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Fair Division under Interval Uncertainty

Ronald R. Yager

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

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FAIR DIVISION UNDER INTERVAL UNCERTAINTY

RONALD R. YAGER
Iona College
715 North Avenue
New Rochelle, NY 10801-18903, USA
E-mail: yager@panix.com

and

VLADIK KREINOVICH
Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
E-mail: vladik@cs.utep.edu

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Revised

It is often necessary to divide a certain amount of money between n participants, i.e., to assign, to each participant, a certain *portion* $w_i \geq 0$ of the whole sum (so that $w_1 + \ldots + w_n = 1$). In some situations, from the fairness requirements, we can uniquely determine these "weights" w_i. However, in some other situations, general considerations do not allow us to uniquely determine these weights, we only know the *intervals* $[w_i^-, w_i^+]$ of possible fair weights. We show that natural fairness requirements enable us to choose unique weights from these intervals; as a result, we present an algorithm for fair division under interval uncertainty.

Keywords: Fair division; Interval uncertainty.

1. Introduction to the Problem

The general problem of fair division. It is often necessary to divide a certain amount of money between n participants, i.e., to assign, to each participant, a certain portion $w_i \geq 0$ of the whole sum (so that $w_1 + \ldots + w_n = 1$).

In some situations, we do not know the exact weights. In some situations, from the fairness requirements, we can uniquely determine the "weights" w_i. However, in some other situations, general considerations do not allow us to uniquely determine these weights, we only know the *intervals* $\mathbf{w}_i = [w_i^-, w_i^+]$ of possible fair weights.

Formulation of the problem. We want to select some values $w_i \in \mathbf{w}_i$ for which $w_1 + \ldots + w_n = 1$, and assign to each participant w_i-th portion of the divided sum. How can we do that *fairly*?

Comment. Before we choose the weights from the given intervals $\mathbf{w}_i$, we must be sure that such a choice is possible, i.e., that the given intervals $\mathbf{w}_i$ are *consistent.* This consistency condition can be easily expressed in terms of a double inequality:

- From $w_i^- \le w_i \le w_i^+$, we conclude that $W^- = w_1^- + \ldots + w_n^- \le w_1 + \ldots + w_n = 1 \le w_1^+ + \ldots + w_n^+ = W^+$. Thus, the given intervals must satisfy the inequality $W^- \le 1 \le W^+$.
- Vice versa, if $W^- \le 1 \le W^+$, then the interval $[W^-, W^+]$ of possible values of $w_1 + \ldots + w_n$ contains 1, and therefore, 1 can be represented as $w_1 + \ldots + w_n$ for some $w_i \in \mathbf{w}_i$.

So, this consistency condition is equivalent to $W^- \le 1 \le W^+$.

2. Towards a Formalization of the Problem

We want to describe a transformation T that maps every finite consistent sequence of intervals $\mathbf{w}_i \subseteq [0,1]$, $1 \le i \le n$, into a sequence of exactly as many real values $w_i \in \mathbf{w}_i$:

$$(\mathbf{w}_1, \ldots, \mathbf{w}_n) \to (w_1, \ldots, w_n)$$

in such a way that for the resulting sequence of real numbers, $w_1 + \ldots + w_n = 1$.

There are some natural properties that we expect from this transformation:

1. First, the distribution must be *fair*, it must not depend on the order in which we presented the participants. A participant who was assigned # 1 could as well be assigned # 5, and vice versa. Therefore, the desired function should not change if we simply swap i-th and j-th participants: If

$$(\mathbf{w}_1, \ldots, \mathbf{w}_{i-1}, \mathbf{w}_i, \mathbf{w}_{i+1}, \ldots, \mathbf{w}_{j-1}, \mathbf{w}_j, \mathbf{w}_{j+1}, \ldots, \mathbf{w}_n) \to$$
$$(w_1, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_{j-1}, w_j, w_{j+1}, \ldots, w_n), \tag{1}$$

then

$$(\mathbf{w}_1, \ldots, \mathbf{w}_{i-1}, \mathbf{w}_j, \mathbf{w}_{i+1}, \ldots, \mathbf{w}_{j-1}, \mathbf{w}_i, \mathbf{w}_{j+1}, \ldots, \mathbf{w}_n) \to$$
$$(w_1, \ldots, w_{i-1}, w_j, w_{i+1}, \ldots, w_{j-1}, w_i, w_{j+1}, \ldots, w_n). \tag{1'}$$

