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Vilem Novak

Irina Perfilieva

Hung T. Nguyen

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

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Research on Advanced Soft Computing and Its Applications

(Introduction to the Special Issue)

Vilém Novák[1], Irina Perfilieva[1],
Hung T. Nguyen[2] and Vladik Kreinovich[3]

[1]Institute for Research and Applications of Fuzzy Modeling (IFARM)
University of Ostrava, 30. dubna 22
701 03 Ostrava 1, Czech Republic
Vilem.Novak@asu.cz Irina.Perfilieva@osu.cz

[2]Department of Mathematical Sciences New Mexico State University,
Las Cruces, NM 88003, USA hunguyen@nmsu.edu

[3]Department of Computer Science, University of Texas at El Paso
El Paso, TX 79968, USA, vladik@cs.utep.edu

Abstract

The main objective for the research presented in this special issue is to advance theoretical basis in soft computing, for the purpose of improving applications.

Why is this theoretical research needed? Because soft computing in general (and intelligent control and decision making in particular) are, in many aspects, still an art. To make this methodology easier to apply, we must use the experience of successful applications of fuzzy control, decision making or classification and extract formal rules that would capture this experience. To be able to do that efficiently, we must understand why some versions of soft computing methodology turned out to be more successful in some practical situations and less successful in others. In other words, to advance the practical success of soft computing methodology, we need further theoretical analysis of soft computing — analysis targeted at enhancing its application abilities.

1 Introduction

1.1 Who Are We?

This issue contains the results presented at the Czech-US Seminar on Current Trends in Soft Computing (June 16–19, 2001, Roznov pod Radhostem, Czech Republic).

This workshop combined the efforts of US-based Rio Grande Institute of Soft Computing (RioSoft) and the Czech-based Institute for Research and Applications of Fuzzy Modeling (IFARM) of the University of Ostrava.

The Institute for Research and Applications of Fuzzy Modeling is an organizational unit at the University of Ostrava. Since its opening in 1996, more than 130 publications has been published by its 9 researchers.

The Rio Grande Institute for Soft Computing is a consortium of researchers from New Mexico State University, New Mexico Highlands University, New Mexico Institute of Mining and Technology, University of New Mexico, and University of Texas at El Paso. It was formed in 1999 with a mission to develop and facilitate the application of innovative soft computing technologies for modeling, prototyping, manufacturing, testing, analysis, and evaluation of processes and systems which have use both in industry and in government.

1.2 How We Got Together

We have known and used each other's results for a long time: we regularly meet at the international conferences, we exchange ideas, results, and problems. In many cases, this exchange of ideas has led to direct influence: e.g., fundamental results on representing of functions by fuzzy logic proven by I. Perfilieva — one of the leaders of the Czech team — in [48] was the main motivation for similar representation results proven by US researchers [27, 36].

Collaboration was boosted during the US visits of Czech researchers. The first boost came when M. Navara from the Czech team visited US in 1998. During this visit, we published a joint paper [57] (see also [10]).

The collaboration was really boosted during the extended US visit of two Czech researchers, V. Novák and I. Perfilieva, in June 2000. They spent a week in Las Cruces and El Paso, giving presentations and working on joint research topics, and then participated – together with several researchers from the US team – in the World Automation Congress in Maui, Hawaii. Due to common interest, this visit was sponsored partly by the Czech granting agency, and partly by the New Mexico State University and by the El Paso-based NASA-sponsored Pan-American Center for Earth and Environmental Studies.

During this visit, we completed a joint paper I. Perfilieva and V. Kreinovich, "A New Universal Approximation Result For Fuzzy Systems, Which Reflects CNF–DNF Duality" (to appear in *International Journal of Intelligent Systems*), and started working on several other joint papers.

Our collaboration was further boosted by a Grant No. W-00016 from the U.S.-Czech Science and Technology Joint Fund. This grant was partly sponsoring several publications [11, 12, 20, 29, 30, 55, 56, 58, 59], and it also provided financial support for the Czech-US Seminar on Current Trends in Soft Computing.

1.3 Motivations for Our Research

In many areas of expertise, such as medicine, geology, etc., human experts are needed. Usually, there are very few top level experts, and it is not physically possible for these few experts to solve all numerous related problems. It is therefore desirable to develop a computer-based system which incorporates the knowledge of the top experts and uses this knowledge either to directly solve the related problems – or, at least, to provide high-level advise to people trying to solve these problems.

