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Interval Mathematics: Algebraic Aspects

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Abstract: Many application-oriented mathematical models deal with real numbers. In real life, due to the inevitable measurement inaccuracy, we do not know the exact values of the measured quantities, we know, at best, the intervals of possible values. It is thus desirable to analyze how the corresponding mathematical results will look if we replace numbers by intervals.

Keywords: interval mathematics

1 Introduction: Why interval mathematics?

Mathematical models normally deal with real numbers. In real life, due to the inevitable measurement inaccuracy, we do not know the exact values of the measured quantities, we know, at best, the intervals of possible values. Thus, with applications in mind, it is desirable to revisit computational formulas and related mathematical results and see what they look like if we use intervals instead of real numbers.

In particular, if we know only the intervals $\mathbf{a}$, $\mathbf{b}$ for two numbers a and b, then the set of possible values of $a + b$ (correspondingly, of $a \cdot b$) also forms an interval. It is natural to call this interval the *sum* $\mathbf{a}+\mathbf{b}$ (corr., product) of the original intervals $\mathbf{a}$ and $\mathbf{b}$.

2 First algebraic aspect: Adding interval data. How do intervals change simple algebraic shapes?

2.1 Simplest shape: (hyper)plane

The simplest possible shape is a *hyperplane,* which can be defined as a solution set for a system of linear equations

$$\sum_{i=1}^{n} a_{ij} \cdot x_j = b_i,$$

i.e., in matrix form, $Ax = b$, where $A = \{a_{ij}\}$, $x = \{x_j\}$, and $b = \{b_i\}$.

2.2 Linear equations with interval coefficients: polytope

A natural idea is to consider the matrix $\mathbf{A} = \{\mathbf{a}_{ij}\}$ and the vector $\mathbf{b} = \{\mathbf{b}_i\}$ with interval coefficients $\mathbf{a}_{ij}$ and $\mathbf{b}_i$. Then, it is natural to define the solution set as the set of all solutions x of the system $Ax = b$ for all $A \in \mathbf{A}$ and all $b \in \mathbf{b}$, i.e., for which $a_{ij} \in \mathbf{a}_{ij}$ and $b_i \in \mathbf{b}_i$ for all i and j.

Definition 2.1. *Let $\mathbf{A}$ be an interval matrix, and let $\mathbf{x}$ be an interval vector. We say that a vector x is a solution for the system $\mathbf{A}x = \mathbf{b}$ of interval linear equations if there exists a matrix $A \in \mathbf{A}$ and a vector $b \in \mathbf{b}$ for which $Ax = b$.*

Theorem 2.1 [1]**.** *For every system of interval linear equations, its set of solutions is a polytope.*

In other words, for every set of interval linear equations, the border of its solution set is piece-wise linear (hyperplanar).

2.3 Symmetric linear equations with interval coefficients: piece-wise quadratic shape

When we consider symmetric interval matrices $\mathbf{A}$, i.e., matrices for which $\mathbf{a}_{ij} = \mathbf{a}_{ji}$, then it is natural to restrict ourselves only to symmetric matrices $A \in \mathbf{A}$.

Definition 2.2. *Let $\mathbf{A}$ be a symmetric interval matrix, and let $\mathbf{x}$ be an interval vector. We say that a vector x is a s-solution for the system $\mathbf{A}x = \mathbf{b}$ of* *interval linear equations if there exists a symmetric matrix $A \in \mathbf{A}$ and a vector $b \in \mathbf{b}$ for which $Ax = b$.*

Theorem 2.2 [2, 3, 6]**.** *For every system of interval linear equations $\mathbf{A}x = \mathbf{b}$, its set of s-solutions has a piece-wise quadratic border.*

In other words, the border of this set consists of finitely many pieces each of which can be described by a quadratic equation:

$$\sum_{i,j} p_{ij} \cdot x_i \cdot x_j + \sum_i q_i \cdot x_i + r = 0.$$

2.4 Linear equations with interval coefficients and general linear dependence between a_{ij} and b_i: arbitrary algebraic sets

The symmetry relation $a_{ij} = a_{ji}$ is a particular case of linear dependence between the coefficients a_{ij}. The corresponding equations $Ax = b$ can be described if we fix the values a_{ij} for $i \le j$, and then take $a_{ji} = a_{ij}$.

