

10-1999

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Comments:

UTEP-CS-99-34.

In: Hrushikesh Mohanty and Chitta Baral (eds.), Trends in Information Technology, *Proceedings of the International Conference on Information Technology ICIT'99, Bhubaneswar, India, December 20-22, 1999*, Tata McGraw-Hill, New Delhi, 2000, pp. 230-233.

Recommended Citation

Von Gundenberg, Juergen Wolff and Kreinovich, Vladik, "Candidate Sets for Complex Interval Arithmetic" (1999). *Departmental Technical Reports (CS)*. 572.

https://scholarworks.utep.edu/cs_techrep/572

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Candidate Sets for Complex Interval Arithmetic

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Abstract

Uncertainty of measuring complex-valued physical quantities can be described by complex sets. These sets can have complicated shapes, so we would like to find a good approximating family of sets. Which approximating family is the best? We reduce the corresponding optimization problem to a geometric one: namely, we prove that, under some reasonable conditions, an optimal family must be shift-, rotation- and scale-invariant. We then use this geometric reduction to conclude that the best approximating low-dimensional families consist of sets with linear or circular boundaries. This result is consistent with the fact that such sets have indeed been successful in computations. It stimulates to study further candidates.

Construction of Optimal Families

A practical problem leading to complex sets.

Many physical quantities are complex-valued: wave function in quantum mechanics, complex amplitude and impedance in electrical engineering, etc.

Due to measurement uncertainty, after measuring a value of a physical quantity, we do not get its *exact* value, we only get a *set* of possible values of this quantity. The shapes of these sets can be very complicated, so we would like to approximate them by simpler shapes from an approximating family. Which family should we choose?

In 1-D case, a similar problem has a simple solution: we choose the family of all (real) intervals. This family has many good properties; in particular, it is closed under point-wise arithmetic operations

$$A \odot B = \{a \odot b \mid a \in A, b \in B\}$$

such as addition, subtraction, and multiplication, which makes this family perfect for the analysis of how these measurement results get processed in a computer.

Unfortunately, for complex sets, no finite-dimensional family containing real intervals is closed under

these operations [Nickel 1980]; moreover, no finite-dimensional family containing real intervals is closed under addition and under multiplication by complex *numbers*. This negative result has a clear geometric meaning, due to the fact that adding a complex number means a shift, and multiplication by a complex number $\rho \cdot \exp(i\theta)$ means rotation by an angle θ and scaling ρ times. So, Nickel's negative result means it is impossible to have a finite-dimensional family of complex sets which would be closed under addition, invariant under shift, rotation, and scaling, and contain real intervals.

Since we cannot have an approximating family which satisfies all desired properties, we must therefore use families which satisfy only some of them. Several families have been proposed: boxes, polygons, circles, ellipsoids, etc. Some families approximate better, some approximate worse. So, an (informal) problem is: which approximating family is the best?

Of course, the more parameters we allow, the better the approximation. So, the question can be reformulated as follows: for a given number of parameters (i.e., for a given dimension of approximating family), which is the best family? In this paper, we formalize and solve this problem.

Formalizing the problem. All proposed families of sets have analytical (or piece-wise analytical) boundaries, so it is natural to restrict ourselves to such families. By definition, when we say that a piece of a boundary is analytical, we mean that it can be described by an equation $F(x, y) = 0$ for some analytical function

$$F(x, y) = a + bx + cy + dx^2 + exy + fy^2 + \dots$$

So, in order to describe a family, we must describe the corresponding class of analytical functions $F(x, y)$.

Since we are interested in finite-dimensional families of sets, it is natural to consider finite-dimensional families of functions, i.e., families of the type

$$\{C_1 \cdot F_1(x, y) + \dots + C_d \cdot F_d(x, y)\},$$

where $F_i(z)$ are given analytical functions, and C_1, \dots, C_d are arbitrary (real) constants. So, the question is: which of such families is the best?

