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On Approximation of Fuzzy Sets by Crisp Sets: From Continuous Control-Oriented Defuzzification To Discrete Decision Making

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Abstract

In this paper, we show that the necessity to make crisp decisions in uncertain (fuzzy) situations leads to the necessity to "approximate" fuzzy sets by crisp sets. We show that seemingly natural approximation ideas – such as using α-cut for a given α – often do not work, and we describe new approximations which not only work, but which are optimal in some reasonable sense.

Keywords: fuzzy sets, decision making, "shadowed" fuzzy sets, approximating fuzzy sets by crisp sets

1 Why There Is a Practical Need To "Approximate" a Fuzzy Set By a Crisp One

1.1 Fuzzy Methods Usually Result in a Fuzzy Recommendation

Fuzzy logic and fuzzy set theory enable us to use experts' uncertain ("fuzzy") knowledge in decision making. The corresponding methods usually generalize known methods of decision making which are based on the crisp (non-fuzzy) knowledge about the environment. These crisp methods enable us to come up with a crisp decision: e.g., whether we should build a plant or not. When we generalize these methods to fuzzy knowledge, as a result, we usually get not a crisp decision, but rather a *fuzzy* decision: e.g., a decision may be "most probably it is better to build a plant".

Formally, in the simplest ("yes"-"no") decision situation, a crisp decision procedure means that for every input x, we decide whether to choose a positive alternative A^+ or a negative alternative A^-. Such a crisp decision can be described by a set S of all the inputs x for which the decision is positive, or, alternatively, by the *characteristic function* $\chi(x)$ of this set, i.e., by a function for which $\chi(x) = 1$ if $x \in S$ and $\chi(x) = 0$ if $x \notin S$.

Similarly, as a result of a fuzzy decision making procedure, for every input x, we generate a *degree* $\mu(x) \in [0, 1]$ to which choosing a positive alternative A^+ is reasonable. Thus, a fuzzy decision making procedure produces a *fuzzy subset* of the set of all inputs, a subset which is characterized by the membership function $\mu(x)$.

1.2 In Many Practical Application, the Fuzzy Recommendation is All We Need

In many practical situations (e.g., in medicine), the main objective of the fuzzy decision making procedure is to help a human decision maker (e.g., a medical doctor). The system presents the value $\mu(x)$ to the human decision maker, and she takes this value into account when making a decision.

1.3 In Automated Decision Making, We Need a Crisp Decision (Which Best Approximates the Fuzzy Recommendation)

However, in some practical problems, there is no time for a human operator to make a decision: e.g., when we plan a robotic mission to a distant planet, the robot must make urgent decisions for which we cannot

wait minutes and hours for the signal to travel to Earth and back. In such situations, we must, for every possible input x, either choose a positive alternative, or choose a negative one. In other words, we must end up with a crisp set. This crisp set should reflect – as close as possible – the fuzzy set produced by the fuzzy decision making system.

Thus, we face a problem of finding, for every fuzzy set $\mu(x)$, a crisp set $\chi(x)$ which is, in some reasonable set, the best approximation for $\mu(x)$.

2 The Best Approximation Should Be an α-Cut

For each input x, the only information that we have about the quality of a positive alternative is the degree $\mu(x)$. Thus, whether we select the positive alternative A^+ or not depends only on the value of $\mu(x)$.

When $\mu(x) = 1$, we select the positive alternative A^+; when $\mu(x) = 0$, we select the negative alternative A^-. If for some alternative x_0, we decided to select the positive alternative A^+, then for every alternative x for which $\mu(x) \geq \mu(x_0)$, it is even more reasonable to select A^+. Similarly, if for some x_0, we selected A^-, then for all alternatives x with $\mu(x) \leq \mu(x_0)$, we should also select A^-. We can describe the corresponding decision making if we use the greatest lower bound α of the set of all values $\mu(x)$ for which we decided on A^+:

- for all inputs x for which $\mu(x) > \alpha$, we select A^+ (i.e., $\chi(x) = 1$); and
- for all inputs x for which $\mu(x) < \alpha$, we select A^- (i.e., $\chi(x) = 0$).

