From Fuzzy Models to Fuzzy Control

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Abstract

Traditional (non-fuzzy) control methodology deals with situations when we know exactly how the system behaves and how it will react to different controls, and we want to choose an appropriate control strategy. This methodology enables us to transform the description of the plant's (system's) behavior into an appropriate control strategy.

In many practical situations, we do not have the exact knowledge of the system's behavior, but we have expert-supplied fuzzy rules which describe this behavior. In such situations, it is desirable to transform these description rules into rules describing control. There exist several reasonable heuristics for such transformation; however, the lack of formal justification restricts their applicability. In this paper, we provide a justification for the most natural of the known heuristics: whenever we have a description rule "if $A(x)$ and $B(u)$ then $C(\dot x)$", and we want to achieve $\dot x = d(x)$, add a control rule "if $A(x)$ and $C(d(x))$, then $B(u)$".

1 Introduction

If we have a crisp model of a system, then traditional control theory enables us to design a controller. In many real-life situations, however, we only have a fuzzy model for a system, formulated in terms of if-then rules. How can we then design a controller?

In some situation, we have justified methods [1, 6], but in general, we have to use heuristics. Probably the most natural heuristic method of extracting control rules from the description rules is as follows: If we have a description rule "$A(x)$ and $B(u)$ imply $C(\dot x)$", then it is natural to use a control rule, according to which "if $A(x)$ and $C(d(x))$, then we must apply control for which $B(u)$" (because, according to the description of the system's evolution, this will lead to the desired property $C(\dot x(t))$ of $\dot x(t)$). In general, when, in the description of a system, we have a description rule of the type

if x_1 is $A_{j,1}$, ..., x_n is $A_{j,n}$, u_1 is $B_{j,1}$, ..., and u_l is $B_{j,l}$, then $\dot x_1$ is $C_{j,1}$, ..., and $\dot x_n$ is $C_{j,n}$,

then we must add the following control rule:

if x_1 is $A_{j,1}$, ..., x_n is $A_{j,n}$, $d(x_1)$ is $C_{j,1}$, ..., and $d(x_n)$ is $C_{j,n}$, then u_1 is $B_{j,1}$, ..., and u_l is $B_{j,l}$.

In this paper, we justify the use of this heuristic.

2 The Notion of Strong Approximation and the Corresponding Universal Approximation Result

By a *fuzzy set*, we mean a function $\mu : R \to [0,1]$; this function will also be called a *membership function*. Fuzzy sets will also be called *natural language terms*. Fuzzy sets will be denoted by capital letters (possibly with indices). A membership function corresponding to a set A will be denoted by $\mu^A(x)$, and a membership function corresponding to a set $A_{j,i}$ will be denoted by $\mu^A_{j,i}(x)$. A membership function that is equal to 1 for $x = a$ and to 0 for all other x will be denoted by $\delta(x-a)$ and called a *crisp number*.

Definition 1. *Assume that two positive integers n and l are given; the integer n will be called the number of variables, and the integer l is called the number of control parameters. We will denote R^n by X or by $\dot X$, and R^l by U.*

By a description rule, we mean a formula of the type

$$A_1(x_1)\,\&\,\ldots\,\&\,A_n(x_n)\,\&\,B_1(u_1)\,\&\,\ldots\,\&\,B_l(u_l) \to C_1(\dot x_1)\,\&\,\ldots\,\&\,C_n(\dot x_n),$$

where A_i, B_j, and C_k are natural language words (i.e., fuzzy sets).

By a fuzzy model, we mean a finite set of description rules. The number of rules in a fuzzy model will be denoted by N, and the j^{th} rule will be denoted by

$$A_{j,1}(x_1)\,\&\,\ldots\,\&\,A_{j,n}(x_n)\,\&\,B_{j,1}(u_1)\,\&\,\ldots\,\&\,B_{j,l}(u_l) \to C_{j,1}(\dot x_1)\,\&\,\ldots\,\&\,C_{j,n}(\dot x_n).$$

Definition 2.

