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Multiplicative Riesz Decomposition on the Ring of Matrices over a Totally Ordered Field

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Multiplicative Riesz Decomposition on the Ring of Matrices over a Totally Ordered Field

by

JULIO CÉSAR URENDA CASTAÑEDA, B.S.

THESIS

Presented to the Faculty of the Graduate School of

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Abstract

The Riesz Decomposition Theorem for lattice ordered groups asserts that when G is an ℓ -group and when a nonnegative element a is bounded by a product of nonnegative elements b_1, \dots, b_n , then a can be decomposed into a product of nonnegative elements b'_1, \dots, b'_n , i.e., $a = b'_1 \cdot \dots \cdot b'_n$, with the property that $b'_i \leq b_i$ for any $i = 1, \dots, n$. In this work we characterize all nonnegative matrices for which this decomposition is possible with respect to *matrix multiplication*. In addition, we show that this result can be applied to ordered semigroups.

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Chapter 1

Introduction

This work extends the Riesz Decomposition Theorem from lattice-ordered groups to lattice-ordered rings, mainly to the lattice-ordered ring of matrices over a totally ordered field. To this end a brief introduction to the theory of lattice-ordered sets, matrix algebra, lattice-ordered groups and lattice-ordered rings will be given.

1.1 Conventions

From now on, we will denote the set of all $m \times n$ matrices over the field \mathbb{F} by $\mathcal{M}_{m \times n}(\mathbb{F})$, and the elements of this set will be denoted by A, B, C, \dots , while the ij^{th} entry of a matrix A will be denoted a_{ij} . In addition, the s^{th} column and the c^{th} row of a matrix A will be denoted by A_s and A^c , respectively.

1.2 Ordered Sets

To begin the study of ordered structures, we have to recall the notion of a partially-ordered set.

Definition 1.2.1. (see, e.g., [6]) A *partially ordered set* (P, \leq) is a set P together with a binary relation \leq on P such that, for all $x, y, z \in P$,

1. $x \leq x$,
2. $x \leq y$ and $y \leq x$ imply $x = y$,
3. $x \leq y$ and $y \leq z$ imply $x \leq z$.

The following examples are standard. For additional information see [6].

Example 1.2.1. The set of all integers, \mathbb{Z} , with \leq , the usual order relation.

Example 1.2.2. The set of all subsets of a set X , i.e., the power set of X , 2^X , ordered by inclusion.

The Examples 1.2.1 and 1.2.2 have additional properties to those described in the definition of a partially-ordered set; namely, for any pair of elements there exists a least upper bound and a greatest lower bound.

Definition 1.2.2. (see, e.g., [6]) Let P be a partially-ordered set and let $x, y \in P$, we say that $s \in P$ is a *least upper bound (supremum)* of x and y if $s \geq x, y$ and for all $u \in P$ such that $x, y \leq u$ then $s \leq u$. A *greatest lower bound (infimum)* is defined dually i.e., replacing every \leq by \geq .

Definition 1.2.3. (see, e.g., [6]) A partially-ordered set L is called a *lattice* if for any $x, y \in L$ a greatest lower bound and a least upper bound of x, y exist (and are in L). They are denoted by $x \wedge y$ and $x \vee y$, respectively.

Under this definition we have the following examples of lattices.

Example 1.2.3. The power set, 2^X of a set X , ordered by inclusion, with $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.

Example 1.2.4. The natural numbers, \mathbb{N} , ordered by divisibility, ($m \preceq n$ if and only if $m|n$) with $m \vee n = \text{lcm}\{m, n\}$ and $m \wedge n = \text{gcd}\{m, n\}$.

Not all partially ordered sets are lattices, as illustrated by the following.

Example 1.2.5. $A = \{a, b, c\}$ and $\leq = \{(a, a), (b, b), (c, c), (a, b), (a, c)\}$, then we can see that (A, \leq) is a partially ordered set, but $b \vee c$ does not exist.

1.3 Ordered Algebraic Structures

A standard reference for basic concepts in ordered algebraic structures is [7].

Definition 1.3.1. A set G which is both a group and a partially-ordered set is a *partially-ordered group* if for all $a, b, c \in G$, $a \leq b$ implies that $ac \leq bc$ and $ca \leq cb$ i.e., the order is compatible with the operation of the group.

Below are a few examples.

Example 1.3.1. The additive group of real numbers \mathbb{R} with the usual order.

Example 1.3.2. The set $\mathbb{R}^{\mathbb{R}}$ with the usual addition of functions and $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$.

Definition 1.3.2. A partially ordered group G on which the underlying order is a lattice is a *lattice-ordered group* or ℓ -group.

Example 1.3.3. (Holland) $A(\Omega)$ the set of all order preserving permutations on a totally ordered set Ω ordered point-wise i.e., $\alpha \leq \beta$ ($\alpha, \beta \in A(\Omega)$) if $\alpha(x) \leq \beta(x)$ for all $x \in \Omega$. Comment: Holland's theorem says that every ℓ -group can be embedded into $A(\Omega)$ for some totally-ordered set Ω (see, e.g, [5] or [10]).

Definition 1.3.3. Let G be a partially ordered group and let e be the identity of G . We define G^{\geq} to be the set of all nonnegative elements of G , in symbols

$$G^{\geq} = \{x \in G \mid x \geq e\}.$$

Definition 1.3.4. A *partially-ordered ring* is a ring R together with a partial order \leq , which makes $(R, +, \leq)$ a partially order abelian group for which for all $x, y \geq 0$ we have $xy \geq 0$.

Definition 1.3.5. A partially-ordered ring R on which the underlying order is a lattice is a *lattice-ordered ring* or ℓ -ring.

Our main example is going to be

Example 1.3.4. The ring of all $n \times n$ matrices over the reals, $\mathcal{M}_{n \times n}(\mathbb{R})$, ordered entry-wise, i.e., $A \leq B$ if and only if $A_{ij} \leq B_{ij}$ for all i, j is an ℓ -ring, see, e.g., [1] or [3].

Chapter 2

The Question About the Riesz-type Decomposition

In this chapter we are going to address the question about a certain Riesz-type decomposition in the framework of two ordered algebraic structures, provide several examples of decompositions in various structures, as well as establish some theorems regarding the decomposition question.

In general, the question about the decomposition is the following. In a partially-ordered algebraic structure A with a binary operation \bullet on A , a special element $e \in A$, if $e \leq a, b_1, \dots, b_k \in A$ with $a \leq b_1 \bullet \dots \bullet b_k$, are there elements $b'_1, \dots, b'_k \geq e$ with $b'_i \leq b_i$ for all $i = 1, \dots, k$ such that $a = b'_1 \bullet \dots \bullet b'_k$?

