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A New Universal Approximation Result For Fuzzy Systems, Which Reflects CNF–DNF Duality

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Abstract

There are two main fuzzy system methodologies for translating expert rules into a logical formula: In Mamdani's methodology, we get a DNF formula (disjunction of conjunctions), and in a methodology which uses logical implications, we get, in effect, a CNF formula (conjunction of disjunctions). For both methodologies, universal approximation results have been proven which produce, for each approximated function $f(x)$, two different approximating relations $R_{\text{DNF}}(x, y)$ and $R_{\text{CNF}}(x, y)$. Since in fuzzy logic, there is a known relation $F_{\text{CNF}}(x) \leq F_{\text{DNF}}(x)$ between CNF and DNF forms of a propositional formula F , it is reasonable to expect that we would be able to prove the existence of approximations for which a similar relation $R_{\text{CNF}}(x, y) \leq R_{\text{DNF}}(x, y)$ holds. Such existence is proved in our paper.

1 Introduction

1.1 Fuzzy control: in brief

Fuzzy control (see, e.g., [9]) is a methodology that translates the expert's if-then rules of the type

$$\text{if } A_i(x) \text{ then } B_i(y), \quad 1 \leq i \leq N, \quad (1)$$

or

$$\text{if } A_{i1}(x_1) \text{ and } \dots \text{ and } A_{in}(x_n) \text{ then } B_i(y), \quad (2)$$

in which the properties $A_i(x)$ and $B_j(y)$ are described by using words from natural languages (such as “ x is small”), into a *control strategy*, i.e., into a function $f : X \rightarrow Y$ describing what exactly control we should apply for a given input $x \in X$. This methodology consists of three major steps:

- first, we *formalize* each “linguistic” property $A_i(x)$ or $B_i(y)$ as a *fuzzy set*, i.e., as a function $A_i : X \rightarrow [0, 1]$ which describes, for each object $x \in X$, to what extent this property holds for this x (e.g., to what extent x is small);
- then, we *combine* these fuzzy sets into a *fuzzy relation*, i.e. a function $R(x, y) : X \times Y \rightarrow [0, 1]$ which describes, for each input $x \in X$ and for each possible output $y \in Y$, to what extent this particular outputs satisfies the expert's rules;
- finally, we apply some *defuzzification procedure* to the *fuzzy relation* $R(x, y)$, and get the desired control strategy, as a function $\tilde{f} : X \rightarrow Y$.

1.2 Mamdani's (DNF) approach

In most practical application of fuzzy control, *Mamdani's* approach is used in the combination (second) step. In this approach, the fuzzy relation $R(x, y)$ is represented by a logical formula

$$(A_1(x) \& B_1(y)) \vee \dots \vee (A_N(x) \& B_N(y)), \quad (3)$$

or as

$$(A_{i1}(x_1) \& \dots \& A_{in}(x_n) \& B_i(y)) \vee \dots \vee (A_{N1}(x_1) \& \dots \& A_{Nn}(x_n) \& B_N(y)) \quad (4)$$

where ‘&’ and ‘ \vee ’ stand for connectives of conjunction and disjunction respectively. In logical terms, we have a disjunction of conjunctions $A_i(x) \& B_i(y)$, i.e., a formula in a Disjunctive Normal Form – DNF.

Then, we select an interpretation: a t-norm $f_{\&} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ for conjunction and a t-conorm $f_{\vee} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ for disjunction (see, e.g.,

[4, 10]), and use these operations in the corresponding formulas (3) and (4), resulting in:

$$R_{\text{DNF}}(x, y) = f_{\vee}(f_{\&}(A_1(x), B_1(y)), \dots, f_{\&}(A_N(x), B_N(y))), \quad (5)$$

$$R_{\text{DNF}}(x, y) = f_{\vee}[f_{\&}(A_{11}(x), \dots, A_{1n}(x_n), B_1(y)), \dots, f_{\&}(A_{N1}(x_1), \dots, A_{Nn}(x_n), B_N(y))]. \quad (6)$$

