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On the Relation Between Two Approaches to Combining Evidence: Ordered Abelian Groups and Uninorms

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Abstract

In this paper, we develop a relationship between two approaches to combining evidence: the Ordered Abelian Group (OAG) approach and the uninorm approach. We show that while there exist uninorms that are not extended OAG's it turns out that for operations which are continuous (in some reasonable sense), these two approaches coincide.

1 Introduction: The Problem of Combining Evidence

In the development of expert systems the problem of combining multiple pieces of evidence about a hypothesis H was recognized early as an important issue [1, 12]. Simply stated, the problem is as follows: If two independent pieces of evidence support the hypothesis H, A with degree a, and B with degree b, then what is our resulting degree of belief in H?

If these degrees a and b are the only information we have, then the resulting resulting degree of belief must depend only on these two values a and b. In mathematical terms, we can say that the resulting degree of belied must be a *function* of two real values a and b. In this paper, we will denote the corresponding function by $a * b$. Which function should we choose?

Several approaches have been proposed for solving this problem. In this paper, we show that two of these approaches:

- Ordered Abelian Groups (OAG) approach [5], and
- the (more recent) uninorm approach [13] –

largely coincide. The fact for an important continuous case, these two approaches – coming from different ideas – coincide, is a good argument in favor of the formulas obtained by using each of these approaches.

To put our result into a proper perspective, we first briefly describe the fuzzy logic approach to combining non-probabilistic uncertain evidence, the approach which was (most probably) historically the first, and which can be viewed as a predecessor of both OAG and uninorm approaches. Then, we mention a different approach presented in the first successful expert system MYCIN, and how this approach was generalized to OAG. After that, we describe the uninorm approach, and formulate and prove our main results.

2 Combining Evidence: Fuzzy Logic Approach

From the logical viewpoint, in our case, H is true if and only if either support from A is true *or* support from B is true. Thus, as the resulting degree of belief $a * b$ in H, we can take the value $f_\vee(a, b)$, where $f_\vee(x, y)$ is a *t-conorm*, an extension of a 2-valued logical "or" operation to $[0, 1]$-valued fuzzy logic (see, e.g., [10, 11]). T-conorms are defined as follows:

Definition 1. *A function $f_\vee : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-conorm if it satisfies the following four conditions for all x, y, z, and t:*

1. $f_\vee(0, x) = x$;
2. $f_\vee(x, y) = f_\vee(y, x)$ *(commutativity);*
3. $f_\vee(x, f_\vee(y, z)) = f_\vee(f_\vee(x, y), z))$ *(associativity);*
4. *if* $x \le z$ *and* $y \le t$*, then* $f_\vee(x, y) \le f_\vee(z, t)$ *(monotonicity).*

T-conorms have been completely classified. In particular, it is known that t-conorms from an important class called "strictly Archimedean" t-conorms are

isomorphic to addition $+$ on the set R^+ of all non-negative real numbers extended by $+\infty$, i.e., there exists a 1-1 monotonic mapping $\varphi : [0, 1] \to R^+$ such that for every x and y,

$$x * y = \varphi^{-1}(\varphi(x) + \varphi(y)). \tag{1}$$

This formula means that for every two inputs x and y, we can compute the value $x * y$ as follows:

- first, we map both values x and y into the set R^+ by computing $x' = \varphi(x)$ and $y' = \varphi(y)$;
- second, we add the values x' and y', thus getting $z' = x' + y' \in R^+$;
- finally, we map the value z' back into the interval $[0, 1]$ by applying the inverse function φ^{-1}: $z = x * y = \varphi^{-1}(z')$.

In particular, to get the "algebraic sum" $x * y = x + y - x * y$, we can use $\varphi(x) = -\ln(1 - x)$, for which $\varphi^{-1}(x) = 1 - e^{-x}$.