Experts can describe their knowledge in terms of statements and rules, but this formulation often comes with uncertainty and ambiguity: experts are often not 100% confident in the statements which form their knowledge, and even when they are, these statements are formulated in terms of words of natural language (such as "large") which do not have precise meaning. To adequately describe the expert knowledge, we must therefore store, in the knowledge base, not only the statements themselves, but also the indication of the degree to which the experts are confident in these statements.

This degree is in most cases characterized by a number from the interval [0,1]. An expert's degree of confidence $d(A)$ in a statement A can be determined, if, e.g., we ask an expert to estimate his/her degree of confidence on a scale from 0 to 10. If s/he selects 8, then we take $d(A) = 8/10$.

Suppose now that we know the degrees of confidence $d(A)$ and $d(B)$ in statements A and B, and we know nothing else about A and B. Suppose also that we are interested in the degree of confidence of the composite statement $A \,\&\, B$. Since the only information available consists of the values $d(A)$ and $d(B)$, we must compute $d(A \,\&\, B)$ based on these values. We must be able to do that for arbitrary values $d(A)$ and $d(B)$. Therefore, we need a *function* that transforms the values $d(A)$ and $d(B)$ into an estimate for $d(A \,\&\, B)$. Such a function is called an *"and"-operation* (*t-norm*). If an "and"-operation $f_\& : [0,1] \times [0,1] \to [0,1]$ is fixed, then we take $f_\&(d(A), d(B))$ as an estimate for $d(A \,\&\, B)$. Similarly, to estimate the degree of confidence in $A \vee B$, we need an *"or"-operation* (*t-conorm*) $f_\vee : [0,1] \times [0,1] \to [0,1]$. A set of truth values (usually, the interval $[0,1]$), endowed with logic-motivated operations like "and" and "or" is called a *fuzzy logic* [3, 40].

The first two pairs of "and" and "or" operations were proposed by L. Zadeh, the father of fuzzy logic, in his original paper [61]: $f_\&(x,y) = \min(x,y)$, $f_\vee(x,y) = \max(x,y)$, and $f_\&(x,y) = x \cdot y$, $f_\vee(x,y) = x + y - x \cdot y$. Later, numerous other operations have been proposed: e.g., "bold and" (Łukasiewicz conjunction) $f_\&(a,b) = \max(a+b-1, 0)$ and "bold or" $f_\vee(a,b) = \min(a+b, 1)$ (Łukasiewicz disjunction).

One of the main applications of fuzzy logic is *fuzzy control* (see, e.g., [28, 42]). In most industrial applications, we want to control the corresponding industrial processes in such a way as to maximize the output within certain (physical and economical) restrictions. When the corresponding mathematical description is linear, we can use well-known optimal control techniques to find the optimal control strategy. In reality, however, most industrial processes are non-linear. For non-linear control problems, the situation is much more complicated: there are good recipes which often work but, alas, there is still no general method of generating an optimal (or even a reasonably good) control.

If for a certain industrial process, no known technique leads to a good quality control, what can we do? Usually, the very fact that this process is actually used in industry means that this process is reasonably well controlled by human controllers. Therefore, if we want to automate this control, we must somehow transform the knowledge of these expert controllers (operators) into an automatic control strategy.

Specifically, our goal is to describe a function which takes the sensor inputs $x_1, \ldots, x_n$ (numbers) and generates the (numerical) value of the control effort u. Unfortunately, expert operators cannot formulate their expertise in these terms. Instead, they describe their control strategy by using uncertain ("fuzzy") statements of the type "if the obstacle is straight ahead, the distance to it is small, and the velocity of the car is medium, press the brakes hard". Fuzzy control is a methodology which translates such statements into precise formulas for control.

If the expert rules are simple if-then rules, then, once we have selected a fuzzy "and"-operation $f_\&(a,b)$ and a fuzzy "or"-operation $f_\vee(a,b)$, we are able to transform an arbitrary set of simple fuzzy if–then rules connecting inputs $x_1, \ldots, x_n$ and the output u into a crisp function $y = f(x_1, \ldots, x_n)$. Since 1970s, this methodology has been successfully used in many practical problems. Fuzzy controllers are used in areas ranging from camcoder control to car control to controlling chemical reactions to controlling the temperature on a Space Shuttle; see, e.g., [39].