1.3 Logical implication (CNF) approach

From the logical viewpoint, it is somewhat more natural to represent the fuzzy relation $R(x, y)$ as a conjunction of implications:

$$(A_1(x) \rightarrow B_1(y)) \& \dots \& (A_N(x) \rightarrow B_N(y)), \quad (7)$$

or

$$((A_{11}(x) \& \dots \& A_{1n}(x_n)) \rightarrow B_1(y)) \& \dots \& ((A_{N1}(x_1) \& \dots \& A_{Nn}(x_n)) \rightarrow B_N(y)). \quad (8)$$

In this case, we select the interpretation: $f_{\&} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ for conjunction and $f_{\rightarrow} : [0, 1] \times [0, 1] \rightarrow [0, 1]$, for implication and use these operations in the corresponding formulas (7) and (8), resulting in:

$$R_{\rightarrow}(x, y) = f_{\&}(f_{\rightarrow}(A_1(x), B_1(y)), \dots, f_{\rightarrow}(A_N(x), B_N(y))), \quad (9)$$

$$R_{\rightarrow}(x, y) = f_{\&}[f_{\rightarrow}(f_{\&}(A_{11}(x), \dots, A_{1n}(x_n)), B_1(y)), \dots, f_{\rightarrow}(f_{\&}(A_{N1}(x_1), \dots, A_{Nn}(x_n)), B_N(y))]. \quad (10)$$

In particular, since in classical logic $A \rightarrow B$ is equivalent to $\neg A \vee B$, and $(A_1(x) \& \dots \& A_n) \rightarrow B$ to $\neg A_1 \vee \dots \vee \neg A_n \vee B$, it makes sense to consider representations of formulas (7) and (8) in the following form

$$(\neg A_1(x) \vee B_1(y)) \& \dots \& (\neg A_N(x) \vee B_N(y)), \quad (11)$$

or

$$(\neg A_{11}(x) \vee \dots \vee \neg A_{1n}(x_n) \vee B_1(y)) \& \dots \& (\neg A_{N1}(x_1) \vee \dots \vee \neg A_{Nn}(x_n) \vee B_N(y)). \quad (12)$$

In logical terms, we have a conjunction of disjunctions $A_i(x) \& B_i(y)$, i.e., a formula in a Conjunctive Normal Form – CNF.

In DNF, we have outside disjunction and inside conjunctions; in CNF, the roles of disjunction and conjunction are reversed: we have outside conjunction and inside disjunctions. In logic, conjunction and disjunction are often called *dual* logical operations; in view of this terminology, CNF and DNF are also often called *dual* forms.

Then, we select the interpretation: a t-norm $f_{\&} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ for conjunction, a t-conorm $f_{\vee} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ for disjunction and a *fuzzy negation* $f_{\neg} : [0, 1] \rightarrow [0, 1]$, and use these operations in the corresponding formulas (11) and (12), resulting in:

$$R_{\text{CNF}}(x, y) = f_{\&}(f_{\vee}(f_{\neg}(A_1(x)), B_1(y)), \dots, f_{\vee}(f_{\neg}(A_N(x)), B_N(y))), \quad (13)$$

$$R_{\text{CNF}}(x, y) = f_{\&}[f_{\vee}(f_{\neg}(A_{11}(x)), \dots, f_{\neg}(A_{1n}(x_n)), B_1(y)), \dots, f_{\vee}(f_{\neg}(A_{N1}(x_1)), \dots, f_{\neg}(A_{Nn}(x_n)), B_N(y))]. \quad (14)$$

1.4 Relation between DNF and CNF approaches

For each of these two methodologies, it is desirable to check that this methodology is *universal*, i.e., that if we use this methodology, then, for an arbitrary control function $f : X \rightarrow Y$, and for an arbitrary accuracy, there exist appropriate if-then rules for which the resulting control strategy represented by $\tilde{f}(x)$ approximates the original control function $f(x)$ within a given accuracy.

There exists many universal approximation results for approximations which are derived from Mamdani-style DNF formulas; first such results were formulated and proved, almost simultaneously, in 1990–92 papers by J. Buckley, Z. Cao, E. Czogala, D. Dubois, M. Grabisch, J. Han, Y. Hayashi, C.-C. Jou, A. Kandel, B. Kosko, J. Mendel, H. Prade, and L.-X. Wang; for a recent survey, see, e.g., [6] and references therein. There also exist several universal approximation results for implication-style CNF formulas [1, 2, 3, 11, 13].