- *By a defuzzification procedure D we mean a mapping that transforms a membership function $\mu(x)$ on a (multi-dimensional) space R^k into an element $D(\mu) \in R^k$ and satisfies the following properties:*
  - *if $\mu(x) = 0$ for all x for which $x_i > a$, then $D_i(\mu) \leq a$;*
  - *if $\mu(x) = 0$ for all x for which $x_i < a$, then $D_i(\mu) \geq a$.*
- *By a centroid defuzzification D^c we mean a mapping that transforms μ into*

$$D^c(\mu) = \frac{\int u \cdot \mu(u)\,\mathrm{d}u}{\int \mu(u)\,\mathrm{d}u}.$$

- *We say that a defuzzification procedure D is consistent if for every membership function $\mu(x)$ which is not identically 0, if $d = D(\mu)$, then $\mu(d) > 0$.*

Comments.

It is easy to check that a centroid defuzzification is an example of a defuzzification in the sense of this definition.

If a membership function is defined only at finitely many points, then integrals degenerate into sums. Thus, if we have a membership function $\mu(x)$ that is different from 0 only for the values $x_1, \ldots, x_n$, then

$$D^c(\mu) = \frac{x_1 \cdot \mu(x_1) + \ldots + x_n \cdot \mu(x_n)}{\mu(x_1) + \ldots + \mu(x_n)}.$$

The necessity for a special requirement that a defuzzification be "consistent" comes from the fact that, when applied to control, centroid defuzzification sometimes leads to counterintuitive and counterproductive recommendations (see, e.g., [7]). For example, if a car is approaching an obstacle, then we can either turn to the left, or turn to the right. Let us take the turning angle u as the control parameter. If the situation is absolutely symmetric, then the corresponding membership function for u is symmetric with respect to changing left and right (i.e., with respect to changing u to $-u$). Because of this symmetry, the angle recommended by the centroid defuzzification based on this membership function is 0, so, our choice will lead us straight ahead to the obstacle. To avoid such recommendations, J. Yen *et al.* recommend, in [7], not to take the centroid value $d^c = D^c(\mu)$ as a desired defuzzification, but first to check whether the value $\mu(d^c)$ is large enough (or at least that it is positive). If it is not, this means that the original membership function consists of several segments, so we should choose one of these segments and apply centroid defuzzification to this section only.

Definition 3. *Assume that $\mathcal{M}$ is a fuzzy model, $f_\&$, $f_\vee$ are $\&-$ and $\vee-$operations (i.e., t-norm and t-conorm), and D is a defuzzification procedure. By the fuzzy input-output function $f^{\mathcal{M}}(x, u, \dot{x})$ corresponding to the fuzzy model $\mathcal{M}$, we mean a mapping which maps every triple of vectors $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_l)$, and $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_n)$ into a number*

$$f^{\mathcal{M}}(x, u, \dot{x}) = f_\vee(p_1^{\mathcal{M}}(x, u, \vec{x}), \ldots, p_N^{\mathcal{M}}(x, u, \dot{x})),$$

where

$$p_j^{\mathcal{M}}(x, u, \dot{x}) = f_\&(r_j(x, u), c_j(\dot{x})),$$

where

$$r_j(x, u, \dot{x}) =$$
$$f_\&(\mu_{j,1}^A(x_1), \ldots, \mu_{j,n}^A(x_n), \mu_{j,1}^B(u_1), \ldots, \mu_{j,l}^B(u_l)),$$

and

$$c_j(\dot{x}) = f_\&(\mu_{j,1}^C(\dot{x}_1), \ldots, \mu_{j,n}^C(\dot{x}_n)).$$

Definition 3′. *Assume that $\mathcal{M}$ is a fuzzy model, $f_\&$, $f_\vee$ are $\&-$ and $\vee-$operations, and D is a defuzzification procedure. By the crisp input-output function corresponding to the fuzzy model $\mathcal{M}$, we mean a mapping which maps every pair (x, u) into a number $c^{\mathcal{M}}(x, u) = D(f^{\mathcal{M}}(x, u, \dot{x}))$, where $f^{\mathcal{M}}$ is the fuzzy input-output function corresponding to $\mathcal{M}$.*