In particular, a real application of fuzzy control of 5 large aluminium melting furnaces has been implemented by IRAFM in the Czech Republic in 1996–98 using their original approach based on fuzzy if-then rules in genuine linguistic form and logical deduction (cf. [45]).

2 Two Problems with the Existing Fuzzy Control Methodologies

In spite of many successes, there is still room for improvement:

- there are many practical problems for which fuzzy control has not been successfully applied;
- there are also many practical problems for which fuzzy control has been applied, but the quality of the resulting controller is still much worse than the quality of a control performed by a skilled operator whose knowledge we try to capture in this fuzzy control.

There are two main reasons why the existing fuzzy control methodology needs improvement:

- First, the existing fuzzy control methodology assumes that all the rules formulated by the expert operators are simple if-then rules. In reality, operators often use more complex linguistic constructions to describe how they control. These constructions may use different *hedges*, i.e., words like "very", "almost", etc. These constructions may use verbs, adjectives, etc. The standard fuzzy control methodology can handle only the most primitive hedges, and often handles them badly. To expand fuzzy control methodologies to the new areas and to improve the quality of fuzzy control, we must learn how to handle such complex linguistic expressions, i.e., to handle *linguistic uncertainty.*
- Second, in addition to well-justified methods and results, the existing intelligent control systems and expert systems use a lot of heuristic techniques. In particular, in most packages and real-life applications, only the simplest t-norms and t-conorms are used. When we use a heuristic method, there is no guarantee that we get optimal results.

  It has been theoretically shown that in many situations, an alternative selection of t-norms and t-conorms can lead to much better quality control; see, e.g., [7, 40] and references therein. Moreover, in many situations, it is known how to select optimal t-norms and t-conorms. To improve the quality of the control, we must therefore analyze different t-norms and t-conorms and learn how to use them in practical problems. We have already done a lot of research in this direction. Some of these results are published in the mathematician-oriented book [40]; the second book devoted to foundations for intelligent control will appear shortly. However, there are still many open problems that require a deep mathematical analysis of unusual structures that naturally appear in the formalization of human reasoning.

At present, there has been some research in both directions outlined above. However, to achieve a real breakthrough in fuzzy control, we must combine these two directions. It so happened that:

- the Czech team has an expertise in extending fuzzy control to more linguistically complex rules, while
- the US team has an expertise in analyzing and optimizing t-norms and t-conorms.

Therefore, if we want to push the successful technology of fuzzy control to a new level, we must collaborate.

3 Research Directions: General Description

In view of the above, we pursued the following directions of joint research:

- First, we further analyzed the possibility of using complex linguistic expressions in intelligent control and soft computing. This direction was led by the Czech team.
- Second, we continued to analyze which t-norms and t-conorms are optimal in different situations. This directions was led by the US team, with an active participation of the Czech team.
- Together, we are working to incorporate all our findings into a single fuzzy control and soft computing methodology.

For the second direction, we need the following:

- to find the *best* hedges, t-norms, t-conorms, etc., we must first describe and analyze all possible t-norms, t-conorms, etc., describe the properties of different operations;
- then, before attempting to find optimal choices, we should analyze the optimal choices that have already been found by using different numerical optimization/approximation techniques and by using such soft computing techniques as neural networks, genetic algorithms, etc.; for this analysis to be successful, we must be able to describe these choices in more understandable terms; in other words, we must achieve a deep understanding of numerical approximations, neural networks, and genetic algorithms;
- next, we must be able to solve the corresponding optimization problem;
- if the result of our optimization is not good enough, we must be able to go beyond the traditional class of t-norms, t-conorms, etc.; there are two possibilities to go beyond the traditional classes:
  - we can use more general *logical values:* instead of using the interval $[0,1]$, we can can more general structures;
  - we can use more general *operations* with logical values: instead of restricting ourselves to commutative associative etc, t-norms and t-conorms, we may want to consider operations which are slightly non-associative, slightly non-commutative, etc.
- finally, from the viewpoint of practical applications, an important issue is computational complexity; a theoretically optimal control strategy is practically useless if computing the control value requires more time than we have.

Since all these directions are important, we have been pursuing them all. To give a better idea of what we are planning to do, let us these directions in more detail.