These results are usually proved separately and provide two different (seemingly unrelated) approximations. In logic, however, CNF and DNF forms are related. In classical (2-valued) logic, every propositional formula F can be represented in both DNF and CNF forms F_{DNF} and F_{CNF} ; for every input x , these forms lead to exactly the same truth value: $F_{\text{CNF}}(x) = F(x) = F_{\text{DNF}}(x)$. In fuzzy logic, each propositional formula can also be transformed (generally, non-equivalently, see [11]) into CNF and DNF forms, so that using $f_{\&} = \min$, $f_{\vee} = \max$, and $f_{\neg}(a) = 1 - a$, we get $F_{\text{CNF}}(x) \leq F_{\text{DNF}}(x)$ (to be more precise, $F_{\text{CNF}}(x) \leq F(x) \leq F_{\text{DNF}}(x)$; see, e.g., [12, 14]). In view of this relation, it is desirable to have a universal approximation result for CNF and DNF formulas which is consistent with this “fuzzy duality”, i.e., in which there is a similar relation between the fuzzy relations $R_{\text{DNF}}(x, y)$ and $R_{\text{CNF}}(x, y)$ which approximate the desired function f . Such a result is presented in this paper.

In proving this duality-related result, we also somewhat generalize the known CNF and DNF universal approximation theorems.

2 General Case: Functions Defined on an Arbitrary Compact Set

Definition 1. Let $k = 1$ or $k = 2$, and let \oplus be a propositional k -ary operation in classical (2-valued) logic (e.g., $\&$, \vee , \neg , \rightarrow). We say that an operation $f_{\oplus} : [0, 1]^k \rightarrow [0, 1]$ is consistent with classical logic if it coincides with \oplus when all its inputs are 0's and 1's (corresponding to “false” and “true”).

Please note that we did not require that f_{\oplus} is continuous (as a function), or that $f_{\&}(a, b)$ is commutative or associative, etc.

Definition 2. Let X be a compact metric space with a metric d_X , Y be a complete metric space with a metric d_Y , $f : X \rightarrow Y$ be a continuous function from X to Y , and $\varepsilon > 0$ be a real number. We say that a fuzzy relation $R : X \times Y \rightarrow [0, 1]$ ε -approximates a function $f : X \rightarrow Y$ if the following two conditions hold:

- for every $x \in X$, $R(x, f(x)) > 0$, and
- for every $x \in X$ and $y \in Y$, if $R(x, y) > 0$, then $d_Y(y, f(x)) \leq \varepsilon$.

Theorem 1. Let operations $f_{\&}$, f_{\vee} , and f_{\neg} be consistent with classical logic. Then, for every compact metric space X , for every continuous function $f : X \rightarrow Y$ into a complete metric space Y , and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which:

- both fuzzy relations R_{DNF} and R_{CNF} (obtained using the interpretation determined by $f_{\&}$, f_{\vee} , and f_{\neg}) ε -approximate f , and
- $R_{\text{CNF}}(x, y) \leq R_{\text{DNF}}(x, y)$ for all x and y .

Comment. For the convenience of the readers, all the proofs are placed in the special Proofs section.

A similar result holds for R_{\rightarrow} instead of R_{CNF} :

Theorem 1'. Let operations $f_{\&}$, f_{\vee} , and f_{\rightarrow} be consistent with classical logic. Then, for every compact metric space X , for every continuous function $f : X \rightarrow Y$ into a complete metric space Y , and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which:

- both fuzzy relations R_{\rightarrow} and R_{DNF} (obtained using the interpretation determined by $f_{\&}$, f_{\vee} , and f_{\rightarrow}) ε -approximate f , and
- $R_{\rightarrow}(x, y) \leq R_{\text{DNF}}(x, y)$ for all x and y .

From the fact that a fuzzy relation $R(x, y)$ ε -approximates a function $f(x)$, we can conclude that the result $\tilde{f}(x) = D(\mu_x)$ of applying, for every $x \in X$,

a defuzzification procedure D (see below) to the corresponding membership function $\mu_x(y) = R(x, y)$ is ε -close to $f(x)$:

Definition 3. By a defuzzification procedure, we mean a mapping $D : [0, 1]^Y \rightarrow Y$ which maps every membership function $\mu : Y \rightarrow [0, 1]$ (which is not identically zero) into an element $D(\mu) \in Y$ for which $\mu(D(\mu)) > 0$.