Symmetry is just one of such relations. It is natural to consider other relations such as antisymmetry, etc. In general, we arrive at the following definition:

Definition 2.3. *By a system of interval linear equations with dependent coefficients, we mean a system of the type*

$$\sum_{j=1}^{n} a_{ij} x_j = b_i,$$

where

$$a_{ij} = a_{ij}^{(0)} + \sum_{\alpha=1}^{p} a_{ij\alpha} f_\alpha,$$

$$b_i = b_i^{(0)} + \sum_{\alpha=1}^{p} b_{i\alpha} f_\alpha,$$

$a_{ij}^{(0)}$, $a_{ij\alpha}$, $b_i^{(0)}$, and $b_{i\alpha}$ are given real numbers, ($1 \le i \le m$, $1 \le j \le n$, $1 \le \alpha \le p$), and coefficients f_α can take arbitrary values from the given intervals $\mathbf{f}_\alpha$.

Definition 2.4. *We say that a vector x is a solution to a given system of linear interval equations with dependent coefficients if it solves a system $Ax = b$ for some values $f_\alpha \in \mathbf{f}_\alpha$.*

To describe the set of such solutions, we must recall two notions: the notion of a semialgebraic set and the notion of a projection:

Definition 2.5. *A set is called semialgebraic if it can be represented as a finite union of subsets, each of which is defined by a finite system of polynomial equations $P_r(x_1, \ldots, x_q) = 0$ and inequalities of the types $P_s(x_1, \ldots, x_q) > 0$ and $P_t(x_1, \ldots, x_q) \ge 0$ (for some polynomials P_i).*

Definition 2.6. *For every set $S \in R^n$, and for every subset $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, n\}$, by a it projection $\pi_I(S)$ we mean the set of all vectors $(x_{i_1}, \ldots, x_{i_q}) \in R^q$ that can be extended to a vector $(x_1, \ldots, x_n) \in S$.*

It is known that a projection of a semialgebraic set is also semialgebraic. Now, we are ready for the main results:

of a system, is also semialgebraic, and that, vice It is known that for every semialgebraic set S, and for every subset $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, n\}$, the corresponding *projection* of S, i.e., the set of all vectors $(x_{i_1}, \ldots, x_{i_q}) \in R^q$ that can be extended to a solution $(x_1, \ldots, x_n)$ of a system, is also semialgebraic, and that, vice

Theorem 2.3 [4]**.** *For every system of interval linear equations with interval coefficients, the set of its solutions is semialgebraic.*

Theorem 2.4 [4]**.** *Every semialgebraic set can be represented as a projection of the set of all solutions of a system of interval linear equations with interval coefficients.*

In this representation, we allow intervals $\mathbf{f}_\alpha$ to be arbitrarily wide. In terms of measurements, wide intervals correspond to low measurement accuracy. It is natural to ask the following question: if we only consider narrow intervals, which correspond to high measurement accuracy, will we still get all possible semialgebraic shapes or a narrower class of shapes?

Definition 2.7 [10]**.** *For a given $\delta > 0$, an interval $\mathbf{x} = [\tilde{x} - \Delta, \tilde{x} + \Delta]$ is called absolutely δ-narrow if $\Delta \le \delta$, and is relatively δ-narrow if $\Delta \le \delta \cdot |\tilde{x}|$.*

Theorem 2.5 [5]**.** *For every $\delta > 0$, every semialgebraic set can be represented as a projection of the solution set of some system of interval linear equations with dependent coefficients, whose intervals are both absolutely and relatively δ-narrow.*

3 Second algebraic aspect: Adding interval operations Interval + rational = algebraic

3.1 Rational functions

The set of rational functions can be defined as the smallest set of functions that contains constants and variables and is closed under arithmetic operations $(+, -, \cdot, /)$.