For the values for which $\mu(x) = \alpha$, we may have $\chi(x) = 1$ or $\chi(x) = 0$. In other words, the crisp set corresponding to the characteristic function $\chi(x)$ is an α-*cut* of the original fuzzy set.

Thus, the above-described informal approximation problem can be described as follows: given a fuzzy set $\mu(x)$, find α for which its α-cut is, in some reasonable set, "the closest" to the original fuzzy set.

3 The Simplest Approach – Using the Same α For All Fuzzy Sets – And Its Drawbacks

In view of the above reformulation of the problem of automatic decision making as a problem of choosing an α-cut it may seem reasonable to just select a certain degree α (e.g., $\alpha = 0.5$ or $\alpha = 0.9$) and use this same value of α for all fuzzy sets $\mu(x)$.

The main drawback of this approach is that it contradicts to the intuitive idea that if two fuzzy sets are close, then the corresponding crisp decision sets should also be close. Indeed, for each $\beta \in [0, 1]$, let us consider a membership function $\mu_\beta(x)$ for which $\mu_\beta(x) = 0$ for $x \leq -1$, $\mu_\beta(x) = \beta$ for all $x \in [-0.9, 0.9]$, $\mu_\beta(x) = 1$ for $x \geq 1$, and $\mu_\beta(x)$ is linear on $[-1, -0.9]$ and on $[0.9, 1]$.

Then, if we pick a value α and select A^+ (i.e., $\chi(x) = 1$) when $\mu(x) \geq \alpha$, then for a sequence of membership functions $\mu_{\alpha-1/N}(x)$, we have $\chi_N(x) = 0$ for all $x \in [-0.9, 0.9]$, but for the limit membership function $\mu_\alpha(x)$, we have $\chi(x) = 1$ for all $x \in [-0.9, 0.9]$. Thus, for large N, the fuzzy sets $\mu_{\alpha-1/N}(x)$ and $\mu_\alpha(x)$ are close (as close as possible), but the resulting crisp functions are as far away as possible: for $x \in [-0.9.0.9]$, we have $\chi_N(x) \equiv 0$ while $\chi(x) \equiv 1$.

Similarly, if we pick a value α and select A^+ (i.e., $\chi(x) = 1$) when $\mu(x) > \alpha$, then for a sequence of membership functions $\mu_{\alpha+1/N}(x)$, we have $\chi_N(x) = 1$ for all $x \in [-0.9, 0.9]$, but for the limit membership function $\mu_\alpha(x)$, we have $\chi(x) = 0$ for all $x \in [-0.9, 0.9]$. Thus, for large N, the fuzzy sets $\mu_{\alpha+1/N}(x)$ and $\mu_\alpha(x)$ are close (as close as possible), but the resulting crisp functions are as far away as possible: for $x \in [-0.9.0.9]$, we have $\chi_N(x) \equiv 1$ while $\chi(x) \equiv 0$.

In other words:

- intuitively, for the transition from $\mu(x)$ to $\chi(x)$, close functions must turn into close ones, i.e., this transition must be *continuous* in some reasonable sense, but
- if we fix the same α for all fuzzy sets $\mu(x)$, we cannot have this continuity.

Due to this continuity requirement, we cannot choose the same α for all fuzzy sets $\mu(x)$; we must select α based on $\mu(x)$.