Proposition 1. If a fuzzy relation $R(x, y)$ ε -approximates a function $f(x)$, then, for every defuzzification procedure D , the result $\tilde{f}(x) = D(\mu_x)$ of applying this defuzzification procedure D to the corresponding membership function $\mu_x(y) = R(x, y)$ is ε -close to $f(x)$, i.e., $d_Y(\tilde{f}(x), f(x)) \leq \varepsilon$.

Corollary 1. Let operations $f_\&$, f_\vee , and f_\neg be consistent with classical logic. Then, for every compact metric space X , for every continuous function $f : X \rightarrow Y$ into a complete metric space Y , and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which, for each defuzzification procedure D , the results $\tilde{f}_{\text{DNF}}(x)$ and $\tilde{f}_{\text{CNF}}(x)$ of defuzzifying the relations R_{DNF} and R_{CNF} (obtained using $f_\&$, f_\vee , and f_\neg) are ε -close to f .

Corollary 1'. Let operations $f_\&$, f_\vee , and f_\rightarrow be consistent with classical logic. Then, for every compact metric space X , for every continuous function $f : X \rightarrow Y$ into a complete metric space Y , for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which, for each defuzzification procedure D , the results $\tilde{f}_{\text{DNF}}(x)$ and $\tilde{f}_{\text{CNF}}(x)$ of defuzzifying the relations R_{DNF} and R_{CNF} (obtained using $f_\&$, f_\vee , and f_\rightarrow) are ε -close to f .

3 Towards a More Realistic Situation: Case When $Y = \mathbb{R}$

In the case when Y is a real line ($Y = \mathbb{R}$), we can use a different class of possible “defuzzification procedures” and still get the same universal approximation result. Namely, we can use the following definition:

Definition 3'. ($Y = \mathbb{R}$) By a defuzzification procedure, we mean a mapping D which maps every non-zero membership function $\mu : \mathbb{R} \rightarrow [0, 1]$ into a real number $D(\mu)$ in such a way that for an arbitrary interval $[a, b]$, if a membership function $\mu(x)$ is equal to 0 for all values x outside an interval $[a, b]$, then $D(\mu) \in [a, b]$.

Comment. Both centroid and center-of-maximum are defuzzification procedures in this sense.

Proposition 1'. ($Y = \mathbb{R}$) If a fuzzy relation $R(x, y)$ ε -approximates a function $f(x)$, then, for every defuzzification procedure D , the result $\tilde{f}(x) = D(\mu_x)$ of applying this defuzzification procedure D to the corresponding membership function $\mu_x(y) = R(x, y)$ is ε -close to $f(x)$, i.e., $d_Y(\tilde{f}(x), f(x)) \leq \varepsilon$.

4 Realistic Case: Functions From \mathbb{R}^n to \mathbb{R}

For the case when $X = \mathbb{R}^n$, we can prove similar results with rules of type (2):

Theorem 2. *Let operations $f_{\&}$, f_{\vee} , and f_{\neg} be consistent with classical logic. Then, for every integer $n > 0$, for every compact set $X \subset \mathbb{R}^n$, for every continuous function $f : X \rightarrow \mathbb{R}$, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (2) for which:*

- both fuzzy relations R_{DNF} and R_{CNF} (obtained using $f_{\&}$, f_{\vee} , and f_{\neg}) ε -approximate f , and
- $R_{\text{CNF}}(x, y) \leq R_{\text{DNF}}(x, y)$ for all $x = (x_1, \dots, x_n) \in X$ and y .

Theorem 2'. *Let operations $f_{\&}$, f_{\vee} , and f_{\rightarrow} be consistent with classical logic. Then, for every integer $n > 0$, for every compact set $X \subset \mathbb{R}^n$, for every continuous function $f : X \rightarrow \mathbb{R}$, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (2) for which:*

- both fuzzy relations R_{\rightarrow} and R_{DNF} (obtained using $f_{\&}$, f_{\vee} , and f_{\rightarrow}) ε -approximate f , and
- $R_{\rightarrow}(x, y) \leq R_{\text{DNF}}(x, y)$ for all $x = (x_1, \dots, x_n) \in X$ and y .