3 Combining Evidence: MYCIN Approach

T-conorms are not always an adequate way of describing the evidence combination. Indeed, according to the properties of a t-conorm, we always have $f_\vee(a, b) \ge a$ and $f_\vee(a, b) \ge b$; as a result, the degree of belief in H coming from two supporting pieces of evidence is larger (or the same) as the degree of belief in H coming from just one piece of evidence. This conclusion makes sense if we only allow *supporting* evidence, but sometimes, we encounter evidence which supports the *negation* $\neg H$ to the hypothesis. Intuitively, if we have two difference pieces of negative evidence, then we should *decrease* our degree of the belief in the hypothesis, i.e., we should have $a * b < a$; on the other hand, if we choose a t-conorm, we get $a * b \ge a$. Thus, for combining pieces of evidence, we need operations which are more general than t-conorms.

First such operations were used already in the historically first expert system MYCIN (see, e.g., [1, 12]). In this system, degrees of belief take values from the interval $[-1, 1]$ instead of the more traditional $[0, 1]$. Here, negative values describe negative evidence, and positive values describe positive evidence. As we move from -1 to 1, we go from the evidence which absolutely 100% supports the negation $\neg H$ to evidence which slightly supports $\neg H$ to evidence which slightly supports H to evidence which absolutely 100% supports H. The combination operation is defined as follows:

$$x * y = \frac{x + y}{1 + x \cdot y}. \tag{2}$$

For this operation, two pieces of positive evidence increase our degree of belief ($x * y > x, y$ if $x, y > 0$), while two pieces of negative evidence decrease our degree of belief ($x * y < x, y$ if $x, y < 0$).

MYCIN's combination operation is defined for all possible pairs (x, y), with one exception of a pair $(-1, 1)$, for which the above formula is not defined. This exception, however, makes perfect sense: the situation when $a = 1$ and $b = -1$ means that we have two pieces of evidence A and B such that A leads to our 100% degree of belief in H, while B leads to a 100% belief in $\neg H$, i.e., in a *rejection* of the hypothesis. In this case, we clearly have a contradiction, both degrees cannot be true, so, instead of trying the combine the two inconsistent pieces of evidence, we should try to analyze and correct the inconsistent degrees of belief a and b.

The MYCIN operation can be easily re-defined on the interval $[0, 1]$ if we map $[-1, 1]$ onto $[0, 1]$ by a mapping $x \to (x+1)/2$. To find $x*y$ for $x, y \in [0, 1]$, we first find the corresponding values $x' = 2x - 1$ and $y' = 2y - 1$ in the interval $[-1, 1]$, then combine x' and y' according to the original MYCIN rule, getting $z' = (x' + y')/(1 + x' \cdot y')$, and then find $z = x * y$ by mapping the resulting value $z' \in [-1, 1]$ back into the interval $[0, 1]$: $z = (z' + 1)/2$. As a result, we get the following operation on the interval $[0, 1]$:

$$x * y = \frac{x \cdot y}{2x \cdot y - x - y} = \frac{x \cdot y}{x \cdot y + (1 - x) \cdot (1 - y)}. \tag{3}$$

4 Combining Evidence: Hájek's OAG Approach

In the early 1980s, Hájek *et al.* generalized MYCIN's operations by describing the corresponding combination operator in general algebraic terms, as an *ordered Abelian group* (OAG) (see, e.g., [5, 6, 7, 8]). Let us give the corresponding definitions.

Definition 2.

- *A set S with a binary operation $*$ is called an Abelian semigroup with zero if this operation is commutative, associative, and has a fixed element g such that $g * x = x$ for all $x \in S$.*

- *An Abelian semigroup with zero is called an Abelian group if every element $x \in S$ has an inverse element y, i.e., an element for which $x * y = g$.*

- *If $\le$ is a linear (total) ordering of the set S, then the Abelian group is called ordered if for every x, y, and z, $x \le y$ implies $x * z \le y * z$.*

For example, for MYCIN's operation, the open interval $(-1, 1)$ is an ordered Abelian group (OAG). To describe the full interval, we must add two new elements ("endpoints") $\perp$ and $\top$, and extend the ordering and the operations to the new set $\{\perp\} \cup S \cup \{\top\}$ as follows: $\perp \le x \le \top$ for all $x \in S$, $\perp \le \top$,

$\perp * x = x * \perp = \perp$ for all $x \neq \top$, and $\top * x = x * \top = \top$ for all $x \neq \perp$. This extension $\widehat{S}$ is called an *extended OAG*.