Comment. Each value $r_j(x, u)$ is a degree to which j-th rule is applicable for the given values x and u. In particular, if all functions $C_{j,i}$ are crisp numbers (i.e., if $\mu_{j,i}^C(\dot{x}) = \delta(\dot{x} - \bar{\dot{x}}_{j,i})$ for some $\bar{\dot{x}}_{j,i}$), all values $\bar{\dot{x}}_{j,i}$ are different, and $D = D^c$, then

$$c_i^{\mathcal{M}} = \frac{\bar{\dot{x}}_{1,i} \cdot r_1(x, u) + \ldots + \bar{\dot{x}}_{N,i} \cdot r_N(x, u)}{r_1(x, u) + \ldots + r_N(x, u)}.$$

This formula is much easier to compute than the formula with integrals; for this reason, it is sometimes used as a fuzzy control methodology even when the words $C_{j,i}$ are not crisp numbers (e.g., we choose the values $\bar{\dot{x}}_{j,i}$ for which $\mu_{j,i}^C(\dot{x})$ is the largest possible and act as if $\mu_{j,i}^C = \delta(\dot{x} - \bar{\dot{x}}_{j,i})$). (It should be noted that, unlike the general result, the above simplified formula does not depend on the choice of an $\vee-$operation.)

Definition 4. *Let $\delta > 0$ be a real number.*

- *By a plant, we mean a pair consisting of a compact set $K \subset X \times U$ and a continuous function $f : X \times U \to \dot{X}$.*
- *We say that a fuzzy model $\mathcal{M}$ δ-approximates a plant (K, f), if for every $(x, u) \in K$, the value of the corresponding crisp input-output function is also δ-close to $f(x, u)$, i.e., $\|c^{\mathcal{M}}(x, u) - f(x, u)\| \leq \delta$. (Here, $\|.\|$ is a standard Euclidean distance.)*

Definition 5. *We say that a fuzzy model $\mathcal{M}$ strongly δ-approximates* a plant (K, f) *if for every* $(x, u, \dot{x})$, *the following two conditions hold:*

- $f^{\mathcal{M}}(x, u, f(x, u)) > 0$, *and*
- *the inequality* $f^{\mathcal{M}}(x, u, \dot{x}) > 0$ *implies that* $\|\dot{x} - f(x, u)\| \leq \delta$.

Comment. In other words, we say that a fuzzy model is an strong δ-approximation to the plant if only values $\dot{x}$ which are δ-close to the actual ones are possible. We call this *strong* approximation because one can prove that a strong δ-approximation implies the (normal) approximation (with a somewhat larger value δ'):

Proposition 1. *If a fuzzy model $\mathcal{M}$ strongly δ-approximates a plant (K, f), then the model $\mathcal{M}$ also δ'-approximates the plant, for $\delta' = \sqrt{n} \cdot \delta$.*

The known universal approximation results show that an arbitrary plant can be approximated by an appropriate fuzzy model. For the desired generation of fuzzy control, we must prove a stronger result: that an arbitrary plant can be *strongly* approximated by an appropriate fuzzy model. Moreover, it is possible to prove that in the approximating fuzzy model, we can restrict ourselves to membership functions of a given type (e.g., triangular, or trapezoidal):

Definition 6.

- *We say that a membership function $\mu_0(x)$ is realistic if it is continuous, positive inside a certain finite interval $[a^-, a^+]$, and equal to outside this interval.*
- *Let $\mu_0(x)$ be a membership function. We say that a function $\mu(x)$ is of type $\mu_0(x)$ if $\mu(x) = \mu_0(a \cdot x + b)$ for some real numbers a and b.*

Theorem 1. *Let D be an arbitrary defuzzification procedure, $f_\&$ be an arbitrary non-nilpotent t-norm, $f_\vee$ be an arbitrary t-conorm, and let $\mu_0(x)$ be a realistic membership function. For every plant (K, f), and for every $\delta > 0$, there exists a fuzzy model $\mathcal{M}$ in which all membership functions are of type μ_0, and which strongly δ-approximates the plant (K, f).*

3 From Fuzzy Model to Fuzzy Control: A Theorem

Let us show that when a fuzzy model is a good approximation, the corresponding fuzzy control rules approximate the appropriate control.

Definition 7. *By a control rule, we mean a formula of the type*

$$A_1(x_1) \,\&\, \ldots \,\&\, A_n(x_n) \rightarrow B_1(u_1) \,\&\, \ldots \,\&\, B_l(u_l),$$

where A_i and B_j are natural language words (i.e., fuzzy sets).