In our universal approximation result, we prove that for every function $f : X \rightarrow Y$, we can select the rules and the membership functions for which we get the desired approximation property.

For Mamdani's (DNF) case, a stronger statement is true: that whatever "realistic" membership function $\mu_0(x)$ we choose, we can always find rules in which all the membership functions $A_{ik}(x_k)$ and $B_i(y)$ are *of the type* μ_0 , i.e., they all have the form $\mu(x) = \mu_0(a \cdot x + b)$ for some real numbers $a \neq 0$ and b (see, e.g., [7, 8]).

From the proof of Theorems 2 and 2', we see that all the membership functions used in the approximation have the same type, i.e., that there is a type μ_0 which provides a universal approximation property both for DNF and for CNF forms. We do not know whether a similar result is true for an arbitrary given type.

5 Proofs

5.1 Proof of Theorem 1

This proof is similar to the original Kosko's proof [5] of a universal approximation result for DNF (i.e., for Mamdani methodology), and to our own proofs from [7, 8, 11].

1°. Let us take $\varepsilon_1 = \varepsilon/2$. Since a function f is continuous on a compact set X , it is also uniformly continuous. Therefore, there exists $\delta > 0$ such that if $d_X(x, x') \leq \delta$, then $d_Y(f(x), f(x')) \leq \varepsilon_1$.

Since X is a compact metric space, there exists a finite δ -net for X , i.e., a finite set of elements $x^{(1)}, \dots, x^{(N)} \in X$ for which, for every $x \in X$, there exists an i for which $d_X(x, x^{(i)}) \leq \delta$. For each of these elements $x^{(i)}$, we can find $y^{(i)} = f(x^{(i)})$. We will show that Theorem 1 holds for N rules of type (1) where for every i ,

- $A_i(x) = 1$ if $d_X(x, x^{(i)}) \leq \delta$, and $A_i(x) = 0$ otherwise;
- $B_i(y) = 1$ if $d_Y(y, y^{(i)}) \leq \varepsilon_1$, and $B_i(y) = 0$ otherwise.

All these fuzzy sets A_i and B_i are crisp: indeed, $A_i(x)$ is a characteristic function of the inequality $d_X(x, x^{(i)}) \leq \delta$, and $B_i(y)$ is a characteristic function of the inequality $d_Y(y, y^{(i)}) \leq \varepsilon_1$.

Since all the $f_{\&}$, f_{\vee} , and f_{\neg} are consistent with classical logic and A_i , B_i are crisp, we conclude that the relations R_{CNF} and R_{DNF} can be obtained using classical logical connectives. Namely,

$$R_{\text{DNF}}(x, y) = 1 \iff \exists i \left(d_X(x, x^{(i)}) \leq \delta \ \& \ d_Y(y, y^{(i)}) \leq \varepsilon_1 \right); \quad (15)$$

$$R_{\text{CNF}}(x, y) = 1 \iff \forall i \left(d_X(x, x^{(i)}) > \delta \vee d_Y(y, y^{(i)}) \leq \varepsilon_1 \right). \quad (16)$$

2°. Let us first show that the relation R_{DNF} ε -approximates the given function f .

2.1°. In accordance with the definition of ε -approximation, we first prove that for every $x \in X$, we have $R_{\text{DNF}}(x, f(x)) > 0$.

Indeed, let x be an arbitrary element of the set X . Since $x^{(1)}, \dots, x^{(N)}$ is a δ -net, there exists an i for which $d_X(x, x^{(i)}) \leq \delta$. Due to our choice of δ , we conclude that $d_Y(f(x), y^{(i)}) \leq \varepsilon_1$. Thus, (15) is true, hence, $R_{\text{DNF}}(x, f(x)) > 0$.

2.2°. Let us now prove that for every $x \in X$ and $y \in Y$, if $R_{\text{DNF}}(x, y) > 0$, then $d_Y(y, f(x)) \leq \varepsilon$.