3.2 What happens if we add intervals: a question

A natural question is: what class of functions will be get if we add to this list the "interval" operation, i.e., an operation that transforms a function f of n variables and given intervals $\mathbf{x}_1, \ldots, \mathbf{x}_n$ into the bounds for the range

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) =$$

$$\{f(x_1, \ldots, x_n) \mid x_1 \in \mathbf{x}_1, \ldots, x_n \in \mathbf{x}_n\}.$$

3.3 Definitions and the main result

Definition 3.1. *By interval-rational functions of several real variables $x_1, ..., x_n$, we mean functions from the following class:*

- *Constants and variables x_i themselves are interval-rational functions.*
- *If f and g are interval-rational functions, then $f + g$, $f - g$, $f \cdot g$, and f/g are interval-rational functions.*
- *If $f(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is an interval-rational function, and $\mathbf{y}_1, \ldots, \mathbf{y}_m$ are intervals, then*
  $$g(x_1, \ldots, x_n) =$$
  $$\sup_{y_1 \in \mathbf{y}_1, \ldots, y_m \in \mathbf{y}_m} f(x_1, \ldots, x_n, y_1, \ldots, y_m)$$
  is an interval-rational function.
- *If $f(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is an interval-rational function, and $\mathbf{y}_1, \ldots, \mathbf{y}_m$ are intervals, then*
  $$g(x_1, \ldots, x_n) =$$
  $$\inf_{y_1 \in \mathbf{y}_1, \ldots, y_m \in \mathbf{y}_m} f(x_1, \ldots, x_n, y_1, \ldots, y_m)$$
  is an interval-rational function.
- *Only functions that are obtained by these operations are interval-rational functions.*

To describe the class of all interval-rational functions, we must recall the definition of an algebraic function:

Definition 3.2. *An analytic function $y = A(x_1, \ldots, x_n)$ is called algebraic if*

$$P(x_1, \ldots, x_n, A(x_1, \ldots, x_n)) = 0$$

for some polynomial $P(x_1, \ldots, x_n, x_{n+1})$ for which the partial derivative $\partial P/\partial x_{n+1}$ is not identically 0 ($\partial P/\partial x_{n+1} \neq 0$). We say that the corresponding polynomial $P(x_1, \ldots, x_{n+1})$ defines the algebraic function $A(x_1, \ldots, x_n)$.

Definition 3.3. *Assume that $\mathcal{V}$ is an open domain in R^n. We say that an algebraic function $A : \mathcal{V} \to R$ can be locally represented by an interval-rational function if for almost every point $(y_1, \ldots, y_n) \in \mathcal{V}$ there exists a neighborhood $\mathcal{U}$ and an interval-rational function $f(x_1, \ldots, x_n)$ such that for all $x = (x_1, \ldots, x_n)$ from $\mathcal{U}$, $f(x) = A(x)$.*

Here, *almost every* is understood in the usual mathematical sense: a property is true for *almost every point* if the set of all points in which it is not true has Lebesgue measure 0.

Theorem 3.1 [8]**.** *Every interval-rational function is algebraic.*

Theorem 3.2 [8]**.** *Every algebraic function can be locally represented by an interval-rational function.*

In other words, if we add an interval substitution operation to the original list of algebraic operations describing rational functions, we get an arbitrary algebraic function.

3.4 A more computer-realistic notion of "almost all" and the corresponding result

In mathematics, "almost all" usually means "all points, except for points from a set of Lebesgue measure 0" (or, "except for points from a set of a small Lebesgue measure"). In the existing computers, however, only rational numbers are represented. The set of all rational numbers is countable and has, therefore, Lebesgue measure 0; so the standard mathematical notion of "almost all" is not very computer-realistic.