When we say “the best”, we mean that on the set of all such families, there must be a relation \geq describing which family is better or equal in quality. This relation must be transitive (if A is better than B , and B is better than C , then A is better than C). This relation is not necessarily asymmetric, because we can have two approximating families of the same quality. However, we would like to require that this relation be *final* in the sense that it should define a unique *best* family A_{opt} (i.e., the unique family for which $\forall B (A_{\text{opt}} \geq B)$). Indeed:

- If none of the families is the best, then this criterion is of no use, so there should be *at least one* optimal family.
- If *several* different families are equally best, then we can use this ambiguity to optimize something else: e.g., if we have two families with the same approximating quality, then we choose the one which is easier to compute. As a result, the original criterion was not final: we get a new criterion ($A \geq_{\text{new}} B$ if either A gives a better approximation, or if $A \sim_{\text{old}} B$ and A is easier to compute), for which the class of optimal families is narrower. We can repeat this procedure until we get a final criterion for which there is only one optimal family.

It is reasonable to require that the relation $A \geq B$ should not change if we add or multiply all elements of A and B by a complex number; in geometric terms, the relation $A \geq B$ should be shift-, rotation- and scale-invariant.

Now, we are ready for the formal definitions.

Definition 1. Let $d > 0$ be an integer. By a d -dimensional family, we mean a family A of all functions of the type $\{C_1 \cdot F_1(x, y) + \dots + C_d \cdot F_d(x, y)\}$, where $F_i(z)$ are given analytical functions, and C_1, \dots, C_d are arbitrary (real) constants. We say that a set is defined by this family A if its border consists of pieces described by equations $F(x, y) = 0$, with $F \in A$.

Definition 2. By an optimality criterion, we mean a transitive relation \geq on the set of all d -dimensional families. We say that a criterion is *final* if there exists one and only one optimal family, i.e., a family A_{opt} for which $\forall B (A_{\text{opt}} \geq B)$. We say that a criterion \geq is *shift-* (corr., *rotation-* and *scale-invariant*) if for every two families A and B , $A \geq B$ implies $TA \geq TB$, where TA is a shift (rotation, scaling) of the family A .

Proposition. ($d \leq 4$) Let \geq be a final optimality criterion which is shift-, rotation-, and scale-invariant, and let A_{opt} be the corresponding optimal family. Then, the border of every set defined by this family A_{opt} consists of straight line intervals and circular arcs.

Comment. This result is in good accordance with numerical experiments, according to which such sets in-

deed provide a good approximation (see, e.g., [Alefeld et al. 1974], [Klatte et al. 1980], [Lerch et al. 1999]).

Proof. This proof is similar to the ones from [Nguyen et al. 1997].

1. Let us first show that the optimal family A_{opt} is itself shift-, rotation-, and scale-invariant.

Indeed, let T be an arbitrary shift, rotation, or scaling. Since A_{opt} is optimal, for every other family B , we have $A_{\text{opt}} \geq T^{-1}B$ (where T^{-1} means the inverse transformation). Since the optimality criterion \geq is invariant, we conclude that $TA_{\text{opt}} \geq T(T^{-1}B) = B$. Since this is true for every family B , the family TA_{opt} is also optimal. But since our criterion is final, there is only one optimal family and therefore, $TA_{\text{opt}} = A_{\text{opt}}$. In other words, the optimal family is indeed invariant.

2. Let us now show that all functions from A_{opt} are polynomials.

Indeed, every function $F \in A_{\text{opt}}$ is analytical, i.e., can be represented as a Taylor series (sum of monomials). Let us combine together monomials $cx^a y^b$ of the same degree $a + b$; then we get

$$F(z) = F_0(z) + F_1(z) + \dots + F_k(z) + \dots,$$

where $F_k(z)$ is the sum of all monomials of degree k . Let us show, by induction over k , that for every k , the function $F_k(z)$ also belongs to A_{opt} .