2.1 Decomposition in Partially Ordered Groups

Example 2.1.1. Let $G = \mathbb{Z} \times \mathbb{Z}$ with $(a, b) \leq (c, d)$ if and only if $(c, d) - (a, b) = (0, n)$ for some $n \geq 0$. We can see that this relation is a partial order on G , but G is not a lattice order since $(0, 0) \wedge (1, 0)$ does not exist. However, the additive decomposition in question can be performed, see e.g., [9], [14], or [16].

Example 2.1.2. Let N be the set of all nonsingular 2×2 matrices, with the order as in Example 1.3.4. We can see that N is a partially ordered group under matrix multiplication; however, the multiplicative decomposition cannot be performed always as the following

illustrates

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If such decomposition exists we would get

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}.$$

With $ad = 1$, $ae = 0$, $bd = 0$ and $be + cf = 2$, which implies that $a \neq 0 \neq d$, $e = 0$, $cf = 2$.

But $0 \leq c, f \leq 1$, contradiction.

2.2 Decomposition in Lattice Groups

The following theorem, which answers the decomposition question in the context of ℓ -groups can also be found in [3].

Theorem 2.2.1. *Let G be a lattice-ordered group, and let $a, b_1, \dots, b_n \in G^{\geq}$ such that*

$$a \leq \prod_{i=1}^n b_i,$$

then there exist $b'_1, \dots, b'_n \in G^{\geq}$ such that $b'_i \leq b_i$ for all i and

$$a = \prod_{i=1}^n b'_i.$$

Proof. By induction on n . If $n = 1$ let $b'_1 = a$ and we have $b'_1 \leq b_1$ and $a = b'_1$. For the inductive step, assume that $b_1, \dots, b_n, b_{n+1} \in G^{\geq}$ and that $a \leq \prod_{i=1}^{n+1} b_i$. Then $e \vee ab_{n+1}^{-1} \leq \prod_{i=1}^n b_i$ and so by the induction hypothesis, there are $b'_1, \dots, b'_n \in G^{\geq}$ with $b'_i \leq b_i$ for all i such that

$$a(a \wedge b_{n+1})^{-1} = e \vee ab_{n+1}^{-1} = \prod_{i=1}^n b'_i.$$

Then, $a = \prod_{i=1}^{n+1} b'_i$ where $b'_{n+1} = a \wedge b_{n+1} \leq b_{n+1}$. □

Example 2.2.1. Let $G = \mathbb{Z}$ with addition and the usual order relation, and let us take a look at $5 \leq 3 + 3 + 2$, in this case we can see that, for example $5 = 1 + 2 + 3$.

2.3 Decomposition in Lattice-Ordered Rings

In case of ℓ -rings the decomposition question can be asked with respect to the multiplication in an similar way. Such decomposition will be called Riesz multiplicative decomposition. In the case of partially-ordered σ -Dedekind complete algebras, the decomposition was introduced by T. Dai in [4].

Example 2.3.1. Let us consider the ring of integers, \mathbb{Z} , with the usual addition, multiplication and natural order. We can see that $0 \leq 2, 2, 3$ and $3 \leq 2 \cdot 2$, but this product is not decomposable. On the other hand, $6 \leq 3 \cdot 3$ can be decomposed into $6 = 3 \cdot 2$.

Example 2.3.2. ([11] Theorem 3.16) Let X be a Hausdorff topological space, and $C(X)$ the usual real algebra of all real continuous functions on X . Then for every two nonnegative functions $f, g \in C(X)$ and a nonnegative function $h \in C(X)$ such that $h \leq fg$ there exists functions $f', g' \in C(X)$, $0 \leq f' \leq f$, $0 \leq g' \leq g$ such that $h = f'g'$. Therefore in $C(X)$ the multiplicative Riesz property is always possible.

In addition we have:

Theorem 2.3.1. *For a totally ordered division ring D the multiplicative Riesz decomposition is always possible.*

Proof. Let $0 \leq a, b_1, \dots, b_n \in D$ with $a \leq b_1 \cdot \dots \cdot b_n$. If $b_i = 0$ for some i , then $b_1 \cdot \dots \cdot b_n = 0$. It follows that $a = 0$ and by setting $b'_j = b_j$ for all $j = 1 \dots n$ we obtain $a = b'_1 \cdot \dots \cdot b'_n = 0$. So we can assume that $b_i > 0$ for all i . Let $b'_i = b_i$ for $1 \leq i < n$ and let $b'_n = b_{n-1}^{-1} \cdot \dots \cdot b_2^{-1} \cdot b_1^{-1} \cdot a$, then $b'_n \leq b_n$; otherwise, $b_{n-1}^{-1} \cdot \dots \cdot b_2^{-1} \cdot b_1^{-1} \cdot a > b_n$ which implies that $a > b_1 \cdot \dots \cdot b_n$, contradiction. Thus we can see that indeed $a = b'_1 \cdot \dots \cdot b'_n$. \square

The class of partially-ordered rings with the property that every invertible, positive element has a positive inverse includes one of the most important, and best known classes of ℓ -rings, the f -rings. Recall that an ℓ -ring R is called an f -ring if for every two elements $a, b \in R$ and a positive element $c \in R$

$$a \wedge b = 0 \quad \text{implies} \quad ac \wedge b = ca \wedge b = 0.$$

It is well-known (see, e.g., [2]) that in an f -ring

$$c \geq 0 \quad \text{implies} \quad c(a \wedge b) = ca \wedge cb.$$

In order to see that an f -ring satisfies the inverse positive property, let $a > 0$ be an invertible element. Then $a(a^{-1} \wedge 0) = aa^{-1} \wedge 0 = 1 \wedge 0 = 0$ since in an f -ring $1 > 0$. Since a is invertible it follows that $a^{-1} \wedge 0 = 0$, and so $a^{-1} > 0$.

In the next chapter we will concentrate on the ring of matrices over a totally ordered field and satisfying $0 < I$. One of the motivations is that this ring has only one lattice order (up to isomorphism, see, e.g., [13]) and the fact that rings of matrices serve as building blocks for various rings, see, e.g., [12].

Chapter 3

Decomposition problem for matrices

In this chapter we will characterize all matrices over a totally ordered field that admit the multiplicative Riesz decomposition, first by identifying all pairs of matrices such that every nonnegative matrix bounded above by their product can be decomposed and then by moving to the general case with an arbitrary number of matrices involved. We will also give several examples, which will illustrate the class of matrices where the decomposition occurs. As part of the motivation, let us mention that in Example 1 of [4] is considered an algebra of $n \times n$ matrices in which the multiplicative Riesz property is satisfied for every two nonnegative matrices.

3.1 Definitions and Examples

Definition 3.1.1. Let $u, v \in \mathbb{F}^n$ be two vectors. We write $u \bowtie v$, if the number of nonzero entries in u is less than or equal to 1 or the number of nonzero entries in v is less than or equal to 1.

Example 3.1.1. Let $u, v, w \in \mathbb{R}^3$ be defined as follows

$$u = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad v = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

then $u \bowtie v$, $v \bowtie u$, $v \bowtie w$, but neither $u \not\bowtie w$ nor $u \not\bowtie u$ so the relation is neither transitive nor reflexive. However, it follows from the definition that " \bowtie " is a symmetric relation.