In real life, when we say that "an algorithm is applied to *real* numbers $x_1, \ldots, x_n$", we usually mean that this algorithm is applied to *rational* numbers $r_1, \ldots, r_n$ that are $\eta-$close to $x_1, \ldots, x_n$, where η

is the computer precision. So, if we fix $\eta > 0$, we can say that an algorithm A works for n real numbers $x_1, \dots, x_n$ if it works fine for all tuples of rational numbers $(r_1, \dots, r_n)$ that are $\eta-$close to $(x_1, \dots, x_n)$.

Now, we have *real-valued* inputs on which the algorithm works fine, and real-valued inputs on which it does not. For real-valued inputs, we can apply Lebesgue measure.

Definition 3.4.

- *Let $\eta > 0$ be a real number. We say that points $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in R^n$ are $\eta-$close if $|x_i - y_i| \le \eta$ for all i.*
- *Let $E \subset R^n$ be a bounded set, and let $\eta > 0$ be a real number. We say that a point $x \in R^n$ $\eta-$possibly belongs to E if there exists a point y that is $\eta-$close to x and that belongs to E.*
- *We say that a bounded set $E \subset R^n$ is $(\eta, \varepsilon)-$small if the set of all points that $\eta-$possibly belong to E has Lebesgue measure $\le \varepsilon$.*
- *We say that a bounded set E is small if for every ε, there exists a η for which the set E is $(\eta, \varepsilon)-$small.*
- *We say that a set $E \subseteq R^n$ (not necessarily bounded) is small if for every $\Delta > 0$, the intersection $E \cap [-\Delta, \Delta]^n$ is small.*
- *We say that a property $P(x)$ holds for computer-realistically almost every x if the set $\{x \mid \neg P(x)\}$ of all x for which P is false is small.*

If a property $P(x)$ is almost always true in this sense, this means, crudely speaking, that for any given ε, we can choose a computer precision η so that for all x except for a set of measure $\le \varepsilon$, we can *guarantee* that $P(x)$ is true even when we only know components x_i with precision η.

We can now reformulate the above Definition 3.3 and Theorems 3.1 and 3.2 in terms of this new computer-realistic definition of "almost all".

Definition 3.5. *Assume that $\mathcal{V}$ is an open domain in R^n. We say that an algebraic function $A : \mathcal{V} \to R$ can be computer-realistically locally represented by an interval-rational function if for computer-realistically almost every point $(y_1, \dots, y_n) \in \mathcal{V}$, there exists a neighborhood $\mathcal{U}$ and an interval-rational function $f(x_1, \dots, x_n)$ such that for all x from $\mathcal{U}$, $f(x) = A(x)$.*

Theorem 3.3 [9]**.** *Every interval-rational function is algebraic.*

Theorem 3.4 [9]**.** *Every algebraic function can be locally represented by an interval-rational function.*

4 Third algebraic aspect: algebraic generalizations of intervals

4.1 Multi-D analogues of intervals: why?

What are the natural multi-D analogues of intervals? One of the main goals of science is to predict what will happen in the world. In other words, to predict the values of different physical quantities. Traditional prediction algorithms take as input the current values $x_1, \dots, x_v$ of some physical variables, and use these value to predict the future state of the Universe. For example, in weather prediction, we take as input the values $x_1, \dots, x_v$ of the meteorological parameters in different points.

There are two main ways to find the values of x_i: by directly measuring them, and by using the ability of an expert to give a reasonable estimate. Measurements are never 100% accurate. Expert estimate are also only approximate. As a result, we never know the exact values of the desired physical quantities $x_1, \dots, x_v$. There are usually several different possibilities that are all consistent with all our measurements and expert estimates.

For example, if the measured value $\tilde{x}_1$ of a temperature x_1 is 35, and the accuracy of this measurement is ± 5, this means that the actual value of x_1 can be equal to any number from the interval $[30, 40]$.

For these different possibilities, the predicting algorithm will lead to slightly different results. Therefore, instead of a *single* predicted result, we get the *set* of possible future values.

Each measurement bring an additional restriction on the set X of all possible values of $x = (x_1, \dots, x_v)$. As a result, the more measurements we take, the more complicated the shape of this set X can be. But we need to process this set in order to get the set of possible future values, and processing odd-shaped sets is computationally very complicated. So, we need to *approximate* these sets by sets belonging to some pre-chosen family of simple sets (e.g., enclose X by a ball, or by an ellipsoid, or by a parallelepiped). What family should we choose?