2. The second property is related to the following fact: two participants should neither gain nor lose simply by joining together. If we know the exact weights w_1 and w_2 of each of the original participants, then the weight of their combination is equal to $w_1 + w_2$. If we do not know the exact weights of each participant, i.e., if we only know the *intervals* of possible values $\mathbf{w}_1 = [w_1^-, w_1^+]$ and $\mathbf{w}_2 = [w_2^-, w_2^+]$ of these weights, then the weight of their combination can take any value $w_1 + w_2$ where $w_1 \in \mathbf{w}_1$ and $w_2 \in \mathbf{w}_2$. This set of possible values is known to be also an interval, with the bounds $[w_1^- + w_2^-, w_1^+ + w_2^+]$. In interval computations (see, e.g., [1,2,3,4,5,6]), this new interval is called the *sum* of the two intervals $\mathbf{w}_1$ and $\mathbf{w}_2$ and denoted by $\mathbf{w}_1 + \mathbf{w}_2$.

Ideally, the division should not change if we simply combine two participants. In other words: If

$$(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \ldots, \mathbf{w}_n) \rightarrow (w_1, w_2, w_3, \ldots, w_n), \tag{2}$$

then

$$(\mathbf{w}_1 + \mathbf{w}_2, \mathbf{w}_3, \ldots, \mathbf{w}_n) \rightarrow (w_1 + w_2, w_3, \ldots, w_n), \tag{2'}$$

3. Finally, small changes in the endpoints w_i^- or w_i^+ should not drastically affect the resulting division. In other words, we want the transformation T to be *continuous* for any given n.

3. Definitions and the Main Result

Definition 1. *We say that a sequence of intervals* $\mathbf{w}_i = [w_i^-, w_i^+] \subseteq [0,1]$, $1 \le i \le n$, *is consistent if* $w_1^- + \ldots + w_n^- \le 1 \le w_1^+ + \ldots + w_n^+$.

Definition 2.

- *By a division scheme under interval uncertainty,* we mean a transformation T *that transforms every consistent finite sequence of intervals* $\mathbf{w}_1, \ldots, \mathbf{w}_n$ *into a sequence of real numbers* $w_i \in \mathbf{w}_i$ *for which* $w_1 + \ldots + w_n = 1$.
- *We say that a division scheme under interval uncertainty is fair if it is continuous and satisfies the conditions* (1)–(2).

Theorem. *There exists exactly one fair division scheme under interval uncertainty, and this fair division scheme transforms each consistent sequence of intervals* $\mathbf{w}_i = [w_i^-, w_i^+]$, $1 \le i \le n$, *into a sequence of real numbers*

$$w_i = \frac{W^+ - 1}{W^+ - W^-} \cdot w_i^- + \frac{1 - W^-}{W^+ - W^-} \cdot w_i^+, \tag{3}$$

where

$$W^- = w_1^- + \ldots + w_n^-, \tag{4}$$

and

$$W^+ = w_1^+ + \ldots + w_n^+. \tag{5}$$

Comment 1. Formula (3) can be re-written in the following equivalent form:

$$w_i = w_i^- + \frac{\Delta w_i}{\Delta W} \cdot (1 - W^-), \tag{3a}$$

where $\Delta w_i = w_i^+ - w_i^-$, and $\Delta W = \Delta w_1 + \ldots + \Delta w_n$. Since $W^- = w_1^- + \ldots + w_n^- \le 1$, what we are doing is essentially adding to the lower endpoint w_i^- for i-th weight an amount proportional to the width $\Delta w_i = w_i^+ - w_i^-$ of the corresponding weight

interval $[w_i^-, w_i^+]$. This width is a natural measure of uncertainty with which we know i-th weight.