Let us first prove that $F_0(z) \in A_{\text{opt}}$. Since the family A_{opt} is scale-invariant, we conclude that for every $\lambda > 0$, the function $F(\lambda z)$ also belongs to A_{opt} . For each term $F_k(z)$, we have $F_k(\lambda z) = \lambda^k F_k(z)$, so

$$F(\lambda z) = F_0(z) + \lambda F_1(z) + \dots \in A_{\text{opt}}.$$

When $\lambda \rightarrow 0$, we get $F(\lambda z) \rightarrow F_0(z)$. The family A_{opt} is finite-dimensional hence closed; so, the limit $F_0(z)$ also belongs to A_{opt} . The induction base is proven.

Let us now suppose that we have already proven that for all $k < s$, $F_k(z) \in A_{\text{opt}}$. Let us prove that $F_s(z) \in A_{\text{opt}}$. For that, let us take

$$G(z) = F(z) - F_1(z) - \dots - F_{s-1}(z).$$

We already know that $F_1, \dots, F_{s-1} \in A_{\text{opt}}$; so, since A_{opt} is a linear space, we conclude that

$$G(z) = F_s(z) + F_{s+1}(z) + \dots \in A_{\text{opt}}.$$

The family A_{opt} is scale-invariant, so, for every $\lambda > 0$, the function

$$G(\lambda z) = \lambda^s F_s(z) + \lambda^{s+1} F_{s+1}(z) + \dots$$

also belongs to A_{opt} . Since A_{opt} is a linear space, the function

$$H_\lambda(z) = \lambda^{-s} G(\lambda z) =$$

$$F_s(z) + \lambda F_{s+1}(z) + \lambda^2 F_{s+2}(z) + \dots$$

also belongs to A_{opt} .

When $\lambda \rightarrow 0$, we get $H_\lambda(z) \rightarrow F_s(z)$. The family A_{opt} is finite-dimensional hence closed; so, the limit $F_s(z)$ also belongs to A_{opt} . The induction is proven.

Now, monomials of different degree are linearly independent; therefore, if we have infinitely many non-zero terms $F_k(z)$, we would have infinitely many linearly independent functions in a finite-dimensional family A_{opt} – a contradiction. Thus, only finitely many monomials $F_k(z)$ are different from 0, and so, $F(z)$ is a sum of finitely many monomials, i.e., a polynomial.

3. Let us prove that if a function $F(x, y)$ belongs to A_{opt} , then its partial derivatives $F_{,x}(x, y)$ and $F_{,y}(x, y)$ also belong to A_{opt} .

Indeed, since the family A_{opt} is shift-invariant, for every $h > 0$, we get $F(x + h, y) \in A_{\text{opt}}$. Since this family is a linear space, we conclude that a linear combination $h^{-1}(F(x + h, y) - F(x, y))$ of two functions from A_{opt} also belongs to A_{opt} . Since the family A_{opt} is finite-dimensional, it is closed and therefore, the limit $F_{,x}(x, y)$ of such linear combinations also belongs to A_{opt} . (For $F_{,y}$, the proof is similar).

4. Due to Parts 2 and 3 of this proof, if any polynomial from A_{opt} has a non-zero part F_k of degree $k > 0$, then it also has a non-zero part $((F_k)_{,x}$ or $(F_k)_{,y}$) of degree $k - 1$. Similarly, it has non-zero parts of degrees $k - 2, \dots, 1, 0$.

So, in all cases, A_{opt} contains a non-zero constant and a non-zero linear function $F_1(x, y) = bx + cy$. We can now use the fact that the family A_{opt} is rotation-invariant; let T be a rotation which transforms (b, c) into the x -axis, then we conclude that

$$F_1(Tz) = b'x \in A_{\text{opt}},$$

and hence $x \in A_{\text{opt}}$. Similarly, $y \in A_{\text{opt}}$. So, the family A_{opt} contains at least 3 linearly independent functions: a non-zero constant, x , and y .

If $d = 3$, then the 3-D family A_{opt} cannot contain anything else, and all the pieces of borders $F(x, y) = 0$ of all the sets defined by this family are straight lines.