4 A Similar Problem Occurs In Fuzzy Control, Where It Is Solved By Using Defuzzification

A similar problem of transforming the fuzzy recommendation into a crisp decision exists in another application area of fuzzy logic and fuzzy set theory: namely, in fuzzy control. Namely, in fuzzy control, for every input x and for every possible control u, we generate the degree $\mu(x, u)$ to which this particular control value u is reasonable for a given input. Then, to generate the actual control $\bar{u}$ which the automated fuzzy controller will apply for a given input x, we use a *defuzzification* procedure, e.g., a *centroid* defuzzification

$$\bar{u}(x) = \frac{\int u \cdot \mu(x, u)\,\mathrm{d}u}{\int \mu(x, u)\,\mathrm{d}u.} \tag{1}$$

5 Why Cannot We Use Known Defuzzification Procedures In Fuzzy Decision Making?

5.1 Centroid Defuzzification Cannot Be Used for Discrete Decision Making

At first glance, it may seem that we can apply the above formula (1) to decision making as well. This impression is, however, erroneous:

- in control, the set of possible decision u is continuous, so averaging described by the formula (1) makes sense;
- on the other hand, in decision making, we only have two possible decisions u: 0 or 1, A^- or A^+; in this case, averaging leads as described by the formula (1) leads to a value intermediate between 0 and 1, and we still face the same problem: how to transform this fuzzy recommendation into a crisp decision.

5.2 A Similar Problem Occurs in Fuzzy Control

This problem is not exclusive for fuzzy decision making, it also sometimes occurs in fuzzy control as well. For example, J. Yen and his collaborators have considered a reasonable situation in which a car is going towards an obstacle on an empty road [4, 5, 6]. To avoid this obstacle, it should either swerve to the left, or to the right. This swerve can be described by a turning angle θ. Due to the symmetry of of the situation, the resulting membership function is symmetric with respect to changing θ to $-\theta$ and, hence, the formula (1) leads to $\bar{\theta} = 0$. In other words, the car should go smack into the obstacle. This crisp recommendation makes no sense, so J. Yen and his coauthors described modifications of the formula (1) which enable us to avoid this counterintuitive recommendation.

5.3 In Discrete Decision Making, the Problem Is Even More Serious Than in Fuzzy Control

In fuzzy control, this problem is rather rare (actually, it was unnoticed for the first decade of fuzzy control), so it is OK to either ignore it, or use some hacked *ad hoc* tools to solve it. In contrast, in fuzzy decision making, this problem is always present, so we better be able to solve it in the best possible manner.

6 How We Can Solve This Problem

A natural way to find the best crisp approximation to a fuzzy set is to formalize the notion of "closeness" ("metric") between two fuzzy sets, and then to select a crisp set which is the closest (in this sense) to the given fuzzy set $\mu(x)$.

What is the natural metric on the set of all fuzzy sets?

7 A Natural Metric on the Class of All Fuzzy Sets: First Idea

7.1 From an Intuitive Idea to a Formula for a Distance Between Two Fuzzy Sets

What does the value $\mu(x)$ mean? One way to assign a numerical value $\mu(x)$ to the degree to which x has a certain property is to poll several experts; then, for every x, we can define $\mu(x)$ as the ratio $\mu(x) = N_\mu(x)/N$, where:

- N is the total number of experts whom we polled, and
- $N_\mu(x)$ is the total number of experts who believe that the given value x satisfies the given property.

To get the entire membership function, we should ask the experts about several values $x_1 < \ldots < x_n$.

With this procedure in mind, it is natural to characterize the difference between two fuzzy sets $\mu(x)$ and $\mu'(x)$ by the total number of experts who disagree on these two sets. For each value x_i, the difference between the values $\mu(x_i)$ and $\mu'(x_i)$ means that different number of experts believe that the corresponding properties are true for x_i: these numbers are $N_\mu(x_i) = N \cdot \mu(x_i)$ and $N_{\mu'}(x_i) = N \cdot \mu'(x_i)$. Thus, the number of experts who disagree on this value x_i is equal to at least $|N_\mu(x_i) - N_{\mu'}(x_i)| = N \cdot |\mu(x_i) - \mu'(x_i)|$. The total number of disagreements can be thus estimated as the sum over all x_i, i.e., as

$$D(\mu, \mu') = \sum_{i=1}^{n} N \cdot |\mu(x_i) - \mu'(x_i)| =$$

$$N \cdot \sum_{i=1}^{n} |\mu(x_i) - \mu'(x_i)|. \tag{2}$$

The more values x_i we take, the more accurate the resulting description of the membership functions. When the values x_i get close, the sum (2) becomes close to the corresponding integral sum:

$$\int |\mu(x) - \mu'(x)| \,\mathrm{d}x \approx \sum_{i=1}^{n} |\mu(x_i) - \mu'(x_i)| \cdot \Delta x.$$