All matrices considered from now on will be $n \times n$ nonnegative matrices.

Definition 3.1.2. Given k matrices B_1, \dots, B_k , we say that the ordered k -tuple (B_1, \dots, B_k) is a multiplicative Riesz k -tuple, or m -Riesz k -tuple, if for every matrix A such that $A \leq B_1 \cdot \dots \cdot B_k$, for every $i = 1, \dots, k$ there exists matrices $B'_i \leq B_i$ such that $A = B'_1 \cdot \dots \cdot B'_k$. In case $k = 2$, we will use the term m -Riesz pair.

We would like to know which pairs of matrices are m -Riesz pairs, so we first take a look at the following examples of 2×2 matrices.

Example 3.1.2. Let $0 < a, b, c, s, t, u, v, w, x \in \mathbb{R}$ with

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \leq \begin{pmatrix} s & 0 \\ t & u \end{pmatrix} \begin{pmatrix} v & 0 \\ w & x \end{pmatrix} = \begin{pmatrix} sv & 0 \\ tv + uw & ux \end{pmatrix}.$$

Notice that

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} s \frac{a}{sv} & 0 \\ t \frac{b}{tv+uw} & u \end{pmatrix} \begin{pmatrix} v & 0 \\ w \frac{b}{tv+uw} & x \frac{c}{ux} \end{pmatrix}$$

and

$$\begin{pmatrix} s & 0 \\ t & u \end{pmatrix} \leq \begin{pmatrix} s \frac{a}{sv} & 0 \\ t \frac{b}{tv+uw} & u \end{pmatrix}, \quad \begin{pmatrix} v & 0 \\ w & x \end{pmatrix} \leq \begin{pmatrix} v & 0 \\ w \frac{b}{tv+uw} & x \frac{c}{ux} \end{pmatrix}.$$

Therefore the pair

$$\left(\begin{pmatrix} s & 0 \\ t & u \end{pmatrix}, \begin{pmatrix} v & 0 \\ w & x \end{pmatrix} \right)$$

is an m -Riesz pair.

However not all pairs of matrices are m -Riesz pairs, as illustrated by the following.

Example 3.1.3. Let $0 < s, t, u, v \in \mathbb{R}$ and

$$B = \begin{pmatrix} s & 0 \\ t & 0 \end{pmatrix}, \quad C = \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix}$$

and let

$$A = \begin{pmatrix} su & 0 \\ 0 & tv \end{pmatrix} < BC = \begin{pmatrix} su & sv \\ tu & tv \end{pmatrix}.$$

If the decomposition existed we would obtain a pair of matrices

$$\begin{pmatrix} s' & 0 \\ t' & 0 \end{pmatrix} \leq \begin{pmatrix} s & 0 \\ t & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u' & v' \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix}$$

where the following equations hold

$$s'u' = su$$

$$s'v' = 0$$

$$t'u' = 0$$

$$t'v' = tv$$

Then $s' = s$, $t' = t$, $u' = u$ and $v' = v$. In order for the first equation to hold. But by the second and third equations we know that either $s' = 0$, $t' = 0$, $u' = 0$ or $v' = 0$ which is impossible. Therefore the decomposition in question does not exist.

3.2 Main results

If we look carefully at the Example 3.1.2 we can see that

$$\begin{pmatrix} s \\ t \end{pmatrix} \bowtie \begin{pmatrix} v & 0 \end{pmatrix}^T \quad \text{and} \quad \begin{pmatrix} 0 \\ u \end{pmatrix} \bowtie \begin{pmatrix} w & x \end{pmatrix}^T$$

and in Example 3.1.3 we notice that

$$\begin{pmatrix} s \\ t \end{pmatrix} \not\bowtie \begin{pmatrix} u & v \end{pmatrix}^T.$$

Thus the decomposition of two matrices seems to depend on the “matching” of the correspondent columns with the appropriate rows of the matrices. In fact we obtain:

Theorem 3.2.1. *The following conditions are equivalent.*

(i) *The pair (B, C) is an m -Riesz pair.*

(ii) *For every $k = 1, \dots, n$, $B_k \bowtie C^k$.*

Proof. (i) \Rightarrow (ii). Let $(d_{ij}) = D = BC$ and suppose that (B, C) is an m -Riesz pair, but there is a number $1 \leq k_0 \leq n$ such that $B_{k_0} \not\bowtie C^{k_0}$. Say $b_{i_0 k_0}, b_{i_1 k_0}, c_{k_0 j_0}, c_{k_0 j_1} > 0$, for some $1 \leq i_0, i_1, j_0, j_1 \leq n$. Let $L = (l_{ij})$ be a matrix defined by:

$$l_{ij} = \begin{cases} b_{i_0 k_0} c_{k_0 j_0} & \text{if } i = i_0 \text{ and } j = j_0 \\ 0 & \text{otherwise} \end{cases}$$

Let us next consider the matrix $A = BC - L$. We have

$$a_{i_0 j_0} = d_{i_0 j_0} - l_{i_0 j_0} = \sum_{k=1}^n b_{i_0 k} c_{k j_0} - b_{i_0 k_0} c_{k_0 j_0} = \sum_{\substack{k=1 \\ k \neq k_0}}^n b_{i_0 k} c_{k j_0} \geq 0.$$

Also for $i \neq i_0$ or $j \neq j_0$,

$$a_{ij} = d_{ij} - l_{ij} = \sum_{k=1}^n b_{ik} c_{kj} - 0 = \sum_{k=1}^n b_{ik} c_{kj} \geq 0.$$

Since (B, C) is an m -Riesz pair, there exist $(b'_{ij}) = B' \leq B$ and $(c'_{ij}) = C' \leq C$ such that $A = B'C'$. We have

$$a_{i_0 j_1} = \sum_{k=1}^n b'_{i_0 k} c'_{k j_1} = \sum_{k=1}^n b_{i_0 k} c_{k j_1}.$$

But this implies that for all $1 \leq k \leq n$, $b'_{i_0 k} = b_{i_0 k}$ and $c'_{k j_1} = c_{k j_1}$, for if there is a k' such that $b'_{i_0 k'} < b_{i_0 k'}$ or $c'_{k' j_1} < c_{k' j_1}$ then

$$a_{i_0 j_1} = \sum_{k=1}^n b_{i_0 k} c_{k j_1} > \sum_{k=1}^n b'_{i_0 k} c'_{k j_1}.$$

Similarly, for all $1 \leq k \leq n$, $b'_{i_1 k} = b_{i_1 k}$ and $c'_{k j_0} = c_{k j_0}$. Since

$$a_{i_0 j_0} = \sum_{k=1}^n b'_{i_0 k} c'_{k j_0} = \sum_{k=1}^n b_{i_0 k} c_{k j_0} > \sum_{\substack{k=1 \\ k \neq k_0}}^n b_{i_0 k} c_{k j_0} = a_{i_0 j_0}$$

which is a contradiction. Therefore (ii) is satisfied.