Indeed, since R_{DNF} is a crisp relation, the only possibility for $R_{\text{DNF}}(x, y) > 0$ is to have $R_{\text{DNF}}(x, y) = 1$, i.e., $R_{\text{DNF}}(x, y)$ to be true. This means that there exists an i for which $d_X(x, x^{(i)}) \leq \delta$ and $d_Y(y, y^{(i)}) \leq \varepsilon_1$. Due to our choice of δ , from $d_X(x, x^{(i)}) \leq \delta$, we can conclude that $d_Y(f(x), f(x^{(i)})) = d_Y(f(x), y^{(i)}) \leq \varepsilon_1$. Thus, from the triangle inequality, we conclude that $d_Y(y, f(x)) \leq d_Y(y, y^{(i)}) + d_Y(y^{(i)}, f(x)) \leq \varepsilon_1 + \varepsilon_1 = \varepsilon$. The statement is proven.

3°. Let us now show that the relation R_{CNF} also ε -approximates the given function f .

3.1°. In accordance with the definition of ε -approximation, we first prove that for every $x \in X$, we have $R_{\text{CNF}}(x, f(x)) > 0$.

Indeed, let x be an arbitrary element of the set X . For every i , we have $d_X(x, x^{(i)}) \leq \delta$ or $d_X(x, x^{(i)}) > \delta$. By definition of δ , the inequality $d_X(x, x^{(i)}) \leq \delta$ implies that $d_Y(f(x), f(x^{(i)})) = d_Y(f(x), y^{(i)}) \leq \varepsilon_1$. Therefore, we have $d_Y(f(x), y^{(i)}) \leq \varepsilon_1$ or $d_X(x, x^{(i)}) > \delta$. Hence, (16) is true, and $R_{\text{CNF}}(x, f(x)) > 0$.

3.2°. Let us show that for every $x \in X$ and $y \in Y$, if $R_{\text{CNF}}(x, y) > 0$, then $d_Y(y, f(x)) \leq \varepsilon$.

Indeed, since R_{CNF} is a crisp relation, the only possibility for $R_{\text{CNF}}(x, y) > 0$ is to have $R_{\text{CNF}}(x, y) = 1$, i.e., $R_{\text{CNF}}(x, y)$ to be true. This means that for every i , either $d_X(x, x^{(i)}) > \delta$ or $d_Y(y, y^{(i)}) \leq \varepsilon_1$. Equivalently, this disjunction means that the inequality $d_X(x, x^{(i)}) \leq \delta$ implies (in the logical sense, where implication is material!) that $d_Y(y, y^{(i)}) \leq \varepsilon_1$.

Since $x^{(1)}, \dots, x^{(N)}$ is a δ -net, there exists an i for $d_X(x, x^{(i)}) \leq \delta$. For this i , we thus have $d_Y(y, y^{(i)}) \leq \varepsilon_1$. On the other hand, due to our choice of δ , from $d_X(x, x^{(i)}) \leq \delta$, we can conclude that $d_Y(f(x), f(x^{(i)})) = d_Y(f(x), y^{(i)}) \leq \varepsilon_1$. Thus, from the triangle inequality, we conclude that $d_Y(y, f(x)) \leq d_Y(y, y^{(i)}) + d_Y(y^{(i)}, f(x)) \leq \varepsilon_1 + \varepsilon_1 = \varepsilon$. The statement is proven.

4°. To complete the proof of the theorem, we must now show that $R_{\text{CNF}}(x, y) \leq R_{\text{DNF}}(x, y)$ for all x and y .

Since both relations $R_{\text{CNF}}(x, y)$ and $R_{\text{DNF}}(x, y)$ are crisp, the desired inequality is equivalent to saying that for every x and y , if $R_{\text{CNF}}(x, y)$ is true, then $R_{\text{DNF}}(x, y)$ should also be true. Indeed, let $R_{\text{CNF}}(x, y)$ hold for some x and y . This means that for every i , the inequality $d_X(x, x^{(i)}) \leq \delta$ implies (in the logical sense) that $d_Y(y, y^{(i)}) \leq \varepsilon_1$. Since $x^{(1)}, \dots, x^{(N)}$ is a δ -net, there exists an i for which $d_X(x, x^{(i)}) \leq \delta$. Thus, for this i , we have $d_Y(y, y^{(i)}) \leq \varepsilon_1$. So, for this i , both inequalities $d_X(x, x^{(i)}) \leq \delta$ and $d_Y(y, y^{(i)}) \leq \varepsilon_1$ are true, hence the formula (16) is true. The statement is proven, and so is the theorem.