Thus,

$$\sum_{i=1}^{n} |\mu(x_i) - \mu'(x_i)| \cdot \Delta x \approx$$

$$\frac{1}{\Delta x} \cdot \int |\mu(x) - \mu'(x)|\, \mathrm{d}x,$$

and

$$D(\mu, \mu') \approx \frac{N}{\Delta x} \cdot \int |\mu(x) - \mu'(x)|\, \mathrm{d}x;$$

the smaller Δx, the closer the left- and right-hand sides.

7.2 Resulting Formula for a Distance Between Two Fuzzy Sets

We are not interested in the absolute values of the distances $D(\mu, \mu')$, only in which pairs are closer and which are more distant. Thus, for our purposes, it is sufficient to multiply all the values $D(\mu, \mu')$ by a constant $\Delta x / N$ and consider the scaled distance

$$d(\mu, \mu') = \int |\mu(x) - \mu'(x)|\, \mathrm{d}x. \qquad (3)$$

7.3 The Approximation Problem Formulated in Terms of This Distance

For this distance (3), we can formulate the above problem: given the fuzzy set $\mu(x)$, find the crisp set $\chi(x)$ (for which $\chi(x) \in \{0, 1\}$ for all x) for which

$$d(\mu, \chi) = \int |\mu(x) - \chi(x)|\, \mathrm{d}x \to \min.$$

7.4 Solving the Corresponding Approximation Problem

This optimization problem is easy to solve: namely, the minimized integral is, in effect, the sum

$$\sum_{i=1}^{n} |\mu(x_i) - \chi(x_i)| \cdot \Delta x \to \min_{\chi(x_1), \ldots, \chi(x_n)}.$$

In this sum, we have n unknowns $\chi(x_1), \ldots, \chi(x_n)$, and each term depends only on one of these n unknowns. Thus, the sum takes the smallest possible value when each of these terms takes the smallest possible value, i.e., when for each x_i, we select, as $\chi(x_i)$, the value 0 or 1 which is the closest to $\mu(x_i)$.

One can easily see that when $\mu(x) > 0.5$, this closest value is $\chi(x) = 1$, and when $\mu(x) < 0.5$, this closest value is $\chi(x) = 0$. Thus, as a solution to the above optimization problem, we get an α-cut with the same value $\alpha = 0.5$ for all fuzzy sets $\mu(x)$.

7.5 Drawbacks of This Solution

We have already observed that this is not a very intuitive choice. Thus, if we want to avoid this choice, we cannot use the distance (3), we must use some a sophisticated formula.

8 A Natural Metric on the Class of All Fuzzy Sets: Second Idea

8.1 New Idea

By definition, the distance between the two sets is equal to 0 when these sets are equal, and grows larger and larger as the two sets become different. Thus, the distance can be viewed as a degree of difference (inequality) between the two fuzzy sets.

In the above text, we tried to directly formalize this notion of inequality, and we got a counterintuitive result.

Let us now, instead, try to formalize inequality indirectly. It is known that the two sets S and S' are equal if and only if $S \subseteq S'$ and $S' \subseteq S$. Correspondingly, the two sets S and S' are different if and only if either $S \not\subseteq S'$ or $S' \not\subseteq S$. Thus, instead of trying to directly formalize the degree to which two fuzzy sets S and S' are different, let us first formalize the degrees to which $S \not\subseteq S'$ or $S' \not\subseteq S$, and then use an "or"-operation (e.g., $\max$) to combine these two degrees into a degree with which $S \neq S'$.