4.2 The necessity to check consistency: motivations

Measuring devices can go wrong; expert estimates can be wrong. As a result, different conditions on x may turn out to be inconsistent. So, before we start processing, it would be nice to check the existing knowledge for possible inconsistencies.

In view of this necessity, it is desirable to choose the family of approximating sets in such a way that this check will not be computationally very complicated.

Let's formulate this necessity in mathematical terms. Assume that we have several pieces of knowledge about x. Each piece of knowledge can be formulated as a set of all the vectors x that are consistent with this particular knowledge. So, we instead of saying that we have several pieces of knowledge, we can say that we have several sets $A, B, \ldots, C \subseteq R^v$. Consistency means that it is possible that a certain value $x \in R^v$ satisfies all these properties, i.e., belongs to all these sets $A, B, \ldots, C$ (in other words, that $A \cap B \cap \ldots \cap C \neq \phi$).

How can we actually check consistency? Knowledge usually comes piece after piece, so a typical situation is as follows. We already have a consistent knowledge base. In other words, we have the pieces of knowledge represented by sets $A_1, \ldots, A_s$, and these pieces of knowledge are consistent. Then, a new piece of knowledge arrives, described by a set A.

A natural way to check consistency is to check whether A is consistent with each of the existing sets A_i. The consistency between A and each of A_i is, of course, necessary for the new knowledge base $\{A_1, \ldots, A_s, A\}$ to be consistent. It is easy to see, however, that this comparison is not always sufficient (the examples of why it is not always so can be easily extracted from the following text). So, we arrive at the following definition:

4.3 Definitions, results, and hypotheses

Definition 4.1. *We say that a family $\mathcal{S}$ of sets allows checking consistency if the following is true: For every s, and for every tuple of sets $A_1 \in \mathcal{S}, \ldots, A_s \in \mathcal{S}, A \in \mathcal{S}$, for which the sets $A_1, \ldots, A_s$ are consistent (i.e., $A_1 \cap \ldots \cap A_s \neq \phi$) and A is consistent with all A_i (i.e., $A \cap A_i \neq \phi$ for all $i = 1, \ldots, s$), all $s+1$ sets $A_1, \ldots, A_s, A$ are consistent (i.e., $A_1 \cap \ldots \cap A_s \cap A \neq \phi$).*

We are interested in measurements, so it is natural to assume that if M is a reasonable approximation to sets that describe the uncertainty of our knowledge, then a set $M + a$ (that is obtained from M by a translation) is also a reasonable approximation. For example, assume that we are measuring time, and as a result we get 35 ± 5 (i.e., a set $M = [30, 40]$). If we now change the starting point for measuring time, e.g., take -5 as the new starting point, then the same result will be expressed as $M + 5 = 40 \pm 5 = [35, 45]$. This new set $M + 5$ must also be a reasonable approximation.

Definition 4.2. *We say that a family $\mathcal{S}$ of sets is translation-invariant if for every $M \in \mathcal{S}$, and for every $a \in R^v$, the set $M + a$ also belongs to $\mathcal{S}$; this set $M + a$ is called a translate of M.*

Theorem 4.1 [11, 12]**.** *Translates of a compact convex set M allow checking consistency if and only if M is a parallelepiped.*

So, a convex compact set M can be a reasonable representation of uncertainty if and only if M is a parallelepiped. This result explains why parallelepipeds are often used to describe uncertainty.

This result is based on the assumption that M is convex. However, we often have knowledge that does not correspond to a convex set. E.g., if we have measured the value of the velocity as 10 ± 1 (i.e., the possible values form an interval $[9, 11]$), and this is the only information that we have about the motion, then the set of all possible values of the velocity components v_1, v_2, v_3 forms a (non-convex) "slice" between two spheres (of radii 9 and 11).