Definition 8. *By a fuzzy controller $\mathcal{C}$, we mean a finite set of control rules. The number of rules in a fuzzy model will be denoted by N, and the j^{th} rule will be denoted by*

$$A_{j,1}(x_1) \,\&\, \ldots \,\&\, A_{j,n}(x_n) \rightarrow$$
$$B_{j,1}(u_1) \,\&\, \ldots \,\&\, B_{j,l}(u_l).$$

Definition 9. *Assume that $\mathcal{C}$ is a fuzzy controller, $f_\&$, $f_\vee$ are $\&-$ and $\vee-$operations, and D is a defuzzification procedure. By the fuzzy input-output function $f^{\mathcal{C}}(x, u)$ corresponding to the fuzzy controller $\mathcal{C}$, we mean a mapping which maps every pair of vectors $x = (x_1, \ldots, x_n)$ and $u = (u_1, \ldots, u_l)$ into a number*

$$f^{\mathcal{C}}(x, u) = f_\vee(p_1^{\mathcal{C}}(x, u), \ldots, p_N^{\mathcal{C}}(x, u)),$$

where

$$p_j^{\mathcal{C}}(x, u) = f_\&(r_j(x), b_j(u)),$$
$$r_j(x) = f_\&(\mu_{j,1}^A(x_1), \ldots, \mu_{j,n}^A(x_n)),$$
$$b_j(u) = f_\&(\mu_{j,1}^B(u_1), \ldots, \mu_{j,l}^B(u_l)).$$

Definition 9′. *Assume that $\mathcal{C}$ is a fuzzy controller, $f_\&$, $f_\vee$ are $\&-$ and $\vee-$operations, and D is a defuzzification procedure. By the crisp input-output function corresponding to the fuzzy controller $\mathcal{C}$, we mean a mapping which maps every vector x into a number $c^{\mathcal{C}}(x) = D(f^{\mathcal{C}}(x, u))$, where $f^{\mathcal{C}}$ is the fuzzy input-output function corresponding to $\mathcal{C}$.*

Definition 10. *Let (K, f) be a plant.*

We say that a state $x \in X$ is possible if $(x, u) \in K$ for some $u \in U$. The set of all possible states will be denoted by X^*.

By a desired trajectory, we mean a continuous function $d : X^* \rightarrow \dot{X}$ such that for every possible x, there exists a $u \in U$ for which $f(x, u) = d(x)$.

Let $\mathcal{M}$ be a fuzzy model consisting of N rules

$$A_{j,1}(x_1) \,\&\, \ldots \,\&\, A_{j,n}(x_n) \,\&$$
$$B_{j,1}(u_1) \,\&\, \ldots \,\&\, B_{j,l}(u_l) \rightarrow$$
$$C_{j,1}(\dot{x}_1) \,\&\, \ldots \,\&\, C_{j,n}(\dot{x}_n),$$

$(1 \leq j \leq N)$, and let $d(x)$ be a desired trajectory. Then, by the corresponding fuzzy controller, we mean a collection of N control rules

$$A_{j,1}(x_1) \,\&\, \ldots \,\&\, A_{j,n}(x_n) \,\&$$
$$C_{j,1}(d_1(x)) \,\&\, \ldots \,\&\, C_{j,n}(d_n(x)) \rightarrow$$
$$B_{j,1}(u_1) \,\&\, \ldots \,\&\, B_{j,l}(u_l).$$

Definition 11. *Let (K, f) be a plant, let $d(x)$ be a desired trajectory for this plant, and let $\delta > 0$ be a real number. We say that a fuzzy controller $\mathcal{C}$ δ-approximates the desired trajectory, if for all $x \in X^*$, we have*

$$\|f(x, c^{\mathcal{C}}(x)) - d(x)\| \leq \delta.$$

Theorem 2. *For every plant (K, f), for every desired trajectory $d(x)$, for every consistent defuzzification procedure D, and for every $\delta > 0$, if a fuzzy model $\mathcal{M}$ strongly δ-approximates a plant, then the corresponding fuzzy controller δ-approximates the desired trajectory.*

4 Auxiliary Result

The example explaining the need for consistent defuzzifications shows that the conclusion of Theorem 2 is not necessarily true if we consider general (possibly inconsistent) defuzzification procedures.