Such structures have been completely classified: It is known that an arbitrary extended OAG is isomorphic to addition $+$ on an extended real line

$$\widehat{R} = \{-\infty\} \cup R \cup \{+\infty\}$$

in the sense that there exists a 1-1 monotonic mapping $\varphi : \widehat{S} \to \widehat{R}$ such that for every x and y,

$$x * y = \varphi^{-1}(\varphi(x) + \varphi(y)). \tag{4}$$

In particular, to get the original MYCIN operation, we can take

$$\varphi(x) = \frac{1}{2} \cdot \ln\left(\frac{1+x}{1-x}\right), \tag{5}$$

with

$$\varphi^{-1}(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \tag{6}$$

For the re-scaled MYCIN-like operation (3), we can take

$$\varphi(x) = -\frac{1}{2} \cdot \ln\left(\frac{1}{x-1}\right),$$

with

$$\varphi^{-1}(x) = \frac{1}{1+e^{-2x}}$$

(which, incidentally, is the activation function typically used in neural networks.)

Notice that the formula (4) is the same as for the strictly Archimedean t-conorms, but there are two differences:

- First, for extended OAG, we need both positive and negative real numbers, while for t-conorms, we only used non-negative real numbers.

- Second, for extended OAG, this formula is always true, while for t-conorms it is only true in a special case (of a strictly Archimedean operation).

5 Combining Evidence: Uninorm Approach

More recently, a new approach to describing combination operations has been proposed in [14] under the name of a *uninorm*. Instead of following a more mathematical path of generalizing the algebraic properties of MYCIN operation, the authors of [14] followed a more foundational path of looking into which part of the standard definition of a t-conorm can be weakened in such a way that it allows MYCIN-type operations as well as usual t-conorms.

Among the conditions from Definition 1, the first condition seems to be the most eligible for changing: this condition makes sense if 0 corresponds to the absence of confirmation, but now 0 stands for the largest negative confirmation, so we have to reformulate this condition by using some value $g \in [0,1]$ which does represent neither positive nor negative confirmation. As a result, we arrive at the following definition:

Definition 3. *A function $f : [0,1] \times [0,1] \to [0,1]$ is called a uninorm if there exists a number $g \in [0,1]$ for which the following four conditions are satisfied for all x, y, z, and t:*

1. $f(g, x) = x$;
2. $f(x, y) = f(y, x)$ *(commutativity);*
3. $f(x, f(y, z)) = f(f(x, y), z))$ *(associativity);*
4. *if $x \le z$ and $y \le t$, then $f(x, y) \le f(z, t)$ (monotonicity).*

In particular, for $g = 0$, we get a standard definition of a t-conorm. (It is worth mentioning that for $g = 1$, we get a definition of a t-norm.) New examples of uninorms correspond to $g \in (0, 1)$.

The structure of uniforms has been studies extensively in [4], where it was proven, among other things, that if $g \in (0, 1)$, then a uninorm cannot be continuous on the whole unit square.

6 Known Relationship Between OAG and Uninorm Approaches

Uninorms have been studied in [2, 3, 4, 13]. It turned out that:

- every extended OAG is a uninorm, but
- the class of uninorms is larger than the class of extended OAG's.

For example, the following operation from [4] is a uninorm but not an extended OAG: $f(x, y) = \max(x, y)$ when $x > 0.5$ or $y > 0.5$, and $f(x, y) = \min(x, y)$ if $x \le 0.5$ and $y \le 0.5$. In this example, $g = 0.5$.

7 New Result: Motivation

The above example can be further generalized: we can take $f(x, y)$ equal to some t-conorm for $x, y \ge g$, to some t-norm for $x, y \le g$, and to either $\min(x, y)$ or $\max(x, y)$ for all other x, y.