8.2 From Idea to Formula

To characterize the degree to which $S \not\subseteq S'$, we will use the same polling idea as above. Namely, when $S \subseteq S'$, this means that for every x_i, the number $N_\mu(x_i)$ of experts believe that x_i has a property S cannot be larger than the number of experts $N_{\mu'}(x_i)$ who believe that x_i has the property S'. Thus, if $N_\mu(x_i) \le N_{\mu'}(x_i)$, then the expert opinions about this value x_i are consistent with the hypothesis that $S \subseteq S'$. If $N_\mu(x_i) > N_{\mu'}(x_i)$, then clearly some experts disagree that $S \subseteq S'$. The number of experts who contribute to the belief that the property $S \not\subseteq S'$ is violated for a given x_i is equal to

$$\max(0, N_\mu(x_i) - N_{\mu'}(x_i)) =$$

$$N \cdot \max(0, \mu(x_i) - \mu'(x_i)).$$

The total number of experts' opinions which contribute to the belief that $S \not\subseteq S'$ is therefore equal to

$$N \cdot \sum_{i=1}^{n} \max(0, \mu(x_i) - \mu'(x_i)).$$

Similarly to the above text, we can conclude that the degree to which $S \not\subseteq S'$ can be characterized by an integral

$$d_{\subseteq}(\mu, \mu') = \int \max(0, \max(\mu(x) - \mu'(x))\, \mathrm{d}x. \quad (4a)$$

Similarly,

$$d_{\supseteq}(\mu, \mu') = \int \max(0, \max(\mu'(x) - \mu(x))\, \mathrm{d}x, \quad (4b)$$

and therefore, the degree $d(\mu,\mu')$ with which two fuzzy sets are different can be characterized as

$$d(\mu,\mu') = \max(d_{\subseteq}(\mu,\mu'), d_{\supseteq}(\mu,\mu')). \quad (4c)$$

8.3 This Formula Can Be Simplified

We are interested in the values $d(\mu,\mu')$ for the case when $\mu' = \chi$ is a characteristic function of a crisp set S, i.e., when for every x, either $\chi(x) = 0$ or $\chi(x) = 1$. In this case, both maximized expressions (4a), (4b) in the formula (4c) can be drastically simplified. Let us start with the first one.

- When $x \in S$, i.e., when $\mu'(x) = \chi(x) = 1$, we have $\mu(x) - \mu'(x) = \mu(x) - 1 \le 0$, and therefore, $\max(0, \max(\mu(x) - \mu'(x)) = 0$. Thus, in computing the first integral (4a), it is sufficient to consider only the values $x \notin S$, i.e., the values x from the complement CS to the set S.
- When $x \notin S$, i.e., when $\chi(x) = \mu'(x) = 0$, then we have $\mu(x) - \mu'(x) = \mu(x)$.

Therefore, the first integral (4a) from the formula (4c) takes the form

$$\int_{CS} \mu(x)\,\mathrm{d}x.$$

Similarly, the second integral (4b) takes the form

$$\int_S (1 - \mu(x))\,\mathrm{d}x.$$

8.4 Resulting Formulation of the Approximation Problem

Thus, the above optimization problem takes the following form: given a fuzzy set $\mu(x)$, find an α-cut S for which

$$d(\mu,\chi_S) =$$
$$\max\left(\int_{CS} \mu(x)\,\mathrm{d}x, \int_S (1 - \mu(x))\,\mathrm{d}x\right) \to \min_S. \quad (5)$$

8.5 Solving the Resulting Approximation Problem

When the set S increases, the first integral I_1 in the formula (5) decreases (because its integration domain shrinks), and the second integral I_2 increases. So:

- If the first integral I_1 is smaller than the second integral I_2, then the value of the maximum is equal to the value of the second integral: $d(\mu,\chi_S) = I_2$. We can decrease S a little bit; then:
  - the value of the second integral I_2 slightly decreases: $I_2' < I_2$, while
  - the value of the first integral I_1 slightly increases, to $I_1' > I_1$.