Comment. We have already mentioned that in crisp (2-valued) propositional logic, CNF and DNF forms represent the same function. It deserves mentioning that although in our proof, the relations $R_{\text{CNF}}(x, y)$ and $R_{\text{DNF}}(x, y)$ are both crisp, they do not necessarily be the same. Indeed, let us consider the simplest case when $X = Y = [0, 1]$ with a normal metric $d_X(x, x') = d_Y(x, x') = |x - x'|$, and $f(x) = x$. Then, $\delta = \varepsilon_1 = \varepsilon/2$. As a δ -net, we can select the points $x^{(i)} = (2i - 1) \cdot \delta$, i.e., $x^{(1)} = \delta$, $x^{(2)} = 3\delta$, etc.; then, $y^{(i)} = x^{(i)}$. Here, for $x = 2\delta$ and $y = 0$, we have $d_X(x, x^{(1)}) \leq \delta$ and $d_Y(y, y^{(1)}) \leq \varepsilon_1$, so $R_{\text{DNF}}(x, y)$

is true. However, the property $R_{\text{CNF}}(x, y)$ is not true, because for $i = 2$, we have $d_X(x, x^{(2)}) \leq \delta$, but $d_Y(y, y^{(2)}) = 3\delta = 3\varepsilon_1 > \varepsilon_1$.

5.2 Proof of Theorem 1'

For this proof, we can use the exact same crisp sets A_i and B_i as in the proof of Theorem 1.

5.3 Proof of Proposition 1

By definition, $\tilde{f}(x) = D(\mu_x)$, where the membership function $\mu_x : Y \rightarrow [0, 1]$ is defined as $\mu_x(y) = R(x, y)$. By the definition of a defuzzification procedure, for every $x \in X$, we have $\mu_x(D(\mu_x)) > 0$, i.e., by definition of μ_x , $R(x, \tilde{f}(x)) > 0$. From the definition of ε -approximation, we can now conclude that $d_Y(\tilde{f}(x), f(x)) \leq \varepsilon$. The proposition is proven.

5.4 Proof of Proposition 1'

By definition, $\tilde{f}(x) = D(\mu_x)$, where the membership function $\mu_x : Y \rightarrow [0, 1]$ is defined as $\mu_x(y) = R(x, y)$. From the definition of ε -approximation, we conclude that if $R(x, y) > 0$, then $d_Y(y, f(x)) = |y - f(x)| \leq \varepsilon$. Thus, if $|y - f(x)| > \varepsilon$, we have $R(x, y) = \mu_x(y) = 0$. Hence, the function $\mu_x(y)$ is equal to 0 outside the interval $[f(x) - \varepsilon, f(x) + \varepsilon]$. By definition of a defuzzification procedure, we can now conclude that the result $\tilde{f}(x)$ of its defuzzification also belongs to the same interval, i.e., that $|\tilde{f}(x) - f(x)| \leq \varepsilon$. The proposition is proven.

5.5 Proof of Theorems 2 and 2'

This proof is similar to the proofs of Theorems 1 and 1'. Indeed, for $X \subset \mathbb{R}^n$, continuity of a function $f : X \rightarrow \mathbb{R}$ with respect to a normal (Euclidean) metric is equivalent to its continuity with respect to the uniform metric $d_X(x, x') = \max_i |x_i - x'_i|$. Thus, from the proofs of Theorems 1 and 1', we conclude that there exist appropriate rules of type (1), with $A_i(x) = 1 \iff d_X(x, x^{(i)}) \leq \delta$ and $A_i(x) = 0$ for all other x , and $B_i(y) = 1 \iff |y - y^{(i)}| \leq \varepsilon_1$ and $B_i(y) = 0$ for all other y . By the definition of the uniform metric d_X , the inequality $d_X(x, x^{(i)}) \leq \delta$ is equivalent to $A_{i1}(x_1) \& \dots \& A_{in}(x_n) = 1$, where $A_{ik}(x_k) = 1 \iff |x_k - x_k^{(i)}| \leq \delta$ and $A_{ik}(x_k) = 0$ for all other x_k . Thus, rules of type (1) can be reformulated in the desired form (2). The theorems are thus proven.

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