So, we arrive at the following open problem:

Open problem. *To describe sets for which translates allow checking consistency.*

There exist non-convex compact sets with this property: e.g., translates of $M = \{0, 1\} \subseteq R^1$ allow checking consistency. However, all such examples that we have constructed so far are disconnected. Therefore, we can formulate the following hypothesis:

Hypothesis 4.1. *If translates of a compact connected set M allow checking consistency, then M is a parallelepiped.*

To help to solve our open problem, let's reformulate it in terms close to those of a well-developed field of math: namely, of homological algebra.

Definition 4.4. *Assume that a set $M \subseteq R^v$ is given, and a positive integer s is fixed. For every*

n, $n-$cochains will be defined as (completely) antisymmetric functions f from $\{1,\ldots,s\}^n$ to a certain subset $M_n \subseteq R^v$. A coboundary operator δ^n transforms an $n-$cochain into an $(n+1)-$cochain as follows:

$$(\delta^n f)(i_0,\ldots,i_n) =$$

$$\sum_{k=0}^{n}(-1)^k f(i_0,\ldots,i_{k-1},\hat{\imath}_k,i_{k+1},\ldots,i_n),$$

where $\hat{\imath}_k$ means that we are skipping $k-th$ variable. We define $M_0 = M_1 = 1$, and $M_{n+1} = M_n - M_n + \ldots$ ($n+1$ times), so that $M_2 = M_1 - M_1 = M - M$.

Comment. This definition is similar to standard definitions from group cohomologies, with the only difference that in our case, the range of the functions defined as cochains is not a group, it is a *subset* of an additive group R^v. Because of that, we have to be careful in our definition to guarantee that the coboundary of a $n-$cochain will be an $(n+1)-$cochain. This is indeed guaranteed by our choice of M_n. Similarly to traditional cohomologies, it is easy to check that δ^n is really a coboundary operator:

Theorem 4.2 [11, 12]**.** *For every $n-$cochain f, $\delta^{n+1}(\delta^n f) = 0$.*

Definition 4.5. *An $n-$cochain is called an $n-$coboundary if it is a coboundary $\delta^{n-1}g$ of some $(n-1)-$cochain g. An $n-$cochain is called an $n-$cocycle if $\delta^n f = 0$.*

Comment. From Theorem 4.2, it follows that *every $n-$coboundary is an $n-$cocycle*. It turns out that the inverse statement is true if and only if translates of M allow checking consistency:

Theorem 4.3 [11, 12]**.** *Let $M \subseteq R^v$. Then, translates of M allow checking consistency if and only if every $2-$cocycle is a $2-$coboundary.*

Comment. In traditional homology theory, both the set of all $n-$cocycles Z^n and the set of all $n-$coboundaries B^n are abelian groups. Since $B^n \subseteq Z^n$, in this traditional theory, the condition that every 2-cocycle is a 2-coubounbdary can be reformulated as $H^2 = 0$, where the factor-group $H^2 = Z^2/B^2$ is called the *second cohomology group*.

4.4 Interval operations in an arbitrary ring

An even more general idea is to consider intervals in an arbitrary partially ordered ring. For such rings, unlike the ring of real numbers, the element-wise sum and the product of intervals is not always an interval. For example, on the ring of all integers, the element-wise product

$$[\underline{a}_1,\overline{a}_1]\cdot[\underline{a}_2,\overline{a}_2] =$$

$$\{a_1\cdot a_2 \mid a_1 \in [\underline{a}_1,\overline{a}_1], a_2 \in [\underline{a}_2,\overline{a}_2]\}$$

of the intervals $[1,2] = \{1,2\}$ and $[1,3] = \{1,2,3\}$ is equal to $[1,2]\cdot[1,3] = \{1,2,3,4,6\}$; this product is *not* an integer-interval. When is it an interval?