4 Extending Fuzzy Control Methodology to More Complex Linguistic Expressions

The prevailing applications of fuzzy control are based on simple control rules, like: "if the distance to the obstacle is small, decelerate fast". In real life, experts use much more sophisticated linguistic constructions to describe their control rules. For example, they use *hedges*, i.e., words like *almost*, *somewhat*, etc. One of the main objectives of the Czech research group is to formalize and use this sophisticated linguistic information in fuzzy control and other applications of fuzzy logic.

This research started with the the monograph [43]. Since the publication of this monograph, the Czech team has extended this approach to a general foundations of fuzzy logic and its applications [47]. Novák has also published a monograph [42] on general theory and applications of fuzzy sets. Other leading persons of the Czech team who are participating in this project are Irina Perfilieva and Jiří Močkoř.

The Czech team has applied their approach to challenging important real-life problems ranging from furnace control to problems related to safety of nuclear power plants.

5 Analysis of All Possible t-Norms, t-Conorms, etc.

The preliminary results of this analysis are described in [40]. This work is mainly done by E. A. Walker and C. Walker from the US team.

A promising special direction is the use of category theory to analyze fuzzy logics and fuzzy sets. Category theory is the foundations of modern algebra and modern mathematics, and it is definitely desirable to apply its rich ideas to the analysis of fuzzy logics. This research has been done separately both by C. Walker from the US team and by J. Močkoř from the Czech team.

6 Understanding the Relation Between Numerical Approximations, Neural Networks, Genetic Algorithms, and Fuzzy Logic

We can approach the problem of finding the optimal fuzzy control as a numerical optimization problem. There exist many techniques for solving this problem, ranging from crisp methods based on numerical optimization and approximation to soft computing techniques such as In control problems, neural networks can learn from patterns and simulate the control decisions of expert controllers. Genetic algorithms can be used to optimize in difficult-to-optimize situations – and fuzzy control is definitely one of such situations.

From the practical viewpoint, numerical optimization techniques, neural networks, and genetic algorithms are very successful in optimization. However, the results of these optimization techniques are often difficult to interpret. Indeed, fuzzy control rules, by definition, are formulated in terms of natural language and are, therefore, easier to understand and to analyze. In contrast, e.g., a neural network is described in terms of weights and connections, which are difficult to grasp and to analyze.

As we have mentioned, heuristic methods are often far from being optimal. To find optimal solutions, we must therefore be able to reformulate numerical approximation techniques, neural networks, and generic algorithms in terms which are easier to analyze, i.e., ideally, in terms similar to the terms in which we reason. In other words, we must find an interpretation of neural networks and genetic algorithms in terms of fuzzy logic.

The attempts to find such an interpretation are an ongoing effort. For approximations, an interpretation in terms of fuzzy logic was pioneered by I. Perfilieva from the Czech team; see, e.g., [44, 49, 50]. Some research in this direction was also done by researchers from the US team; see, e.g., [7, 24, 33].

For neural networks and genetic algorithms, some preliminary unpublished results, by A. Di Nola and V. Kreinovich, have been discussed at the workshop. These results – obtained mainly by the US team – are also based on the earlier mathematical foundational results by I. Perfilieva from the Czech team [48].

7 Finding Optimal t-Norms and t-Conorms for Different Optimization Problems

The explaination of the current empirically optimal selections in various areas of soft computing, ranging from the choice of t-norms and t-conorms in fuzzy logic to the choice of activation functions in neural networks, is given in [26].

In particular, for t-norms and t-conorms, the following results hold:

- If we are looking for the *smoothest* control, then the best choice is to use $f_\&(a,b) = a \cdot b$ and $f_\vee(a,b) = \min(a,b)$ [7, 14, 15, 54].
- If we are looking for the control that is *most robust* (i.e., least sensitive to the inaccuracy with which we measure the membership functions), then, depending on what exactly we are looking for, we can get two different results:
  - if we are looking for the control that is the most robust *in the the worst case*, then the best choice is to use $f_\&(a,b) = \min(a,b)$ and $f_\vee(a,b) = \max(a,b)$ [25, 31, 34, 32, 40];
  - if we are looking for the control that is the most robust *in the average*, then the best choice is to use $f_\&(a,b) = a \cdot b$ and $f_\vee(a,b) = a + b - a \cdot b$ [25, 32, 35, 40];
  - instead of minimizing the *average* error, we can try to minimize the corresponding *entropy* [4, 5, 13, 51, 52, 53]:
    - if we use the *average* entropy (in some reasonable sense), we get the same pair of optimal functions as for average error;
    - for an appropriately defined *worst-case* entropy the optimal operations are $f_\&(a,b) = \min(a,b)$ and $f_\vee(a,b) = a + b - a \cdot b$.
- If we are looking for the model that is the *fastest to compute*, then the best choice is to use $f_\&(a,b) = \min(a,b)$ and $f_\vee(a,b) = \max(a,b)$ [16].
- Finally, if, in control applications, we are looking for the *most stable* control for a given system, then the best choice is to use $f_\&(a,b) = \min(a,b)$ and $f_\vee(a,b) = a + b - a \cdot b$ [13, 14, 15, 54].