Comment 2. Alternatively, we can represent the formula (3) in another equivalent form:

$$w_i = w_i^+ - \frac{\Delta w_i}{\Delta W} \cdot (W^+ - 1), \tag{3b}$$

Since $W^+ = w_1^+ + \ldots + w_n^+ \geq 1$, what we are doing is essentially subtracting to the upper weight w_i^+ an amount proportional to the width $\Delta w_i = w_i^+ - w_i^-$ of the corresponding weight interval $[w_i^-, w_i^+]$.

Examples:

- If all the weight intervals coincide, we get $w_i = 1/n$ for all i, in good accordance with the notion of fairness.

- If we only know the *upper* bounds w_i^+ for the weights, i.e., if $w_i^- = 0$ for all i, then
$$w_i = \frac{w_i^+}{w_1^+ + \ldots + w_n^+}.$$

- If we only know the *lower* bounds w_i^- for the weights, i.e., if $w_i^+ = 1$ for all i, then
$$w_i = \frac{n-1}{n - W^-} \cdot w_i^+ + \frac{1}{n - W^-}.$$

4. Proof of the Theorem

1. Let us first make a comment that will be used in the following proof. Due to symmetry (1′), if two of n intervals coincide, i.e., if $\mathbf{w}_i = \mathbf{w}_j$, then the resulting values w_i and w_j must be equal too.

2. We want to prove that the transformation T is described by the formula (3) for all consistent sequences of intervals $\mathbf{w}_i$. To prove it, let us first start by showing that this is true for intervals $\mathbf{w}_i = [w_i^-, w_i^+]$ with *rational* endpoints.

Since all the endpoints are rational, we can reduce them to a common denominator. Let us denote this common denominator by N; then each of the endpoints w_i^- and w_i^+ has the form m/N for a non-negative integer m. Let us denote the corresponding numerators by m_i^- and m_i^+; then, we have $w_i^- = m_i^-/N$ and $w_i^+ = m_i^+/N$ (where $m_i^- = N \cdot w_i^-$ and $m_i^+ = N \cdot w_i^+$).

Each interval $\mathbf{w}_i = [m_i^-/N, m_i^+/N]$ can be represented as a sum of m_i^- degenerate intervals $[1/N, 1/N]$ and $m_i^+ - m_i^-$ non-degenerate intervals $[0, 1/N]$. Totally, we get $m_1^- + \ldots + m_n^- = N \cdot (w_1^- + \ldots + w_n^-) = N \cdot W^-$ degenerate intervals $[1/N, 1/N]$ and $N \cdot (W^+ - W^-)$ non-degenerate intervals $[0, 1/N]$. So, if we know how the transformation T transforms the resulting "long list" of $N \cdot W^- + N \cdot (W^+ - W^-) = N \cdot W^+$ intervals, we will be able to use the property (2′) and find the result of applying T to the original set of intervals.

What is the result of applying T to this long list? This long list contains intervals of two types, and intervals of each type are identical. We have already proven (in Part 1 of this proof) that if two intervals from the list are equal, then the corresponding values of w_i are equal too. Thus:

- the transformation T maps all degenerate intervals $[1/N, 1/N]$ into one and the same value $\alpha \in [1/N, 1/N]$, i.e., into the value $w_i = 1/N$;
- similarly, the transformation T maps all non-degenerate intervals $[0, 1/N]$ into one and the same value; we will denote this value by β.

So, we get the mapping

$$\left(\left[\frac{1}{N}, \frac{1}{N}\right], \ldots, \left[\frac{1}{N}, \frac{1}{N}\right], \left[0, \frac{1}{N}\right], \ldots, \left[0, \frac{1}{N}\right]\right) \to \left(\frac{1}{N}, \ldots, \frac{1}{N}, \beta, \ldots, \beta\right). \quad (6)$$