Extended OAG are discontinuous for $f(0, 1)$ and $f(1, 0)$. All known examples of uninorms that are not extended OAGs have one common property: they are

discontinuous not only for $f(0,1)$ and $f(1,0)$, but also for some other values. For example, the above example is discontinuous when $x < 0.5$ and $y = 0.5$: then, for small $\varepsilon \to 0$, $f(x,0.5) = \min(x,0.5) = x$, while $f(x,0.5+\varepsilon) = \max(x,0.5+\varepsilon) = 0.5+\varepsilon \to 0.5 \neq x$.

In this paper, we show that this additional discontinuity is a general feature of all such examples: namely, we prove that every "maximally continuous" uninorm is an extended OAG.

8 New Result: Formulation

Definition 4. *We say that a uninorm is maximally continuous if it is continuous at all the points (x,y) except for the points $(0,1)$ and $(1,0)$.*

Proposition. *Every maximally continuous uninorm with $g \in (0,1)$ is an extended OAG.*

Comment. From this Proposition, one can easily deduce the result from [4] that a uninorm with $g \in (0,1)$ cannot be continuous for all x and y. Indeed, if a uninorm is continuous everywhere, then by the Proposition, it is an extended OAG, thus defined by the formula (4), but this formula is not continuous at the point (0,1) – a contradiction which shows that everywhere continuous uninorms are impossible for $g \in (0,1)$.

Corollary. *An arbitrary maximally continuous uninorm with $g \in (0,1)$ is isomorphic to addition $+$ on an extended real line $\widehat{R} = \{-\infty\} \cup R \cup \{+\infty\}$ in the sense that there exists a 1-1 monotonic mapping $\varphi : [0,1] \to \widehat{R}$ such that for every x and y,*

$$f(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y)). \tag{7}$$

Comment. This corollary generalizes a result from [9], where a similar statement was proven under the additional condition that the uninorm is strictly monotonic.

9 New Result: Proofs

1°. Let us first show that for the operation $x*y = f(x,y)$ every number $x \in (0,1)$ has an inverse element $y \in (0,1)$ for which $x*y = g$. To prove the existence of this y, we will consider three possible cases: $x = g$, $x < g$, and $x > g$.

1.1°. When $x = g$, we can take $y = g \in (0,1)$, then $x*y = g*g = g$ by the definition of a uninorm.

1.2°. Let us now consider the case when $x < g$. In this case, due to property *1.*, we have $g*x = x$. Due to $x < g$, we conclude that $g*x < g$.

Let us now show, by reduction to a contradiction, that $1 * x \geq g$. Indeed, assume that $1 * x < g$. By *1.*, we have $1 * g = 1$, so by monotonicity, we have $1 * 1 \geq 1 * g = 1$, hence $1 * 1 = 1$. Since $1 * x < g$ and $1 * 1 = 1 > g$, and the function $f(1, y) = 1 * y$ is continuous for $y > 0$, we can apply the Intermediate Value Theorem and conclude that $1 * t = g$ for some $t \in (x, 1)$. Then, by associativity, we must have $t * (1 * 1) = (t * 1) * 1$, but:

- we already know that $1 * 1 = 1$, hence $t * (1 * 1) = t * 1 = g$ (by our choice of t);
- on the other hand, by our choice of t, we have $t * 1 = g$, hence $(t * 1) * 1 = g * 1$, and by the property *1.*, we have $g * 1 = 1$.

So, $t * (1 * 1) = g \neq (t * 1) * 1 = 1$. This contradiction shows that our assumption $1 * x < g$ is impossible and thus, $1 * x \geq g$.

So, $g * x < g$, $1 * x \geq g$, and since the function $y \to f(y, x)$ is continuous for all y, we can apply the Intermediate Value Theorem and conclude that $y * x = g$ for some $y \in (g, 1]$.

If $y = 1$, then we would have $1 * x = g$, which, as we have already shown, leads to a contradiction. Thus, $y \in (g, 1) \subset (0, 1)$.

1.3°. Finally, let us consider the remaining case when $x > g$. In this case, due to property *1.*, we have $g * x = x$. Due to $x > g$, we conclude that $g * x > g$.