(ii) \Rightarrow (i). Let $A \leq BC$. Suppose that for every $1 \leq s \leq n$, $B_s \bowtie C^s$. Our goal is to define matrices B' and C' such that $A = B'C'$ and $0 \leq B' \leq B$ and $0 \leq C' \leq C$. To this end let $D = BC$ and put:

$$f_{ij} = \begin{cases} a_{ij}/d_{ij} & \text{if } a_{ij} > 0 \\ 0 & \text{if } a_{ij} = 0 \end{cases}$$

Since $A \leq D$, f_{ij} is well-defined for every $1 \leq i, j, \leq n$. The table below defines the matrices B' and C' : for every $1 \leq s \leq n$, the last two columns of the table define the s^{th} column of B' and the s^{th} row of C' , respectively. Since for every s , $B_s \bowtie C^s$, the rows of the table exhaust all possibilities according to the number of nonzero entries in the B_s and in C^s .

Table 3.1: Possibilities for B_s and C^s

	$\Xi(B_s)$	$\Xi(C^s)$	B'_s	$(C')^s$
1	*	0	$\mathbf{0}$	$\mathbf{0}$
2	0	*	$\mathbf{0}$	$\mathbf{0}$
3	1	> 1	B_s	$c'_{sj} = c_{sj}f_{rj}$
4	≥ 1	1	$b'_{is} = b_{is}f_{ik}$	C^s

Here $\Xi(\mathbf{v})$ denotes the number of nonzero elements in the vector \mathbf{v} ; the “*” means any number between 0 and n ; r is the only index such that $b_{rs} > 0$; k is the only index such that $c_{sk} > 0$; and $1 \leq i, j \leq n$.

We will use the table to show that for every $1 \leq i, j, s \leq n$,

$$b'_{is}c'_{sj} = b_{is}c_{sj}f_{ij}. \quad (\dagger)$$

In cases marked (1) and (2) the equality (\dagger) is trivial. In case (3) for all $i \neq r$, $b_{is} = b'_{is} = 0$ and so (\dagger) holds, whereas for $i = r$ we have $b'_{is}c'_{sj} = b_{is}c_{sj}f_{rj} = b_{is}c_{sj}f_{ij}$. Case (4) is similar

to (3).

Since every $f_{ij} \leq 1$, we have $0 \leq B' \leq B$ and $0 \leq C' \leq C$. To complete the proof we show that $A = B'C'$. If $a_{ij} > 0$ then $d_{ij} > 0$, and by (†) we have

$$\begin{aligned}
(B'C')_{ij} &= \sum_{s=1}^n b'_{is} c'_{sj} \\
&= \sum_{s=1}^n b_{is} c_{sj} f_{ij} \\
&= \left(\sum_{s=1}^n b_{is} c_{sj} \right) f_{ij} \\
&= \left(\sum_{s=1}^n b_{is} c_{sj} \right) \frac{a_{ij}}{d_{ij}} \\
&= \left(\sum_{s=1}^n b_{is} c_{sj} \right) \frac{a_{ij}}{\sum_{s=1}^n b_{is} c_{sj}} \\
&= a_{ij},
\end{aligned}$$

and if $a_{ij} = 0$ then $f_{ij} = 0$, so

$$\begin{aligned}
(B'C')_{ij} &= \sum_{s=1}^n b'_{is} c'_{sj} \\
&= \sum_{s=1}^n b_{is} c_{sj} f_{ij} \\
&= 0 \\
&= a_{ij}.
\end{aligned}$$

Thus for every $1 \leq i, j \leq n$, $a_{ij} = (B'C')_{ij}$, therefore $A = B'C'$.

□

Example 3.2.1. The multiplicative Riesz property is not symmetric for $n > 1$, i.e., if

(B, C) is an m -Riesz pair, it does not follow that (C, B) is:

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

By Theorem 3.2.1 (B, C) is an m -Riesz pair, but CB is not. Moreover, for $n > 2$ we can construct the following:

$$D = \begin{pmatrix} B & O \\ O & O \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} C & O \\ O & O \end{pmatrix}.$$

By Theorem 3.2.1 (D, E) is an m -Riesz pair, but ED is not.

Corollary. *Let B and C be matrices such that (B, C) is an m -Riesz pair, U_B and U_C be the matrices that are obtained by replacing every nonzero entry of B and C respectively by 1. Then the pair (U_B, U_C) is an m -Riesz pair.*

Proof. We have $B_s \bowtie C^s$ for all $1 \leq s \leq n$. Then we have $(U_B)_s \bowtie (U_C)^s$ for any $1 \leq s \leq n$. □

Example 3.2.2. Let

$$B = \begin{pmatrix} 3 & 0 & 2 \\ 6 & 5 & 4 \\ 9 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 8 & 0 & 0 \\ 7 & 14 & 21 \\ 0 & 0 & 19 \end{pmatrix}$$

thus we get that

$$U_B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad U_C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

by Corollary 3.2 above (B, C) and (U_B, U_C) m -Riesz pair.

Corollary. *If B and C form an m -Riesz pair, then B', C' constitute an m -Riesz pair for any $B' \leq B$ and $C' \leq C$.*

Proof. Let B and C be an m -Riesz pair, $B' \leq B$, $C' \leq C$ and let $1 \leq k \leq n$ then $B_k \bowtie C^k$, which implies that $B'_k \bowtie C'^k$. \square

Now that we have considered pairs of matrices we will see that it is also possible to characterize all m -Riesz 3-tuples of matrices.

Theorem 3.2.2. *Let A, X, Y and Z matrices such that $A \leq XYZ$. Then there exists matrices X', Y' and Z' with $X' \leq X$, $Y' \leq Y$ and $Z' \leq Z$ so that $A = X'Y'Z'$ if and only if for all $1 \leq k \leq n$*

$$(XY)_k \bowtie Z^k \quad \text{and} \quad X_k \bowtie (YZ)^k$$

Proof. \Rightarrow Let A, X, Y, Z, X', Y' and Z' as in the hypothesis where either $(XY)_k \not\bowtie Z^k$ or $X_k \not\bowtie (YZ)^k$ for some k . Then we have of the following cases.

Case 1. Let $B = XY$, $C = Z$, $B' = X'Y'$ and $C' = Z'$. Then $A \leq BC$, $B' \leq B$, $C' \leq C$ and $A = B'C'$, but $B_k \not\bowtie C^k$ contradicting Theorem 3.2.1.

Case 2. Let $B = X$, $C = YZ$, $B' = X'$ and $C' = Y'Z'$. Then $A \leq BC$, $B' \leq B$, $C' \leq C$ and $A = B'C'$, but $B_k \not\bowtie C^k$ contradicting Theorem 3.2.1.

