$\frac{1}{V'(t_\pm)} \cdot \ln(\Delta t),$$

hence for $\Delta t \to 0$, this integral tends to infinity.

If we express t in terms of the new variable t_1, i.e., take $t = T^{-1}(t_1)$, we get $\dfrac{\partial c}{\partial t} \cdot V(t) = \dfrac{\partial c_1}{\partial t_1}$, where $c_1(t_1, s) \stackrel{\rm def}{=} c(T^{-1}(t_1), s)$ is the value $c(t, s)$ expressed in terms of the new time coordinate $t_1 = T(t)$ (for which $t = T^{-1}(t_1)$).

Thus, the equation (16) takes the simplified form

$$\frac{\partial c_1}{\partial s} = F_1(t_1) \cdot c_1(t_1, s) - \frac{\partial c_1}{\partial t_1}, \tag{18}$$

where $F_1(t_1) \stackrel{\rm def}{=} F(T^{-1}(t_1))$ is the value $F(t)$ expressed in terms of the new time coordinate $t_1 = T(t)$.

We can simplify the equation (18) even further if we introduce a new variable $s_1 \stackrel{\rm def}{=} t_1 - s$. In terms of this variable, $s = t_1 - s_1$, and the kernel $c_1(t_1, s)$ takes the form $c_2(t_1, s_1) \stackrel{\rm def}{=} c_1(t_1, t_1 - s_1)$. Vice versa, we have $s_1 = t_1 - s$; thus,

$$c_1(t_1, s) = c_2(t_1, t_1 - s). \tag{19}$$

For this expression (19),

$$\frac{\partial c_1(t_1, s)}{\partial s} = -\frac{\partial c_2(t_1, t_1 - s_1)}{\partial s_1} = \frac{\partial c_2}{\partial s_1}_{(t_1, t_1 - s_1)} \tag{20}$$

and

$$\frac{\partial c_1(t_1, s)}{\partial t_1} = \frac{\partial c_2(t_1, t_1 - s)}{\partial t_1} = \frac{\partial c_2}{\partial t_1} + \frac{\partial c_2}{\partial s_1} \tag{21}$$

Thus, substituting the formulas (19), (20), and (21) into the equation (19), we conclude that

$$\frac{\partial c_2}{\partial s_1} = F_1(t_1) \cdot c_2(t, s) \frac{\partial c_2}{\partial t_1} + \frac{\partial c_2}{\partial s_1}. \tag{22}$$

Canceling equal terms $\dfrac{\partial c_2}{\partial s_1}$ in both sides, we get a simplified formula $0 = F_1(t_1) \cdot c_2(t_1, s_1) + \dfrac{\partial c_2}{\partial t_1}$, i.e., $\dfrac{\partial c_2}{\partial t_1} = -F_1(t_1) \cdot c_2(t_1, s_1)$. For each value s_1, the auxiliary function $a_{s_1}(t_1) \stackrel{\rm def}{=} c_2(t_1, s_1)$ satisfies the equation $\dfrac{da_{s_1}}{dt_1} = -F_1(t_1) \cdot a_{s_1}(t_1)$. Moving all the terms depending on a_{s_1} into the left-hand side and all the other terms into the right-hand side, we conclude that $\dfrac{da_{s_1}}{a_{s_1}} = -F_1(t_1) \cdot dt_1$. Integrating both sides of this equation, we get

$$\ln(a_{s_1})(t_1) = L(t_1) + C, \tag{23}$$

where $L(t_1) \stackrel{\rm def}{=} -\int F_1(t_1) \cdot dt_1$, the integration constant C may be different from different values s_1: $C = C(s_1)$. Applying $\exp$ to both side of the equality (23), we conclude that

$$c_2(t_1, s_1) = a_{s_1}(t_1) = \exp(L(t_1) + C(s_1)) =$$

$$\exp(L(t_1)) \cdot \exp(C(s_1)).$$

Substituting $s_1 = t_1 - s$ into this formula, we get

$$c_1(t_1, s) = c_2(t_1, t_1 - s) = \exp(L(t_1)) \cdot \exp(C(t_1 - s)).$$

Finally, substituting $t_1 = T(t)$ into this formula, we get

$$c(t, s) = c_1(T(t), s) = \exp(L(T(t)))) \cdot \exp(C(T(t) - s)).$$

One can easily check that this transformation can be reduced to the convolution (fuzzy transform) $A(t - s)$ with $A(x) \stackrel{\rm def}{=} \exp(C(x))$, with reduction described by the formulas $m(t) = \exp(L(T(t)))$ and $p(t) = T(t)$.

Thus, for the interval I containing the given point t_g, we get the desired t_g-local reduction.

Thus, for both cases, the theorem is proven.

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