These optimization results are in good accordance with the general group-theoretic approach that enables us to classify techniques that are optimal relative to arbitrary reasonable criteria [1, 14, 15, 27, 54].

This approach is a general approach to optimization under uncertainty. In practical applications, we often need to make a selection in the situation in which we do not have a complete knowledge. For example, when we design an "optimal" image processing system for a rover going to a distant planet – but we do not know what kind of images to expect. We want to optimize message processing algorithms on the Internet – but these algorithms will then be hardwired and use for several years, and we do not know how exactly Internet will change and how what type of message will be routed during these years.

In traditional mathematical optimization problems of optimization without uncertainty, the relative quality of different alternatives is described by an objective function. In optimization under uncertainty, we do not have an objective function. We know that there is some preference relation — or, in mathematical terms, a partial pre-order.

In many practical problems, we also have natural symmetry operations so that the preference relation should naturally be invariant with respect to these symmetries. It turns out that in many practical situations, this invariance is sufficient to find — if not the optimal solution, but at least a small class of possibly optimal solutions. In our book [26], we have shown that this approach explains many heuristic methods in computer science, such as a choice of an activation function in neural networks, congestion-avoiding routing algorithms, etc. Since then, we have successfully used this idea in many other problems including imaging (e.g., visible shapes of extraterrestrial bodies can be naturally explained within this approach).

8 Analyzing More General Sets of Logical Values

Traditional fuzzy control methodology is mainly oriented towards values from the interval $[0,1]$. In many real-life situations, when the traditional approach does not work well, it is reasonable to consider more general sets of logical values. The research into such general algebraic description is mainly done by the Czech team, with the participation of A. Di Nola.

A specific case of sets of logical values was also analyzed by the US team. Specifically, the need for more general logics comes from the fact that just like experts are not sure about the statement S, they are also not sure about their own degrees of belief $d(S)$. Thus, instead of a single number $d(S)$, we can consider several possible numbers d, with degrees $d_2(d)$ describing to what extent these numbers are adequate descriptions of the original expert's uncertainty. This "second-order" approach has several successful applications. In principle, it is possible to go further and consider the fact that the degrees $d_2(d)$ are also not given precisely, so we seem to need the third-, fourth-order etc, approaches. However, in practice, such theoretically possible approaches turned out to be not useful. This fact can be explained if we take the multiresolutional character of reasoning into consideration:

- On the one hand, every "first-order" and "second-order" logic, in which the set of degree of belief is an ordered set, can be naturally described as a limit of an interval-related multiresolutional procedure [8, 9, 23, 60].
- On the other hand, if degrees come from words, then the third order is no longer necessary [11].

It is natural to select a continuous approach which best reflects the multiresolutional character of human reasoning, i.e., in which there is a qualitative difference between different pairs of degrees. A natural way to describe this difference in continuous case is to use the approach of non-standard analysis, with the actual infinitesimal elements (= lexicographic ordering). The optimal selection of such logics is described in [18, 37].

Eventually, we plan to combine these two directions.

9 Analyzing Possible Non-Associative Operations

Since $A\&B$ and $B\&A$ mean the same thing, it is natural to require that our degree of belief in $A\&B$ be the same as our degree of belief in $B\&A$, i.e., that the and-operation is commutative. Similarly, from the fact that $A\&(B\&C)$ and $(A\&B)\&C$ mean the same thing, we can conclude that the and-operation is associative. So, & is a semigroup operation. Similarly, "or"-operation $\vee$ is a semigroup operation, etc. So, we have several related semigroup (and other) operations on the same ordered set.