  For a sufficiently small increase in S, we still have $I_1' < I_2'$. Thus, for the new set $S' \supseteq S$, we have $d(\mu,\chi_{S'}) = I_2'$ and therefore, $d(\mu,\chi_{S'}) = I_2' < d(\mu,\chi_S) = I_2$. Hence, the minimum in (5) cannot be attained when $I_1 < I_2$.
- Similarly, when $I_1 > I_2$, we can slightly increase S and get a new set S' for which $d(\mu,\chi_{S'}) < d(\mu,\chi_S)$. Hence, the minimum in (5) cannot be attained when $I_1 > I_2$.

Thus, the minimum is attained when $I_1 = I_2$, i.e., when

$$\int_{CS} \mu(x)\,\mathrm{d}x = \int_S (1 - \mu(x))\,\mathrm{d}x. \quad (6)$$

Comment. This metric was first proposed in [3]; in that paper, for simple fuzzy sets, explicit formulas are given for the optimal crisp approximation.

9 Possibility of "Undecided"

9.1 Possibility

In the above text, we assumed that for every input x, we have to make a certain decision: either chose A^+ or A^-. In reality, in many practical situations, we can select a third option: claiming that there is not enough information for making a meaningful decision, and further tests are necessary.

9.2 Towards a Formalization of This Possibility

For one of such inputs x, it may later turn out that it is best to choose A^+, i.e., that $\chi(x) = 1$. It may also later turn out that it is best to choose A^-, i.e., that $\chi(x) = 0$. It may also turn out that the best strategy is picking A^+ with a certain probability $p \in [0,1]$. This possibility can be described by saying that $\chi(x) = p$ is also a possible value of $\chi(x)$. In other words, for such x, the set of possible values of $\chi(x)$ is the entire interval $[0,1]$.

9.3 Formal Definition: the Notion of a "Shadowed" Set

To take this possibility into consideration, the authors of [3] introduce a new notion of a *shadowed set*. This set is characterized by a characteristic function which maps every input x into either 0 (meaning that we choose A^-) or 1 (meaning that we choose A^+), or the interval $[0,1]$ (meaning that we do not know what to choose). The set of inputs x for which $\chi)(x) = [0,1]$ is called the *shadow* of the shadowed set.

9.4 Describing Distance Between Shadowed Fuzzy Sets: First Idea

How can we describe the distance for such sets? The fact that $\chi(x) = [0,1]$ means that later on, the actual (unknown) value of $\chi(x)$ may turn out to be any value from the interval $[0,1]$. Thus, a shadowed set is not a single membership function, but a whole *set* of membership functions. Normally, a distance between a point and a set is defined as the smallest possible distance between this points and points from this set:

$$d(a, B) = \inf_{b \in B} d(a, b).$$

Thus, if we fix a formula for the distance between two membership functions (e.g., the formula (3)), then we can naturally define a distance between a membership function μ and a shadowed set χ as the smallest possible distance between μ and different functions $\mu' \in \chi$:

$$d(\mu, \chi) = \min_{\mu' \in \chi} \int |\mu(x) - \mu'(x)| \, \mathrm{d}x. \qquad (7)$$

9.5 Minimizing the Corresponding Distance

Similarly to the above discussion of the formula (3), the minimum is attained when the term $|\mu(x) - \mu'(x)|$ is the smallest for every x. When $\chi(x) = [0,1]$, then the minimum of this term is attained when $\mu'(x) = \mu(x)$, and this minimum is equal to 0. Thus, in computing the expression (6), it is sufficient to take into consideration only the values x for which $\chi(x) = 0$ or $\chi(x) = 1$. For such values, we terms get the following values:

- when $\chi(x) = 0$, we have $|\mu(x) - \mu'(x)| = |\mu(x)| = \mu(x)$;
- when $\chi(x) = 1$, we have $|\mu(x) - 1| = |\mu(x) - 1| = 1 - \mu(x)$.