\Leftarrow Conversely, let $A \leq XYZ$ with $(XY)_k \bowtie Z^k$ and $X_k \bowtie (YZ)^k$ for all k , then Theorem 3.2.1 there existed F' and G' such that $F' \leq XY$, $Z' \leq Z$, $X' \leq X$ and $G' \leq YZ$ with $A = F'Z' = X'G'$ where we describe X', G', F' and Z' using Theorem 3.2.1.

Table 3.2: Possibilities for X_s and $(YZ)^s$

	$\Xi(X_s)$	$\Xi((YZ)^s)$	X'_s	$(G')^s$
1_s	*	0	$\mathbf{0}$	$\mathbf{0}$
2_s	0	*	$\mathbf{0}$	$\mathbf{0}$
3_s	1	> 1	X_s	$G'_{sj} = (YZ)_{sj} f_{\mu j}$
4_s	≥ 1	1	$x'_{is} = x_{is} f_{i\lambda}$	$(YZ)^s$

Table 3.3: Possibilities for $(XY)_t$ and Z^t

	$\Xi((XY)_t)$	$\Xi(Z^t)$	F'_t	$(Z')^t$
1_t	*	0	$\mathbf{0}$	$\mathbf{0}$
2_t	0	*	$\mathbf{0}$	$\mathbf{0}$
3_t	1	> 1	$(XY)_t$	$z'_{tj} = z_{tj}f_{\sigma j}$
4_t	≥ 1	1	$F'_{it} = (XY)_{it}f_{i\nu}$	Z^t

Here μ, λ, σ and ν are the only indices such that $x_{\mu s} > 0$, $(YZ)_{s\lambda} > 0$, $(XY)_{\sigma t} > 0$ and $z_{t\nu} > 0$ respectively, and defining Y' as follows

$$y_{st} = \begin{cases} y_{st}f_{\mu\nu} & \text{if } 3_s \text{ and } 4_t \\ y_{st} & \text{otherwise} \end{cases}$$

We claim that $A = X'Y'Z'$. For this we will show that for any $1 \leq i, s, t, j \leq n$

$$x'_{is}y'_{st}z'_{tj} = x_{is}y_{st}z_{tj}f_{ij} \quad (\dagger)$$

where f_{ij} is defined as in Theorem 3.2.1. For short we denote the case k_s and l_t for $1 \leq k, l \leq 4$ by (k_s, l_t) . So let $1 \leq i, s, t, j \leq n$, in cases $(1_s, *)$, $(2_s, *)$, $(*, 1_t)$ and $(*, 2_t)$ the equality (\dagger) is trivial. In case $(3_s, 3_t)$, first note that $\mu = \sigma$ and for all $i \neq \mu$, $x'_{is} = x_{is} = 0$ so (\dagger) holds, whereas for $i = \mu$, $x'_{is} = x_{is}, y'_{st} = y_{st}$ and $z'_{tj} = z_{tj}f_{ij}$, so $x'_{is}y'_{st}z'_{tj} = x_{is}y_{st}z_{tj}f_{ij}$. With case $(3_s, 4_s)$ if $i \neq \mu$ or $j \neq \nu$ we have $x'_{is} = x_{is} = 0$ or $z'_{tj} = z_{tj} = 0$ respectively so (\dagger) holds, whereas $i = \mu$ and $j = \nu$, $x'_{is} = x_{is}$, $y'_{st} = y_{st}f_{ij}$ and $z'_{tj} = z_{tj}$, so $x'_{is}y'_{st}z'_{tj} = x_{is}y_{st}z_{tj}f_{ij}$. The case $(4_s, 3_t)$ yields (\dagger) provided that $y_{st} = 0$. Note that $y_{st} > 0$ is impossible in $(4_s, 3_t)$, because it would imply that $y_{st}z_{tm} > 0$ and $y_{st}z_{tm'} > 0$ for $m \neq m'$ which indicates that $(YZ)_{sm}, (YZ)_{sm'} > 0$ which is impossible. Finally, the case $(4_s, 4_t)$ is similar to $(3_s, 3_t)$.

In all the cases $x'_{is}y'_{st}z'_{tj} = x_{is}y_{st}z_{tj}f_{ij}$. Now, if $a_{ij} = 0$ then

$$\begin{aligned}
(X'Y'Z')_{ij} &= \sum_{1 \leq s, t \leq n} x'_{is}y'_{st}z'_{tj} \\
&= \sum_{1 \leq s, t \leq n} x_{is}y_{st}z_{tj}f_{ij} \\
&= \left(\sum_{1 \leq s, t \leq n} x_{is}y_{st}z_{tj} \right) f_{ij} \\
&= 0 = a_{ij}.
\end{aligned}$$

If $a_{ij} > 0$ then

$$\begin{aligned}
(X'Y'Z')_{ij} &= \sum_{1 \leq s, t \leq n} x'_{is}y'_{st}z'_{tj} \\
&= \sum_{1 \leq s, t \leq n} x_{is}y_{st}z_{tj}f_{ij} \\
&= \left(\sum_{1 \leq s, t \leq n} x_{is}y_{st}z_{tj} \right) f_{ij} \\
&= \left(\sum_{1 \leq s, t \leq n} x_{is}y_{st}z_{tj} \right) \left(\frac{a_{ij}}{\sum_{1 \leq s, t \leq n} x_{is}y_{st}z_{tj}} \right) \\
&= a_{ij}
\end{aligned}$$

□

Example 3.2.3. Let

$$X = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix}.$$

We have

$$XY = \begin{pmatrix} 30 & 36 & 42 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad YZ = \begin{pmatrix} 0 & 14 & 0 \\ 0 & 32 & 0 \\ 0 & 50 & 0 \end{pmatrix}.$$

Hence $X_s \bowtie (YZ)^s$ and $(XY)_s \bowtie Z^s$ for all $1 \leq s \leq n$, thus (X, Y, Z) is an m -Riesz 3-tuple.

Example 3.2.4. Let

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have

$$XY = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad YZ = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 0 \end{pmatrix}.$$

Hence $(XY)_s \bowtie Z^s$ for all $1 \leq s \leq n$, but $X_1 \not\bowtie (YZ)^1$; therefore (X, Y, Z) is not an m -Riesz 3-tuple.

Corollary. *Let A, B_1, \dots, B_k be matrices ($k \geq 3$) such that $A \leq B_1 \cdot \dots \cdot B_k$. Then there exist matrices $B'_i \leq B_i$ for all i such that $A = B'_1 \cdot \dots \cdot B'_k$ if and only if $(B_1)_l \bowtie (B_2 \cdot \dots \cdot B_k)^l$ and $(B_1 \cdot \dots \cdot B_{k-1})_l \bowtie (B_k)^l$ for all $1 \leq l \leq n$.*

Proof. We take $X = B_1, Y = B_2 \cdot \dots \cdot B_{k-1}$ and $Z = B_k$, apply Theorem 3.2.2, and take induction by k . □

Chapter 4

Further consequences of the decomposition property

In this chapter we will discuss some further consequences of the matrix decomposition theorems discussed in Chapter 3, and some applications to the theory of partially ordered semigroups.