From the condition that $w_1 + \ldots + w_n = 1$, we conclude that

$$\frac{1}{N} + \ldots + \frac{1}{N} \ (N \cdot W^- \text{ times}) + \beta + \ldots + \beta \ (N \cdot (W^+ - W^-) \text{ times}) = 1,$$

i.e., that

$$\frac{N \cdot W^-}{N} + \beta \cdot N \cdot (W^+ - W^-) = 1;$$

hence,

$$\beta = \frac{1 - W^-}{N \cdot (W^+ - W^-)}. \quad (7)$$

If we apply the property $(2')$ to the formula (6), then we can conclude that

$$(\ldots, \mathbf{w}_i, \ldots) =$$

$$\left(\ldots, \left[\frac{1}{N}, \frac{1}{N}\right] + \ldots + \left[\frac{1}{N}, \frac{1}{N}\right] (m_i^- \text{ times}) + \right.$$

$$\left.\left[0, \frac{1}{N}\right] + \ldots + \left[0, \frac{1}{N}\right] (m_i^+ - m_i^- \text{ times}), \ldots\right) \to$$

$$\left(\ldots, \frac{1}{N} + \ldots + \frac{1}{N} \ (m_i^- \text{ times}) + \beta + \ldots + \beta \ (m_i^+ - m_i^- \text{ times}), \ldots\right) =$$

$$(\ldots, w_i, \ldots),$$

where

$$w_i = m_i^- \cdot \frac{1}{N} + (m_i^+ - m_i^-) \cdot \beta. \quad (8)$$

Substituting the formula (7) instead of β, we conclude that

$$w_i = \frac{m_i^-}{N} + \frac{m_i^+ - m_i^-}{N} \cdot \frac{1 - W^-}{W^+ - W^-}. \quad (9)$$

By definition of the numbers m_i^-, we conclude that $m_i^-/N = w_i^-$ and that $(m_i^+ - m_i^-)/N = (m_i^+/N) - (m_i^-/N) = w_i^+ - w_i^-$. Therefore, (9) takes the form

$$w_i = w_i^- + (w_i^+ - w_i^-) \cdot \frac{1 - W^-}{W^+ - W^-}.$$

Grouping together terms proportional to w_i^-, we conclude that

$$w_i = w_i^- \cdot \left(1 - \frac{1 - W^-}{W^+ - W^-}\right) + w_i^+ \cdot \frac{1 - W^-}{W^+ - W^-}, \tag{10}$$

and finally, subtracting the two fractions in (10), we get the desired result (3).

3. We have shown that the formula (3) holds for all intervals with rational endpoints. Since the transformation T is continuous, and since every interval can be represented as a limit of intervals with rational endpoints, we can conclude, by tending to a limit, that this formula is true for *all* intervals. The theorem is proven.

5. Comment

Sometimes, the interval uncertainty is *fictitious*: e.g., if we have only two participants, and we know the exact weight w_1 for one of them (i.e., $\mathbf{w}_1 = [w_1, w_1]$), then, although we may be given a non-degenerate interval $\mathbf{w}_2$ for the second probability, we know that, due to the equality $w_1 + w_2 = 1$, the only possible value of this second weight is $w_2 = 1 - w_1$. In general, if we have n intervals $[w_i^-, w_i^+]$, $1 \le i \le n$, then, due to the condition $w_1 + \ldots + w_n = 1$, we have $w_i = 1 - (w_1 + \ldots + w_{i-1} + w_{i+1} + \ldots + w_n)$; therefore, the actual value of w_i must lie between $1 - (w_1^+ + \ldots + w_{i-1}^+ + w_{i+1}^+ + \ldots + w_n^+)$ and $1 - (w_1^- + \ldots + w_{i-1}^- + w_{i+1}^- + \ldots + w_n^-)$. As a result, for each i from 1 to n, only the values from the "reduced" interval $\mathbf{w}'_i = [w_i'^-, w_i'^+]$ are possible, where:

$$w_i'^- = \max\left(w_i^-, 1 - \sum_{j \ne i} w_i^+\right) \text{ and } w_i'^+ = \min\left(w_i^+, 1 - \sum_{j \ne i} w_i^-\right). \tag{11}$$

For example, if we start with a sequence $\mathbf{w}_1 = [0, 1]$, $\mathbf{w}_2 = [0, 0.5]$, we get new intervals $\mathbf{w}'_1 = [0.5, 1]$ and $\mathbf{w}'_2 = [0, 0.5]$. Here, the interval $\mathbf{w}'_1$ is narrower than the original interval $\mathbf{w}_1 = [0, 1]$. In such situations, when one of these new intervals is narrower than the original one, this means that a part of the original uncertainty is "fictitious".

