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Abstract

In the 1960s, A.N. Kolmogorov described the main reason why a mathematical correct solution to a system of differential equations may be not physically possible: Traditional mathematical analysis tacitly assumes that all numbers, no matter how large or how small, are physically possible. From the engineering viewpoint, however, a number like $10^{10^{10}}$ is not possible, because it exceeds the number of particles in the Universe. In this paper, we extend Kolmogorov's ideas from discrete objects to continuous objects known with given accuracy ε, and show how this extension can clarify the analysis of dynamical systems.

1 Problem

In the 1960s, A.N. Kolmogorov described the main reason why a mathematical correct solution to a system of differential equations may be not physically possible (see, e.g., [6]):

- Traditional *mathematical* analysis tacitly assumes that all numbers, no matter how large or how small, are physically possible.
- From the *engineering* viewpoint, however, a number like $10^{10^{10}}$ is not possible, because it exceeds the number of particles in the Universe.

In particular, solutions to the corresponding systems of differential equations which lead to some numbers – or statements that we can predict the outcome of a chaotic system – are mathematically OK, but physically meaningless.

Very small numbers are also physically meaningless. For example, mathematically, there is a positive probability that a kettle, placed on a cold stove, will boil by itself. From the viewpoint of a working physicist, this probability is so small that this is absolutely impossible.

How can we describe this in precise terms?

2 Main Idea

Our main idea is that it is important to list the *observables* $o_1, \ldots, o_n$ of the system, and to impose physical restrictions on the accuracy and range of these observables.

In particular, two states $o = (o_1, \ldots, o_n)$ and $o' = (o'_1, \ldots, o'_n)$ in which the values of these observables are close – e.g., for which

$$d(o, o') \stackrel{\text{def}}{=} \sqrt{(o_1 - o'_1)^2 + \ldots + (o_n - o'_n)^2} \le \varepsilon$$

for some threshold value ε – cannot be physically distinguished.

In effect, what we are doing is introducing discreteness into a system. Clearly, such a discreteness makes the description of a physical system more realistic. However, as it is well known, discreteness makes the analysis of the system more computationally difficult: e.g., continuous optimization problems are easy to solve while the corresponding discrete optimization problems are difficult to solve (NP-hard; see, e.g., [10]). The complexity of analyzing discrete system is one of the main reasons why it is much easier to describe a solid body as a continuous media than to describe it as a collection of discrete atoms/molecules.

Let us show that our particular version of discreteness not only makes the models more realistic, but also often simplifies the analysis. We will show it on the example of the *reversibility* problem, a known problem of the foundations of physics:

- On the one hand, Newton's equations are reversible – they remain the same if we replace time t by $-t$.
- On the other hand, it is well known that many physical processes are not physically reversible – e.g., if we place an ink drop in water, it will spread around, but it is physically impossible for the ink in water to gather into a single drop.

From the above viewpoint, there is no contradiction here. Namely, to reverse an operation, we must be able to guarantee the resulting state based on the initial setting. According to our approach, when we set up an initial state $o(t_0)$, we cannot distinguish between ε-close states. Different initial states $o(t_0)$ lead, generally speaking, to different resulting states $o(t)$ at a future moment of time $t > t_0$. So, the only situation when we can guarantee the result is when different resulting states are also physically indistinguishable, i.e., when $d(o(t_0), o'(t_0)) \le \varepsilon$ implies $d(o(t), o'(t)) \le \varepsilon$. The transition from the drop-of-ink to spread-ink has this property, while the inverse transition does not.

In general, from the purely mathematical viewpoint, we can predict the behavior of an arbitrary dynamical system. However, in real life, for many systems, we cannot do that, because predicting the state $o(t)$ with a given accuracy ε_0 requires, for large t, that we know the initial state $o(t_0)$ with an impossible accuracy $\varepsilon_0 \cdot c^{-(t-t_0)}$ (for some $c > 1$). The only case when we can meaningfully predict the behavior of the system is when $d(o(t_0), o'(t_0)) \le \varepsilon$ implies $d(o(t), o'(t)) \le \varepsilon$. In other cases, we have a *chaos-type* phenomenon.

Similarly, the only case when we can reconstruct the past state is when $d(o(t), o'(t)) \le \varepsilon$ implies $d(o(t_0), o'(t_0)) \le \varepsilon$.