4.1 Right and Left m -Riesz Matrices

All matrices considered will be $n \times n$ nonnegative matrices.

Definition 4.1.1. We say that a matrix C is a *right m -Riesz matrix* if for all matrices A and B such that $A \leq BC$ there is a matrix $B' \leq B$ such that $A = B'C$

Theorem 4.1.1. *A matrix C is a right m -Riesz matrix if and only if for all $1 \leq s \leq n$, $\Xi(C^s) \leq 1$ where $\Xi(\mathbf{v})$ denotes the number of nonzero elements in the vector \mathbf{v} .*

Proof. (\Rightarrow). Let C be a right m -Riesz matrix and let s be such that $\Xi(C^s) > 1$. Then there are two different indices p, q such that $c_{sp}, c_{sq} > 0$. Let $B = (b_{ij})$ be a matrix defined by:

$$b_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = s \\ 0 & \text{otherwise} \end{cases}$$

Then the only nonzero entries of BC are in the first row; in fact, $(BC)_{1p} = c_{sp}$ and

$(BC)_{1q} = c_{sq}$, now define $A = (a_{ij})$ to be:

$$a_{ij} = \begin{cases} 0 & \text{if } i = 1 \text{ and } j = p \\ (BC)_{ij} & \text{otherwise} \end{cases}$$

Thus $A < BC$ since C is a right m -Riesz matrix, $B' < B$. Then it follows that $b_{1s} < 1$, which in turn implies that $a_{1q} = b_{1s}c_{sq} < BC_{1q}$, which is a contradiction.

(\Leftarrow) By the proof from Theorem 3.2.1 we can see set $C' = C$ in all cases of table 3.1. \square

Example 4.1.1. Let

$$C_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 4.1.1 we can see that C_1 is a right m -Riesz matrix, but C_2 is not.

Lemma 4.1.2. For A, B matrices, $A \leq B$ if and only if $A^T \leq B^T$.

Proof. (\Rightarrow) Let $1 \leq i, j \leq n$ then $(A^T)_{ij} = A_{ji} \leq B_{ji} = (B^T)_{ij}$ therefore $A^T \leq B^T$.

(\Leftarrow) Replace every A^T and B^T by A and B , respectively and vice-versa. \square

Definition 4.1.2. We say that a matrix B is a *left m -Riesz matrix* if for all matrices A and C such that $A \leq BC$ there is a matrix $C' \leq C$ such that $A = BC'$.

Theorem 4.1.3. A matrix B is a left m -Riesz matrix if and only if for all $1 \leq s \leq n$, $\Xi(B_s) \leq 1$ where $\Xi(\mathbf{v})$ denotes the number of nonzero elements in the vector \mathbf{v} .

Proof. Apply Lemma 4.1.2 and Theorem 4.1.1. \square

Example 4.1.2. Let

$$B_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 4 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 4.1.3 we can see that B_1 is a left m -Riesz matrix, but B_2 is not.

Definition 4.1.3. A matrix D is an m -Riesz matrix if it is both a left and right m -Riesz matrix.

Theorem 4.1.4. A matrix R is a m -Riesz matrix if and only if for $1 \leq s \leq n$, $\Xi(R_s) \leq 1$ and $\Xi(R^s) \leq 1$.

Proof. Immediate from Theorems 4.1.3 and 4.1.1. □

Example 4.1.3. Let

$$R_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}.$$

By Theorem 4.1.4 we can see that R_1 is an m -Riesz matrix, but R_2 is not.

Recall that a generalized permutation matrix is an $n \times n$ matrix having precisely one nonzero-entry in every row and in every column, see e.g., [15].

Corollary. Every generalized permutation matrix is an m -Riesz matrix.

Proof. A generalized permutation matrix satisfies the conditions of Theorem 4.1.4. □

4.2 A remark about ordered semigroups

Definition 4.2.1. A *semigroup* is a set S together with a binary operation \cdot on S such that for all $a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Example 4.2.1. The set of all nonnegative $(n \times n)$ real matrices forms a semigroup (see [1]).

Definition 4.2.2. A *partially-ordered semigroup* is a semigroup S together with a partial order \leq , such that for all $a, b, c \in S$ with $a \leq b$ we have $ac \leq bc$ and $ca \leq cb$.

Example 4.2.2. If G is a group and $L(G)$ is the lattice of all normal subgroups of G , then $(L(G), \cdot, \subseteq)$ is a partially-ordered semigroup where $H \cdot K = [H, K]$ the commutator (see, e.g., [3]).

4.3 An Application to Ordered Semigroups

We can see in [1] that the set of all nonnegative elements of a partially ordered ring R forms a partially-ordered semigroup with matrix multiplication. The following theorem provides an excellent example of an ordered semigroup arising from the m-Riesz matrices.

Theorem 4.3.1. *The set \mathcal{R} of all m-Riesz matrices under matrix multiplication is a partially-ordered semigroup.*

Proof. We do not need to prove that the operation is associative since in general matrix multiplication is associative. Also the operation preserves the order relation since \mathcal{R} is a subset of the set of nonnegative matrices. It remains to show that the operation is closed. So let $R, T \in \mathcal{R}$ and let $1 \leq p \leq n$. If $(RT)_s = (RT)^s = 0$ there is nothing to prove. On the other hand, if for some $(RT)_s \neq 0$ then there is i with $(RT)_{is} > 0$ which implies that there is a unique k with $R_{ik} > 0$ and $T_{ks} > 0$. But then i also must be unique. The case where $(RT)^s \neq 0$ is similar. Therefore \mathcal{R} satisfies Definition 4.2.2. \square

The following example shows that \mathcal{R} is not a lattice.

Example 4.3.1. Let

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then

$$R \vee S = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Therefore \mathcal{R} is not lattice.

Chapter 5

Concluding Remarks

5.1 Future Work and Open Questions

Theorem 4.3.1 opens up a perspective of researching the Riesz Decomposition Property for a wider class of partially-ordered semigroups. Any advance in this direction will lead to a better understanding of the multiplicative Riesz Decomposition in the context of partially-ordered rings.

Also, it would be desirable to obtain the equivalent conditions for the decomposition of more than two matrices similar to those established in Theorem 3.2.1.

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Appendix A

Algorithms

In this appendix we will present two algorithms: one that checks whether a k -tuple of real matrices is an m -Riesz k -tuple, and another one that produces the actual decomposition. We choose a Java implementation with the aid of the Jama a matrix package for Java. Jama is available at <http://math.nist.gov/javanumerics/jama/> along with documentation and source code.