There are two possible approaches to such "fictitious" uncertainty:

- In the above text, we assumed that we can have an arbitrary sequence of interval weights which is consistent in the sense of Definition 1. In particular, we may have a sequence $\mathbf{w}_1 = [w_1, w_1]$, $\mathbf{w}_2 = [1 - w_1, 1]$, which includes "fictitious" weights. For this case, the above Theorem justifies the use of formulas (3)–(5).

- Alternatively, we can first *reduce* the original sequence of weight intervals to a new sequence (11), and then apply the formulas (3)–(5) to the resulting sequence $\mathbf{w}'_1, \ldots, \mathbf{w}'_n$. The justification of this second approach is provided by the following: if, in Definition 2, we restrict ourselves only to *reduced* sequences of interval weights, then our proof of the Theorem shows, in effect, that thus restricted mapping is described by the same formulas (3)–(5).

6. From Fair Division Under Interval Uncertainty to Fair Division Under Fuzzy Uncertainty

In the above text, we have described how to come up with a fair division in the situation when for each participant i, we know the *interval* $\mathbf{w}_i$ of i-th possible fair weights. Sometimes, we do not know these intervals either; instead, experts describe what is fair by using *words* from natural language, e.g., by saying that "in a fair distribution, i-th participant should get around one half of the sum". Such words can be naturally formalized by using *fuzzy logic* (see, e.g.,[5]). So, instead of *intervals* $\mathbf{w}_i$ of possible values of the weights, we have *fuzzy numbers* W_i which describe possible portions; how can we then describe a fair division?

To come up with such a fair division scheme, we can use the known fact that a fuzzy number can be naturally described in terms of intervals: namely, to describe a fuzzy number W_i, it is sufficient, for each level $\alpha \in (0,1]$, to describe an α-cut $W_i^{(\alpha)}$ of W_i. For a fuzzy number W_i, this α-cut is an interval[5].

So, for each $\alpha \in (0,1]$, we get a sequence of intervals $W_1^{(\alpha)}, \ldots, W_n^{(\alpha)}$. By applying the above fair division scheme to these intervals, we get a sequence of real numbers $w_1^{(\alpha)}, \ldots, w_n^{(\alpha)}$ for which $\sum\limits_{i=1}^{n} w_i^{(\alpha)} = 1$. We want to transform these values $w_i^{(\alpha)}$, which correspond to different levels α, into a single sequence w_i. One way to do it is to take an *average* over different values of α, i.e., to take

$$w_i = \int_0^1 w_i^{(\alpha)}\, d\alpha.$$

The motivation of taking the average over α is given, e.g., in[7], where it is also shown that this averaging provides a uniform explanation of notion ranging from sigma-count as a measure of cardinality of a fuzzy set to center-of-gravity defuzzification. In our particular application, the additional advantage of taking the average is that since, for each α, we have $w_1^{(\alpha)} + \ldots + w_n^{(\alpha)} = 1$, after integration, we have $w_1 + \ldots + w_n = 1$, i.e., the resulting sequence of real numbers is indeed a division.

This property remains true if, instead of the difficult-to-use theoretical integral formula for w_i, we use a natural approximate formula

$$w_i^{\approx} = \sum_{k=1}^{m} w^{(\alpha_k)} \cdot (\alpha_k - \alpha_{k-1}),$$

where $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_{m-1} < \alpha_m = 1$ are the levels α for which we actually compute the vectors $w_i^{(\alpha)}$. For this natural approximation, from $w_1^{(\alpha)} + \ldots + w_n^{(\alpha)} = 1$, we can also conclude that $w_1^{\approx} + \ldots + w_n^{\approx} = 1$, i.e., that the resulting sequence of approximate weights $w_i^{\approx}$ is also a division.

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