In that example, we had *two different* possible control to achieve a given goal. One can show that if we only consider goals for which there is *only one* control, then a similar results holds for arbitrary (not necessarily consistent) defuzzification procedures:

Definition 12. *Let (K, f) be a plant. We say that a desired trajectory $d(x)$ describes a unique control if for every $x \in X^*$, there is only one value u for which $f(x, u) = d(x)$.*

Theorem 3. *Let (K, f) be a plant, and let $d(x0$ be its desired trajectory which describes a unique control. Then, for every defuzzification procedure D, and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if a fuzzy model $\mathcal{M}$ strongly δ-approximates a plant, then the corresponding fuzzy controller ε-approximates the desired trajectory.*

5 Future Work: Extending These Results to Other Control Situations

In this paper, we analyzed the simplest control situations, in which the only goal is to follow a given trajectory. We have shown that for such situations, there is a natural strategy for fuzzy control, and that this strategy can be mathematically justified.

In practice, in addition to the control situation with the above follow-the-trajectory goal, there exist many different control situations, with different control objectives. In many such situations, there also have natural fuzzy heuristics for transforming the fuzzy model of a controlled system into a fuzzy controller; let us give a few examples.

1. One of the control goals may be saving the fuel (or, in more general terms, energy necessary for control).

In terms of control u, this requirement means that we would like, if possible, to make the value $\|u\|$ of the control u to be small. It is natural to interpret "small" as a fuzzy term, with the corresponding membership function $\mu_{\text{small}}(x)$, and to consider the resulting requirements "$\|u\|$ is small" as one of the conditions for control. Formally, this means that instead of the standard fuzzy input-output function $f^{\mathcal{C}}(x, u)$ of the fuzzy controller, we consider a new function $f^{\mathcal{C}}_{\text{new}}(x, u) = f_{\&}(f^{\mathcal{C}}(x, u), \mu_{\text{small}}(\|u\|))$.

2. Instead of minimizing the fuel, we can consider a more general control problem of minimizing a certain characteristic which depends on the state and on the control.

In terms of traditional control theory, we can consider the problem of minimizing the characteristic $\int \mathcal{A}(x(t), u(t))\,\mathrm{d}t$, for some function $\mathcal{A} : X \times U \to R$. In common-sense terms, we can express this requirement by saying that for all moment of time t, the quantity $\mathcal{A}(t, u)$ should be small. This new requirement can also be naturally described in fuzzy terms: namely, instead of the standard fuzzy input-output function $f^{\mathcal{C}}(x, u)$ of the fuzzy controller, we consider a new function

$$f^{\mathcal{C}}_{\text{new}}(x, u) = f_{\&}(f^{\mathcal{C}}(x, u), \mu_{\text{small}}(\mathcal{A}(x, u))).$$

3. In some control situations (e.g., in controlling a plane or a spaceship), we need to follow the given trajectory exactly, because any deviation may be dangerous. However, in some practical situations, deviations are permissible; we can use this non-uniqueness of the desired trajectory to satisfy some additional goals.

Let us give two examples of such situations.

3.1. In many control situations, e.g., in controlling the elevator or a train, we want the control to be *smooth*.

The common-sense meaning of the word "smooth" is that the state should not change too fast, i.e., that the rates $\dot{x}_i$ with which the state changes should be small. This smallness condition can also be naturally described in fuzzy terms, and it can be added to the rules which describe the given trajectory. To be more precise, we know that $\dot{x}$ should be approximately equal to $d_0(x)$, and that $\dot{x}$ should be small. In this case, we can consider the corresponding fuzzy rules "$\dot{x}$ is close to $d_0(x)$" and "$\dot{x}$ is small" as a fuzzy model. Then, as the actual desired trajectory $d(x)$ to be used in the fuzzy controller design, we can take a crisp input-output function of this fuzzy model. This function takes into consideration both the approximate trajectory and the requirement that the control be smooth.

3.2. In some control situations, we want the control to reach a certain state $x^{(0)}$ by a certain moment of time t_0. A natural way to describe this requirements in commonsense terms is as follows: if we are close to t_0 in time, then the corresponding state x should be close to $x^{(0)}$.