4.5 Definitions and the Main Result

Definition 4.5. *A ring K is a set of elements with two binary operations $+$ and $\cdot$ (called addition and multiplication) that satisfies the following three properties:*

- *K is an Abelian (commutative) group under addition;*
- *multiplication is associative;*
- *the right- and left-distributive laws hold, i.e.,*

$$a\cdot(b+c) = a\cdot b + a\cdot c$$

and

$$(a+b)\cdot c = a\cdot c + b\cdot c.$$

We say that a ring has no (proper) divisors of zero if from $a\cdot b = 0$, it follows that $a = 0$ or $b = 0$.

Definition 4.6. *By an ordered ring, we mean a ring K with a partial order $<$ such that:*

- *there exists an element a for which $a > 0$;*
- *if $a < b$, then for every $c \in K$, we have $a + c < b + c$;*
- *if $a > 0$ and $b > 0$, then $a\cdot b > 0$;*

An order is called consistent if the following two properties hold:

- *if $a > 0$ and $a\cdot b > 0$ then $b > 0$;*
- *if $b > 0$ and $a\cdot b > 0$ then $a > 0$.*

An order is called linear or total if for every two elements a and b, either $a < b$, or $a = b$, or $a > b$.

Comment. In the following text, we will only consider consistently ordered rings with no divisors of zero.

Definition 4.7. *Let K be an ordered ring with no divisors of 0. By an interval, we mean a set $[\underline{a}, \overline{a}] = \{a \in K \mid \underline{a} \le a \le \overline{a}\}$, where $\underline{a} \le \overline{a}$. The element $\underline{a}$ is called the lower endpoint of this interval, and the element $\overline{a}$ is called its upper endpoint. The sum and product of two intervals are defined element-wise.*

We want to reformulate the desired property — that the element-wise sum and element-wise product of two intervals are always intervals — in purely algebraic terms.

This property is true for the ring of real numbers and it is not true for the ring of integers. From the algebraic viewpoint, the main difference between these two rings is as follows:

- in the ring of real numbers, multiplication is *invertible* in the sense that for every two real numbers $x \ne 0$ and a, there exists a number q ($q = a/x$) for which $x \cdot q = a$;
- on the other hand, multiplication in the ring of all *integers* is *not* invertible: for example, $2 \ne 0$, but there exists no integer q for which $2 \cdot q = 3$.

In view of this difference, it is natural to conjecture that the desired interval property is equivalent to invertibility of multiplication. This first guess turns out to be wrong: there are examples of ordered rings with the above interval property in which multiplication is not invertible. These example enable to correct the original conjecture into an exact theorem, according to which the interval property is equivalent to a special property called *almost invertibility.*

To introduce this new notion, let us first recall the formal definition of *invertibility*:

Definition 4.8. *Let K be a ring. We say that multiplication is invertible if for every two elements a and $x \ne 0$, there exist:*

- *an element q_l for which $q_l \cdot x = a$; and*
- *an element q_r for which $x \cdot q_r = a$.*

If a ring has a unit element e (i.e., with an element for which $e \cdot a = a \cdot e = a$ for all a), then a ring is invertible if and only if it is a *field* (commutative or non-commutative). In some algebra textbooks, a field is defined as a *commutative* ring with a property that non-zero elements form a group. In such books, a *non-commutative* ring with this property is called a *skew field*, or a *s-field*.

Not every ring is invertible (for example, as we have already mentioned, the ring of all integers is not invertible). There can be two reasons for that:

- In some cases, multiplication is not invertible in the *original* ring, but we can still easily *add intermediate* elements and make it invertible. For example, in the ring of integers, we do not have a number q for which $2 \cdot q = 3$, but we have $2 \cdot 1 < 3$ and $2 \cdot 2 > 3$, so, we can add an extra element 3/2 for which $1 < 3/2 < 2$. If we add all such elements, we will thus expand the ring of integers to the field of all rational numbers.
- It can also be that an element x is "infinitely smaller" than a in the sense that whatever element q we take, we always have $x \cdot q < a$.