If $d = 4$, then we cannot have any cubic or higher order terms in A_{opt} , because then, due to Part 3, we would have both this cubic part *and* a (linearly independent) quadratic part, and the total dimension of A_{opt} would be at least $3 + 2 = 5$. So, all functions from A_{opt} are quadratic. Since $\dim(A_{\text{opt}}) = 4$, and the dimension of 0- and 1-D parts is 3, the dimension of possible parts of second degree is 1. Since A_{opt} is rotation-invariant, the quadratic part $dx^2 + exy + fy^2$ must be also rotation-invariant (else, we would have two linearly independent quadratic terms in A_{opt} : the original expression and its rotated version). Thus, this quadratic part must be proportional to $x^2 + y^2$.

Hence, every function $F \in A_{\text{opt}}$ has the form

$$F(x, y) = a + bx + cy + d(x^2 + y^2),$$

and therefore, all the pieces of borders $F(x, y) = 0$ of all the sets defined by this family are either straight lines or circular arcs. The proposition is proven.

Properties of Sub-Families

In this section we recall that the usually used implementations of complex interval arithmetic, e.g. axis parallel rectangles, circles, and circular sectors are subfamilies of A_{opt} and collect their properties.

In general we can describe a family of sets by choosing a specific number of boundary curves of the type:

$$\mathcal{C} : a + bx + cy + d(x^2 + y^2)$$

Interesting for practical applications are those which lead to closed bounded sets, are characterized by a few parameters and have efficient implementations of the arithmetic operations. Since no family can be closed for addition and multiplication multispect combinations of several representations are favorable, see [Lerch et al. 1999]

Circles

- 1 boundary curve: $d \neq 0$
- 3 real parameters needed: $a/d, b/d, c/d$
- equivalent representation: 1 point, 1 radius
- properties
 - shift-invariant
 - rotation-invariant
 - scale-invariant
 - addition closed
 - multiplication not closed
 - multiplicative inversion closed
 - intersection not closed

Rectangles

- 4 boundary curves: two vertical lines $d = 0, c = 0, b \neq 0$ and two horizontal lines $d = 0, b = 0, c \neq 0$
- 4 real parameters needed: $a_1/b_1, a_2/b_2, a_3/c_3, a_4/c_4$
- equivalent representation: 2 points
- properties
 - shift-invariant
 - not rotation-invariant
 - scale-invariant
 - addition closed
 - multiplication not closed
 - multiplicative inversion not closed
 - intersection closed

Sectors

- 4 boundary curves: two straight lines $d = 0, a = 0, c \neq 0$ and two circles $c = 0, b = 0, d \neq 0$ origin centered
- 4 real parameters needed: $b_1/c_1, b_2/c_2, a_3/d_3, a_4/d_4$
- equivalent representation: 2 real intervals (radius, angle)
- properties
 - not shift-invariant

- rotation-invariant
- scale-invariant
- addition not closed
- multiplication closed
- multiplicative inversion closed
- intersection not closed

Parallelograms

- 4 boundary curves: 4 straight lines $d = 0, c \neq 0$ with $b_1/c_1 = b_2/c_2$ and $b_3/c_3 = b_4/c_4$
- 6 real parameters needed: $a_1, a_2, a_3, a_4, b_1/c_1, b_3/c_3$
- equivalent representation: 3 points (1 corner and 2 edges)
- properties
 - shift-invariant
 - rotation-invariant
 - scale-invariant
 - addition not closed
 - multiplication not closed
 - multiplicative inversion not closed
 - intersection not closed

The operations like addition or membership test are more efficient than for sectors. A generalization to parallelepipeds or zonotopes is used in dynamic systems to avoid the wrapping effect, see [Kühn 1998], [Lohner 1988]

Acknowledgments. This work was supported in part by NASA under cooperative agreement NCC5-209, by NSF grants No. DUE-9750858 and CDA-9522207, by United Space Alliance, grant No. NAS 9-20000 (PWO C0C67713A6), by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-95-1-0518, by the National Security Agency under Grant No. MDA904-98-1-0564, and by Nortel Co.

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