In terms of control, this requirement can be reformulated as follows: if a moment of time t is close to t_0, then the control should bring us closer to the state $x^{(0)}$, i.e., e.g., if we are far away from $x^{(0)}$, then the rate with which x approaches $x^{(0)}$, should be reasonably large. This rate can be expressed in terms of x

and $\dot{x}$ and thus, it can serve as an additional rule for determining the control $d(x)$ to be used in designing a fuzzy controller $\mathcal{C}$.

In test examples, these natural fuzzy heuristics seem to work; however, in contrast to the main heuristic for which we have provided a justification, we do not know of any precise mathematical justifications of these additional heuristics. Providing such justifications is, thus, an important open problem.

6 Ideas of the Proofs

The proof of Theorem 1 is similar to the proof of Theorem 2 from [3].

In order to prove Theorem 2, let us first prove the relationship between fuzzy input-output functions corresponding to the fuzzy model and to the corresponding fuzzy controller. Indeed, according to the definition of a fuzzy input-output function $f^{\mathcal{M}}(x,u,\dot{x})$ of a fuzzy model, we have

$$f^{\mathcal{M}}(x,u,\dot{x}) = f_{\vee}(p_1^{\mathcal{M}}(x,u,\vec{x}),\ldots,p_N^{\mathcal{M}}(x,u,\dot{x})),$$

where

$$p_j^{\mathcal{M}}(x,u,\dot{x}) = f_{\&}(\mu^A_{j,1}(x_1),\ldots,\mu^A_{j,n}(x_n),$$
$$\mu^B_{j,1}(u_1),\ldots,\mu^B_{j,l}(u_l),$$
$$\mu^C_{j,1}(\dot{x}_1),\ldots,\mu^C_{j,n}(\dot{x}_n)).$$

Similarly, when we apply, to the corresponding fuzzy controller $\mathcal{C}$, the definition of the fuzzy input-output function $f^{\mathcal{C}}(x,u)$ of a fuzzy controller, we conclude that $f^{\mathcal{C}}(x,u) = f_{\vee}(p_1^{\mathcal{C}}(x,u),\ldots,p_N^{\mathcal{C}}(x,u))$, where

$$p_j^{\mathcal{C}}(x,u) = f_{\&}(\mu^A_{j,1}(x_1),\ldots,\mu^A_{j,n}(x_n),$$
$$\mu^C_{j,1}(d_1(x)),\ldots,\mu^C_{j,n}(d_n(x)),$$
$$\mu^B_{j,1}(u_1),\ldots,\mu^B_{j,l}(u_l)).$$

Comparing the above expressions for $p_j^{\mathcal{M}}(x,u,\dot{x})$ and $p_j^{\mathcal{C}}(x,u)$, we conclude that for every j, we have

$$p_j^{\mathcal{C}}(x,u) = p_j^{\mathcal{M}}(x,u,d(x)),$$

and therefore, that

$$f^{\mathcal{C}}(x,u) = f^{\mathcal{M}}(x,u,d(x)).$$

Now, we are ready to prove the theorem. By the definition of a consistent defuzzification, we have $f^{\mathcal{C}}(x,c^{\mathcal{C}}(x)) > 0$. Due to the above relationship between the fuzzy input-output functions $f^{\mathcal{C}}(x,u)$ and $f^{\mathcal{M}}(x,u,d(x))$. this inequality implies that $f^{\mathcal{M}}(x,c^{\mathcal{C}}(x),d(x)) > 0$, i.e., that $f^{\mathcal{M}}(x,u,\dot{x}) > 0$ for $u = c^{\mathcal{C}}(x)$ and $\dot{x} = d(x)$.

We know that the fuzzy model $\mathcal{M}$ is a strong δ-approximation to the given plant (K,f). By definition of a strong δ-approximation, this means that if $f^{\mathcal{M}}(x,u,\dot{x}) > 0$, then $\|\dot{x} - f(x,u)\| \le \delta$. We have proven the inequality $f^{\mathcal{M}}(x,u,\dot{x}) > 0$ for $u = c^{\mathcal{C}}(x)$ and $\dot{x} = d(x)$; therefore, for these u and $\dot{x}$, we can conclude that $\|\dot{x} - f(x,u)\| = \|d(x) - f(x,c^{\mathcal{C}}(x))\| \le \delta$. This inequality is exactly what we mean by saying that the fuzzy controller $\mathcal{C}$ δ-approximates the desired trajectory. The theorem is proven.

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