Let us now show, by reduction to a contradiction, that $0 * x \leq g$. Indeed, assume that $0 * x > g$. By *1.*, we have $0 * g = 0$, so by monotonicity, we have $0 * 0 \leq 0 * g = 0$, hence $0 * 0 = 0$. Since $0 * x > g$ and $0 * 0 = 0 < g$, and the function $f(0, y) = 1 * y$ is continuous for $y < 1$, we can apply the Intermediate Value Theorem and conclude that $0 * t = g$ for some $t \in (0, x)$. Then, by associativity, we must have $t * (0 * 0) = (t * 0) * 0$, but:

- we already know that $0 * 0 = 0$, hence $t * (0 * 0) = t * 0 = g$ (by our choice of t);
- on the other hand, by our choice of t, we have $t * 0 = g$, hence $(t * 0) * 0 = g * 0$, and by the property *1.*, we have $g * 0 = 0$.

So, $t * (0 * 0) = g \neq (t * 0) * 0 = 0$. This contradiction shows that our assumption $0 * x > g$ is impossible and thus, $0 * x \leq g$.

So, $0 * x \leq g$, $g * x > g$, and since the function $y \to f(y, x)$ is continuous for all y, we can apply the Intermediate Value Theorem and conclude that $y * x = g$ for some $y \in [0, g)$.

If $y = 0$, then we would have $0 * x = g$, which, as we have already shown, leads to a contradiction. Thus, $y \in (0, g) \subset (0, 1)$.

The existence of the inverse element is now proven for all $x \in (0, 1)$.

2°. Let us now show, by reduction to a contradiction, that if $x \in (0, 1)$ and $y \in (0, 1)$, then $x * y \in (0, 1)$. To make this conclusion, we need to prove that the values $x * y = 0$ and $x * y = 1$ are impossible.

2.1°. If $x * y = 0$, then, due to monotonicity, $x * 0 \leq x * y = 0$ and $x * 0 = 0$. By 2°, there exists an element $x^{-1} \in (0, 1)$ which is inverse to x, i.e., for which $x^{-1} * x = g$. Due to associativity, $x^{-1} * 0 = x^{-1} * (x * y) = (x^{-1} * x) * y = g * y = y$ and at the same time, $x^{-1} * 0 = x^{-1} * (x * 0) = (x^{-1} * x) * 0 = g * 0 = 0$. Thus, $y = 0$, which contradicts to our assumption that $y \in (0, 1)$. This contradiction shows that $x * y \neq 0$.

2.2°. If $x * y = 1$, then, due to monotonicity, $x * 1 \geq x * y = 1$ and $x * 1 = 1$. By 2°, there exists an element $x^{-1} \in (0, 1)$ which is inverse to x, i.e., for which $x^{-1} * x = g$. Due to associativity, $x^{-1} * 1 = x^{-1} * (x * y) = (x^{-1} * x) * y = g * y = y$ and at the same time, $x^{-1} * 1 = x^{-1} * (x * 1) = (x^{-1} * x) * 1 = g * 1 = 1$. Thus, $y = 1$, which contradicts to our assumption that $y \in (0, 1)$. This contradiction shows that $x * y \neq 1$.

Thus, $x * y \neq 0$ and $x * y \neq 1$, so $x * y \in (0, 1)$.

3°. Thus, the operation $*$ is defined on the set $(0, 1)$, it is commutative, associative, and monotonic on this set, and every element has an inverse. Hence, it is an OAG.

It is known that every OAG has the form $x * y = \varphi^{-1}(\varphi(x) + \varphi(y))$ for an appropriate function φ. From this representation, we can conclude that for $x \in (0, 1)$, $\varepsilon * x \to 0$ as $\varepsilon \to 0$, hence due to continuity, $0 * x = 0$ for all $x \in (0, 1)$. Due to monotonicity, we also get $0 * 0 = 0$. Similarly, we get $1 * x = 1$ for all $x > 0$. Thus, $*$ is indeed an extended OAG. The proposition is proven.

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