Thus,

$$d(\mu, \chi) =$$

$$\int_{x:\chi(x)=0} \mu(x) \, \mathrm{d}x + \int_{x:\chi(x)=1} (1 - \mu(x)) \, \mathrm{d}x. \qquad (8)$$

9.6 Solution to the Corresponding Optimization Problem and Its Drawbacks

This expression makes sense, but its minimization does not: its minimum is attained (and equal to 0) when for every x, we have $\chi(x) = [0,1]$, i.e., when the shadow coincides with the entire universe of discourse.

9.7 Re-Defining Distance Between Shadowed Sets: From a New Idea to a Definition

Since our goal is to make a decision, such a super-cautious shadowed set, in which the answer is always "I do not know", does not make much sense. It therefore makes sense to require that the shadowed set is close to the original fuzzy set *and* the shadow is not too large. This ideal requirement is not satisfied if either the shadowed set is not equal to the original fuzzy set *or* the shadow is large. Thus, if we use $\max$ to describe "or", we can take, as a degree to which this ideal is not satisfied, the maximum of the degree $d(\mu, \chi)$ to which the shadowed set χ is different from μ and the size

$$\int_{x:\chi(x)=[0,1]} \mathrm{d}x$$

of the shadow. In other words, we want to find a shadowed set χ for which

$$\max \left(d(\mu, \chi), \int_{x:\chi(x)=[0,1]} \mathrm{d}x \right) \to \min_{\chi}.$$

9.8 Solving the Corresponding Optimization Problem

When we increase the shadow, the first of the maximized terms decreases, the second increases, and vice versa. Thus, similarly to the non-shadowed case, we can conclude that the minimum is attained when the two terms are equal, i.e., when

$$\int_{x:\chi(x)=0} \mu(x) \, \mathrm{d}x + \int_{x:\chi(x)=1} (1 - \mu(x)) \, \mathrm{d}x =$$

$$\int_{x:\chi(x)=[0,1]} \mathrm{d}x. \qquad (9)$$

Comment. This metric was also first proposed in [3]; in that paper, for simple fuzzy sets, explicit formulas are given for the optimal shadowed approximation.

10 Conclusion

In decision making problems under fuzzy uncertainty, traditional fuzzy technique leads to a fuzzy recommendation like "it is somewhat reasonable to make a decision A". This fuzzy recommendation can be described by a fuzzy set (membership function), which assigns, to every possible situation x, the degree $\mu(x)$ to which the decision A is reasonable in this particular situation. Such a fuzzy representation is of great help to a human decision maker (e.g., to a medical doctor), but if we want to produce an automatic decision making system, we must transform the fuzzy recommendation $\mu(x)$

into a crisp recommendation $\chi(x)$, which, for every x, either recommends to make the decision A ($\chi(x) = 1$), or to make the opposite decision ($\chi(x) = 0$). This crisp recommendation can be therefore described as a crisp set S (for which $\chi(x)$ is a characteristic function). We would like to make this crisp recommendation to reflect the original fuzzy recommendation $\mu(x)$ in the best possible way. In other words, we would like to select a crisp set S which is, in some reasonable sense, the optimal approximation to the original fuzzy set $\mu(x)$. In this paper, we show that the optimal approximation is the α-cut S for which

$$\int_{CS} \mu(x)\,\mathrm{d}x = \int_S (1-\mu(x))\,\mathrm{d}x. \qquad (6)$$

If we allow the possibility of not making any decision in some situations, then we arrive at the necessity to approximate the original fuzzy set by a "shadowed" set, i.e., a set in which a characteristic function can take three possible values: "yes" ($\chi(x) = 1$), "no" ($\chi(x) = 0$), and "undecided" ($\chi(x) = [0, 1]$). The optimal approximation is given by a combination of α-cuts which satisfies the formula (9).

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