This fact is well known in computer science, and researchers in AI have been using the circa 1950s-1960s classification results for ordered semigroups to design and enhance their systems. Some of these results have led to drastic improvements in the quality of the corresponding intelligent control systems and and expert systems.

Let us also remark that the use of "and" and "or" connectives in natural language is much more intricate than can be captured by simple associative and commutative operations used till now (cf., e.g. [17]). This is another, very important motivation for study of non-associative operations.

However, there is still a gap between formalisms — in which algebraic properties like associativity are always true — and actual operations used by human experts in which all these properties are only approximately true. There are many empirical examples of non-associative, non-commutative, etc. operations. However, in our mathematical description of uncertainty in human reasoning, we stick with the associative operations simply because not much is known about non-associative ones. To bridge this gap, we need a deep mathematical analysis of the empirical operations, with the hope of extracting new weakened properties (something like "approximately associative") that would enable us to get reasonable mathematical results. We have already started this work, and obtained interesting mathematical results. These results are not yet at the level of deep general theorems, but even at the current weak level, they have interesting applications: for example, we are now able to have an explanation for the empirical 7 ± 2 rule, according to which experts usually use between 5 and 9 degrees of belief to describe their uncertainty.

Specifically, it is known that for given $p_1 = p(S_1)$ and $p_2 = p(S_2)$, possible values of $p(S_1 \,\&\, S_2)$ form an interval $\mathbf{p} = [p^-, p^+]$, where $p^- = \max(p_1 + p_2 - 1, 0)$ and $p^+ = \min(p_1, p_2)$; and possible values of $p(S_1 \vee S_2)$ form an interval $\mathbf{p} = [p^-, p^+]$, where $p^- = \max(p_1, p_2)$ and $p^+ = \min(p_1 + p_2, 1)$ (see, e.g., a survey [38] and references therein). So, in principle, we can use such interval estimates and get an interval $\mathbf{p}(C)$ of possible values of $p(C)$. Sometimes, this idea leads to meaningful estimates, but often, it leads to a useless $\mathbf{p}(C) = [0, 1]$ [38, 41]. In such situations, it is reasonable, instead of using the entire interval $\mathbf{p}$, to select a point within this interval as a reasonable estimate for $p(S_1 \,\&\, S_2)$ (or, correspondingly, for $p(S_1 \vee S_2)$).

Since the only information we have, say, about the unknown probability $p(S_1 \,\&\, S_2)$ is that it belongs to the interval $[p^-, p^+]$, it is natural to select a *midpoint* of this interval as the desired estimate:

$$f_{\&}(p_1, p_2) \stackrel{\text{def}}{=} \frac{1}{2} \cdot \max(p_1 + p_2 - 1, 0) + \frac{1}{2} \cdot \min(p_1, p_2);$$

$$f_{\vee}(p_1, p_2) \stackrel{\text{def}}{=} \frac{1}{2} \cdot \max(p_1, p_2) + \frac{1}{2} \cdot \min(p_1 + p_2, 1).$$

This midpoint selection is not only natural from a common sense viewpoint; it also has a deeper justification. Namely, in accordance of our above discussion, for $n = 2$ statements S_1 and S_2, to describe the probabilities of all possible Boolean combinations, we need to describe $2^2 = 4$ probabilities $x_1 = p(S_1 \,\&\, S_2)$, $x_2 = p(S_1 \,\&\, \neg S_2)$, $x_3 = p(\neg S_1 \,\&\, S_2)$, and $x_4 = p(\neg S_1 \,\&\, \neg S_2)$; these probabilities should add up to 1: $x_1 + x_2 + x_3 + x_4 = 1$. Thus, each probability distribution can be represented as a point $(x_1, \ldots, x_4)$ in a 3-D simplex $s = \{(x_1, x_2, x_3, x_4) \mid x_i \geq 0 \,\&\, x_1 + \ldots + x_4 = 1\}$. We know the values of $p_1 = p(S_1) = x_1 + x_2$ and $p_2 = p(S_2) = x_1 + x_3$, and we are interested in the values of $p(S_1 \,\&\, S_2) = x_1$ and $p(S_1 \vee S_2) = x_1 + x_2 + x_3$. It is natural to assume that *a priori*, all probability distributions (i.e., all points in a simplex s) are "equally possible", i.e., that there is a uniform distribution ("second-order probability") on this set of probability distributions. Then, as a natural estimate for the probability $p(S_1 \,\&\, S_2)$ of $S_1 \,\&\, S_2$, we can take the conditional mathematical expectation of this probability under the condition that the values $p(S_1) = p_1$ and $p(S_2) = p_2$:

$$E(p(S_1 \,\&\, S_2) \mid p(S_1) = p_1 \,\&\, p(S_2) = p_2) =$$

$$P(x_1 \mid x_1 + x_2 = p_1 \,\&\, x_1 + x_3 = p_2).$$

The problem is that these operations are non-associative. Why is this a problem? If we are interested in estimating the degree of belief in a conjunction of three statements $S_1 \,\&\, S_2 \,\&\, S_3$, then we can either apply the "and" operation to p_1 and p_2 and get an estimate $f_\&(p_1, p_2)$ for the probability of $S_1 \,\&\, S_2$ and then, we apply the "and" operation to this estimate and p_3, and get an estimate $f_\&(f_\&(p_1, p_2), p_3)$ for the probability of $(S_1 \,\&\, S_2) \,\&\, S_3$. Alternatively, we can get start by combining S_2 and S_3, and get an estimate $f_\&(p_1, f_\&(p_2, p_3))$. Intuitively, we would expect these two estimates to coincide, but, e.g., $(0.4 \,\&\, 0.6) \,\&\, 0.8 = 0.2 \,\&\, 0.8 = 0.1$, while $0.4 \,\&\, (0.6 \,\&\, 0.8) = 0.4 \,\&\, 0.5 = 0.2 \neq 0.1$.

How can we solve this problem? Since we know that the numerical values are only an approximation, we can analyze how non-associative the above operations can be. If the difference is below the natural resolution level, then, from the practical point of view, the above operations are as good as associative ones. The following is true [2, 20]:

$$\max_{a,b,c} |f_\&(f_\&(a, b), c) - f_\&(a, f_\&(b, c))| = \frac{1}{9};$$

$$\max_{a,b,c} |f_\vee(f_\vee(a, b), c) - f_\vee(a, f_\vee(b, c))| = \frac{1}{9}.$$

Each word describing a degree of belief is a "granule" covering the entire sub-interval of values. Thus, non-associativity is negligible if the corresponding realistic "granular" degree of belief have granules of width $\geq 1/9$. One can fit no more than 9 granules of such width in the interval $[0, 1]$. This may explain why humans are most comfortable with ≤ 9 items to choose from – the famous "7 plus minus 2" law; see, e.g., [21, 22].

10 Reducing Computational Complexity of the Existing Techniques

In addition to a purely mathematical problem of developing new techniques, we face the problem of decreasing the computational complexity of the corresponding algorithms. Most problems of optimization under uncertainty are NP-hard even if consider only the degenerate case of interval uncertainty [6]. It is therefore important to develop new methods for interval and fuzzy computations. Since the main reason why the corresponding computational problems are so complex is the presence of constraints that limit possible combinations of values of different variables, our plan is to concentrate on computations under such constraints, i.e., on *constrained* interval and fuzzy arithmetic. We have started doing a joint research in this direction [11, 57], and we plan to continue this work.

An important approach to reducing computational complexity is the approach of *granularity*. In contrast to data processing, where all the real numbers come from measurements, in fuzzy systems, the degrees of belief come from expert estimates. Although we use real numbers to describe such degrees, but in reality, an expert cannot describe his degree with too large an accuracy: hardly anyone can distinguish between, say, degree 0.81 and 0.82. So, although we use real numbers, in reality, we should combine these real numbers into finitely many *granules* (clusters), and reduce the computational complexity by using, instead of the entire real number, only the index of the granule to which it belongs. Granules do not have to be crisp sets, they can be fuzzy sets as well.

This granular approach have been partly pursued by researchers from both teams. At the workshop, V. Novák described a new approach in which granules are characterized by membership functions whose description (in a fixed formal language) is short (see [46]). This promising approach to granularity seems to be closely related to Algorithmic Information Theory (theory of Kolmogorov complexity) [19] where a complexity of a string is defined as a shortest length of its description in a certain formal language.

11 Future Plans

We plan continue our collaboration in designing new application-oriented theoretical foundations of soft computing methodology, and thus, in finding new more successful applications of this methodology.

Our immediate goal is to improve control applications in the areas in which we already have an experience of applying fuzzy control: furnace control and safety of nuclear power stations. We also plan to develop new practical applications.

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