A.1 Checking Algorithm

```
/**
 * Determines if product of matrices {B_1,...,B_k} admits a decomposition.
 * @param product {B_1,...,B_k}
 * @return true if {B_1,...,B_k} is decomposable; false otherwise
 */
public static boolean isDecomposable(Matrix[] product){
    int factors = product.length;
    int n = product[0].getColumnDimension();
    if(factors == 1){
        return true;
    }
    else if(factors == 2){
        Matrix B = product[0];
        Matrix C = product[1];
```

```

int nonzeroCol = 0;
int nonzeroRow = 0;
for(int s = 0; s < n; s++){
    nonzeroCol = 0;
    nonzeroRow = 0;
    for(int l = 0; l < n; l++){
        if(B.get(s, l) != 0){
            nonzeroCol++;
        }
        if(C.get(l, s) != 0){
            nonzeroRow++;
        }
        if(nonzeroCol > 1 && nonzeroRow > 1){
            return false;
        }
    }
}
return true;
}
else{
    // 3 or more factors
    Matrix[] rPair,lPair,rProd,lProd;
    rPair = new Matrix[2];

    rProd = new Matrix[factors - 1]; // subproduct {B_2,...,B_k}
    for(int i = 0; i < rProd.length; i++){
        rProd[i] = product[i+1];
    }
}

```



```

    }
    rPair[0] = product[0];
    rPair[1] = multiMultiply(rProd);

    lPair = new Matrix[2];

    lProd = new Matrix[factors - 1]; // subproduct {B_1,...,B_{k-1}}
    for(int i = 0; i < lProd.length; i++){
        lProd[i] = product[i];
    }
    lPair[0] = multiMultiply(lProd);
    lPair[1] = product[factors - 1];
    return (isDecomposable(lPair) && isDecomposable(rPair));
} //end if-then-else
} //end isDecomposable

```

Example A.1.1. Let

$$W = \begin{pmatrix} 1.000 & 0.000 & 4.000 & 1.000 & 0.000 & 0.000 \\ 1.000 & 0.000 & 6.000 & 0.000 & 2.000 & 0.000 \\ 3.000 & 1.000 & 7.000 & 0.000 & 5.000 & 0.000 \\ 5.000 & 1.000 & 4.000 & 0.000 & 4.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 1.000 & 2.000 & 3.000 & 4.000 & 5.000 & 6.000 \end{pmatrix}$$

$$X = \begin{pmatrix} 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 3.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 2.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 4.000 & 0.000 \\ 7.000 & 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0.000 & 1.000 & 0.000 & 8.000 & 1.000 \\ 2.000 & 2.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 3.000 & 3.000 & 0.000 & 0.000 \\ 0.000 & 4.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 5.000 & 0.000 & 6.000 & 4.000 \end{pmatrix}$$

$$Z = \begin{pmatrix} 0.000 & 0.000 & 3.000 & 5.000 & 9.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \\ 1.000 & 4.000 & 6.000 & 7.000 & 9.000 \\ 0.000 & 0.000 & 0.000 & 3.000 & 0.000 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{pmatrix}$$

here the following piece of code

```
//P1 is {W,X}
System.out.println("Is (W,X) decomposable? " +
    MatrixDecomposition.isDecomposable(P1));
```

produced

```
Is (W,X) decomposable? true
```

and this one

```
//P2 is {Y,Z}
```

```
System.out.println("Is (Y,Z) decomposable? " +
MatrixDecomposition.isDecomposable(P2));
```

yielded

```
Is (W,X) decomposable? false
```

as expected from Theorem 3.2.2.

A.2 Decomposition Algorithm

```
/** Given nonnegative matrices A, B_1, B_2, ... B_k where
 * A <= B_1*B_2*...*B_k, returns nonnegative matrices B'_1,B'_2,...,B'_k
 * such that A = B'_1*B'_2*...*B'_k and B'_i <= B_i for 1 <= i <= k.
 * @param A n x n matrix
 * @param product collection of n x n matrices {B_1,B_2,...,B_k}
 * that will be decomposed
 * @return {B'_1,B'_2,...,B'_k}
 */
public static Matrix[] matrixRieszDecomposition(Matrix A, Matrix[]
product){

    Matrix[] decompProd = new Matrix[product.length];://{B'_1,...,B'_k}
    Matrix F = divMatrix(A, multiMuliply(product));
    int n = A.getColumnDimension();
    int factors = product.length;
    //initialize decompProd
    for(int i = 0; i < factors; i++){
        decompProd[i] = (Matrix) product[i].clone();
    }//end for
```

```

//go over the possible number of factors
if(factors == 1){
    decompProd[0] = (Matrix) A.clone();
}
else if( factors == 2){
    for(int s = 0; s < n; s++){
        //go over all possibilities for product[0], product[1]
        Matrix B = product[0];
        Matrix C = product[1];
        Matrix Bprime = decompProd[0];
        Matrix Cprime = decompProd[1];
        //number of nonzero entries in the s-th column of B
        int nonZerosCol = 0;
        int r = -1; //unique index such that B_{rs} > 0
        int k = -1; //unique index such that C_{sk} > 0
        //number of nonzero entries in the s-th row of C
        int nonZerosRow = 0;

        //count number of nonzero entries in the s-th column of B
        for(int i = 0; i < n; i++){
            if(B.get(i, s) > 0){
                nonZerosCol++;
                r = i;
            }//end if
        }//end for

        //count number of nonzero entries in the s-th row of C

```

```

for(int j = 0; j < n; j++){
    if(C.get(s, j) > 0){
        nonZerosRow++;
        k = j;
    }// end if
} //end for

if(nonZerosCol == 1 && nonZerosRow > 1){
    //get unique nonzero entry of column s of B
    for(int j = 0; j < n; j++){
        Cprime.set(s, j, C.get(s, j)*F.get(r, j));
    }// end for
}

else if(nonZerosCol >= 1 && nonZerosRow == 1){
    //get unique nonzero entry of row c of C
    for(int i = 0; i < n; i++){
        Bprime.set(i, s, B.get(i, s)*F.get(i, k));
    } //end for
}

else{
    for(int l = 0; l < n; l++){
        Bprime.set(l, s, 0);
        Cprime.set(s, l, 0);
    } // end for
} // end if-then-else
}

} else{ //3 or more factors

```

```

Matrix X,Y1,Y2,Z;
Matrix[] X_YZ, XY_Z;
X = product[0]; //first element of the factors
Z = product[factors - 1]; // last element of the factors
Matrix[] rFactors, lFactors;
rFactors = new Matrix[factors - 1]; // subproduct {B_2,...,B_k}
for(int i = 0; i < rFactors.length; i++){
rFactors[i] = product[i+1];
}
lFactors = new Matrix[factors - 1]; // subproduct {B_1,...,B_{k-1}}
for(int i = 0; i < lFactors.length; i++){
lFactors[i] = product[i];
}
Y1 = multiMultiply(rFactors);
Y2 = multiMultiply(lFactors);
Matrix[] leftProd = new Matrix[2];
leftProd[0] = X;
leftProd[1] = Y1;
Matrix[] rightProd = new Matrix[2];
rightProd[0] = Y2;
rightProd[1] = Z;
//decompose {B_1,B_2*...*B_k}
X_YZ = matrixRieszDecomposition(A,leftProd);
//decompose {B_1*...*B_{k-1},B_k}
XY_Z = matrixRieszDecomposition(A,rightProd);
//construct B'_2
for(int s = 0; s < n; s++){
for(int t = 0; t < n; t++){