Rings in which this second reason is the only obstacle to invertibility will be called *almost invertible*:

Definition 4.9. *Let K be an ordered ring.*

- *We say that an element $x \ne 0$ is left-infinitely smaller than an element a (and denote it by $x \ll_l a$), if $y \cdot x < a$ for all $y \in K$.*
- *We say that an element $x \ne 0$ is right-infinitely smaller than an element a (and denote it by $x \ll_r a$), if $x \cdot y < a$ for all $y \in K$.*

Definition 4.10. *Let K be an ordered ring. We say that multiplication is almost invertible (and, respectively, that the ring K is almost invertible) if for every a and $x \ne 0$, the following two properties hold:*

- *if x is not left-infinitely smaller than a, then there exists an element q for which $q \cdot x = a$.*
- *if x is not right-infinitely smaller than a, then there exists an element q for which $x \cdot q = a$.*

Now, we are ready to formulate the main result of this section:

Theorem 4.4 [7]. *For every consistently ordered ring K with no divisors of 0, the following two conditions are equivalent to each other:*

i) *The product of every two intervals is also an interval.*

ii) *The ring K is totally (linearly) ordered and almost invertible.*

If one of these conditions is satisfied, then the sum of every two intervals is also an interval.

For the rings in which there are no infinitely smaller elements, this result can be further simplified:

Definition 4.11. *We say that a ring is weakly Archimedean if none of its elements is left- or right-infinitely smaller than any other.*

Many rings are weakly Archimedean, including the ring of all real numbers, the ring of all integers, and many others. A weakly Archimedean ring is almost invertible if and only if it is invertible. Hence, we get the following corollary:

Corollary 4.1 [7]**.** *For every weakly Archimedean consistently ordered ring K with no divisors of 0, the following two conditions are equivalent to each other:*

i) *The product of every two intervals is also an interval.*

ii) *The ring K is linearly ordered and invertible.*

If one of these conditions is satisfied, then the sum of every two intervals is also an interval.

We have already mentioned that a ring with a unit element is invertible if and only if it is a field. Hence, we get the following second corollary:

Corollary 4.2 [7]**.** *For every weakly Archimedean consistently ordered ring K with a unit element e and with no divisors of 0, the following two conditions are equivalent to each other:*

i) *The product of every two intervals is also an interval.*

ii) *The ordered ring K is a linearly ordered field.*

If one of these conditions is satisfied, then the sum of every two intervals is also an interval.

In short, this result says that for interval arithmetic to be possible in a ring, this ring has to be a *field*. It is worth mentioning that, somewhat in contrast to this result, the intervals themselves do not form a field; for example, the set of intervals does not even have a distributivity property.

4.6 Auxiliary results

We have found the conditions under which *both the product and the sum* of the two intervals form an interval. It may be interesting to consider the case when *only addition* is defined, and find out when, in this case, the sum of any two intervals is also an interval.

Definition 4.12. *By an ordered (Abelian) group, we mean an Abelian group G with a partial order $<$ such that:*

- *there exists an element a for which $a > 0$;*
- *if $a < b$, then for every $c \in K$, we have $a + c < b + c$.*

To formulate our result, we will need the following property:

Definition 4.13. *We say that an order is* 2-2-*separating if for every four elements $\underline{a}, \underline{b}, \overline{a}, \overline{b}$, if each of the two elements $\underline{a}$ and $\underline{b}$ is smaller than or equal to each of the elements $\overline{a}, \overline{b}$ (i.e., if $\underline{a} \le \overline{a}$, $\underline{a} \le \overline{b}$, $\underline{b} \le \overline{a}$, and $\underline{b} \le \overline{b}$), then there exist an element c that lies in between, i.e., for which $\underline{a} \le c \le \overline{a}$ and $\underline{b} \le c \le \overline{b}$.*

Comments.

- This definition is close to the notion of TR(2,2) (Riesz) ordered groups.
- Every group in which an order forms a lattice has this property.
- If a group G is a compact topological group in which the order is (in some reasonable sense) consistent with topology, then this property is equivalent to G being a lattice.

Theorem 4.5 [7]**.** *Let G be an ordered Abelian group. Then, the sum of every two intervals is also an interval if and only if the order on the group G is* 2-2-*separating.*

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