```

```

//number of nonzero entries in the s-th and t-th column of
//X and XY respectively
int nonZerosCol_s = 0, nonZerosCol_t = 0;
int mu = -1; //unique index such that  $X_{\{\mu s\}} > 0$ 
int nu = -1; //unique index such that  $Z_{\{t nu\}} > 0$ 
//number of nonzero entries in the s-th and t-th row of
//YZ and Z respectively
int nonZerosRow_s = 0, nonZerosRow_t = 0;

//count number of nonzero entries in the s-th column of X
for(int i = 0; i < n; i++){
    if(X_YZ[0].get(i, s) > 0){
        nonZerosCol_s++;
        mu = i;
    }
}
} //end for

//count number of nonzero entries in the s-th row of YZ
for(int j = 0; j < n; j++){
    if(X_YZ[1].get(s, j) > 0){
        nonZerosRow_s++;
    }
}
} //end for

//count number of nonzero entries in the t-th column of
//XY
for(int i = 0; i < n; i++){

```

```

if(XY_Z[0].get(i, t) > 0){
    nonZerosCol_t++;
}
} //end for

//count number of nonzero entries in the t-th row of YZ
for(int j = 0; j < n; j++){
if(XY_Z[1].get(t, j) > 0){
    nonZerosRow_t++;
    nu = j;
}
} //end for

if((nonZerosCol_s == 1 && nonZerosRow_s > 1) &&
(nonZerosCol_t >= 1 && nonZerosRow_t == 1)){
if(factors > 3){
    for(int l = 0; l < n; l++){
        decompProd[1].set(s, l, decompProd[1].get(s,l)*F.get(mu,nu));
    } //end for
}
else{
    decompProd[1].set(s, t, decompProd[1].get(s,t)*F.get(mu,nu));
} //end if-then-else
} //end if
} //end for
} // end for
decompProd[0] = X_YZ[0]; //get B'_1
decompProd[factors - 1] = XY_Z[1]; // get B'_k

```



```
}
```

```
return decompProd;
```

```
}// end matrixRieszDecomposition
```

Example A.2.1. Let

$$A = \begin{pmatrix} 0.000 & 3.000 & 0.000 & 5.000 & 0.000 & 3.000 \\ 0.000 & 0.000 & 0.000 & 6.000 & 0.000 & 4.000 \\ 1.000 & 3.000 & 4.000 & 15.000 & 4.000 & 0.000 \\ 20.000 & 14.000 & 7.000 & 3.000 & 9.000 & 6.000 \\ 0.000 & 0.000 & 0.000 & 12.000 & 0.000 & 0.000 \\ 0.000 & 3.000 & 0.000 & 15.000 & 0.000 & 0.000 \end{pmatrix}$$

$$B = \begin{pmatrix} 1.000 & 5.000 & 0.000 & 0.000 & 0.000 & 8.000 \\ 2.000 & 0.000 & 0.000 & 0.000 & 0.000 & 6.000 \\ 3.000 & 0.000 & 0.000 & 0.000 & 1.000 & 0.000 \\ 4.000 & 0.000 & 0.000 & 7.000 & 0.000 & 0.000 \\ 5.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 6.000 & 5.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix}$$

$$C = \begin{pmatrix} 0.000 & 0.000 & 0.000 & 8.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 1.000 & 4.000 & 5.000 & 0.000 & 0.000 & 0.000 \\ 4.000 & 6.000 & 7.000 & 0.000 & 7.000 & 9.000 \\ 1.000 & 3.000 & 4.000 & 6.000 & 9.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 4.000 \end{pmatrix}$$

then

$$BC = \begin{pmatrix} 0.000 & 5.000 & 0.000 & 8.000 & 0.000 & 32.000 \\ 0.000 & 0.000 & 0.000 & 16.000 & 0.000 & 24.000 \\ 1.000 & 3.000 & 4.000 & 30.000 & 9.000 & 0.000 \\ 28.000 & 42.000 & 49.000 & 32.000 & 49.000 & 63.000 \\ 0.000 & 0.000 & 0.000 & 40.000 & 0.000 & 0.000 \\ 0.000 & 5.000 & 0.000 & 48.000 & 0.000 & 0.000 \end{pmatrix}$$

and the decomposition algorithm produced

$$B' = \begin{pmatrix} 0.625 & 3.000 & 0.000 & 0.000 & 0.000 & 0.750 \\ 0.750 & 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \\ 1.500 & 0.000 & 0.000 & 0.000 & 1.000 & 0.000 \\ 0.375 & 0.000 & 0.000 & 7.000 & 0.000 & 0.000 \\ 1.500 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 1.875 & 3.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix}$$

$$C' = \begin{pmatrix} 0.000 & 0.000 & 0.000 & 8.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 2.857 & 2.000 & 1.000 & 0.000 & 1.286 & 0.857 \\ 1.000 & 3.000 & 4.000 & 3.000 & 4.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 4.000 \end{pmatrix}$$

$$B'C' = \begin{pmatrix} 0.000 & 3.000 & 0.000 & 5.000 & 0.000 & 3.000 \\ 0.000 & 0.000 & 0.000 & 6.000 & 0.000 & 4.000 \\ 1.000 & 3.000 & 4.000 & 15.000 & 4.000 & 0.000 \\ 20.000 & 14.000 & 7.000 & 3.000 & 9.000 & 6.000 \\ 0.000 & 0.000 & 0.000 & 12.000 & 0.000 & 0.000 \\ 0.000 & 3.000 & 0.000 & 15.000 & 0.000 & 0.000 \end{pmatrix}$$

we can see that $A \leq BC$, $A = B'C'$, $B' \leq B$ and $C' \leq C$.

Curriculum Vitae

Julio César Urenda Castañeda was born on December 20, 1985. The first son of Raymundo Urenda Moreno and Amalia Castañeda Bonilla, he graduated from Centro de Bachillerato Tecnológico Industria y de Servicios No. 114, Ciudad Juárez, Chihuahua, México, in the spring of 2000. He entered The University of Texas at El Paso in fall 2003. While pursuing his bachelor's degree in Computer Science he worked as a Mathematics tutor at ACES, and as Peer Leader at the Computer Science department, both at the The University of Texas at El Paso. He received his bachelor's degree in Computer Science in the summer of 2007.

In the fall of 2007, he entered the Graduate School of The University of Texas at El Paso. While pursuing a master's degree in Mathematics he worked as a Teaching Assistant for the Mathematical Sciences department at The University of Texas at El Paso and as a Mathematics Instructor for Universidad Autónoma de Ciudad Juárez.

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