There are systems for which we can both predict the future and reconstruct the past. In such systems, $d(o(t), o'(t)) \le \varepsilon \leftrightarrow d(o(t_0), o'(t_0)) \le \varepsilon$. In geometry, it has been proven (see, e.g., [1, 2]) that the corresponding transformation $o(t_0) \to o(t)$ is *linear*; moreover, it is an isometry (a composition of shifts and rotations). Thus, the only true reversible systems are linear ones.

This restriction is not as restrictive as it may sound, because, e.g., in quantum mechanics, dynamics is always linear – and so, via quantization, we can represent every system as linear.

Within this idea, how can we describe the properties of a physical state? For example, suppose that we have some way to describe the complexity $K(o)$ of a mathematical state (e.g., as its Kolmogorov complexity [12] or as its entropy). In this case, physically indistinguishable (i.e., ε-close) states may have different complexity, so, instead of the *actual* complexity of an observable state, we can only talk about the *bounds* on this complexity. For each state o, we can have indistinguishable states $o' \sim o$ with arbitrary high complexity, so the upper bound is simply ∞, and the only meaningful characteristic is the lower bound $K_\varepsilon(o) \stackrel{\text{def}}{=} \min\{K(o') \,|\, d(o, o') \le \varepsilon\}$ (for Kolmogorov complexity, a similar characteristic was first proposed in [13]). For chaotic systems, thus defined complexity $K_\varepsilon(o(t))$ increases with time t; for stabilizing systems, it decreases; for physically reversible (linear) systems, it remains $\approx$ const.

It is worth mentioning that our approach is related to A. Trautman's work [14] that describes a similar discrete version of spinors, and to Kaneko's coupled map lattices approach [5] in which subsystems form a discrete geometry.

3 Towards an Accurate Formalization of This Idea

How can we formalize our solution to the reversibility problem? In effect, our main idea is that the probability of ink molecules accidentally getting together is so small that from the physical viewpoint, it is simply impossible.

A similar situation occurs if we flip a coin 100 times in a row, and get heads all the time. In this case, a mathematician knowledgeable in probability may say that it is still possible that the coin is fair – since in this case, the probability of 100 heads in a row is small (2^{-100}) but positive. However, an engineer (and any person who uses common sense) would say that the coin is not fair, because if it is was a fair coin, then this *abnormal* event would be impossible.

At first glance, it looks like this idea has a natural formalization: if a probability of an event is small enough (say, $\le p_0$ for some very small p_0), then this event cannot happen. For example, the probability that a fair coin falls heads 100 times in a row is 2^{-100}, so, if we choose $p_0 \ge 2^{-100}$, then we will be able to conclude that such an event is impossible.

The problem with this approach is that *every* sequence of heads and tails has exactly the same probability. So, if we choose $p_0 \ge 2^{-100}$, we will thus exclude all possible sequences of heads and tails as physically impossible. However, anyone can toss a coin 100 times, and thus prove that some sequences are physically possible.

"Abnormal" means something unusual, rarely happening: if something is rare enough, it is not typical ("abnormal"). Let us describe what, e.g., an abnormal

height may mean. If a person's height is ≥ 6 ft, it is still normal (although it may be considered abnormal in some parts of the world). Now, if instead of 6 pt, we consider 6 ft 1 in, 6 ft 2 in, etc, then sooner or later we will end up with a height h such that everyone who is higher than h will be definitely called a person of abnormal height. We may not be sure what exactly value h experts will call "abnormal", but we are sure that such a value exists.

Let us express this idea is general terms. We have a *Universe of discourse*, i.e., a set U of all objects that we will consider. Some of the elements of the set U are abnormal (in some sense), and some are not. Let us denote the set of all elements that are *typical* (*not abnormal*) by T. On the set U, we have a decreasing sequence of sets $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ with the property that $\cap A_n = \emptyset$. In the above example, U is the set of all people, A_1 is the set of all people whose height is ≥ 6 ft, A_2 is the set of all people whose height is ≥ 6 ft 1 in, A_2 is the set of all people whose height is ≥ 6 ft 2 in, etc. We know that if we take a sufficiently large n, then all elements of A_n are abnormal (i.e., none of them belongs to the set T of not abnormal elements). In mathematical terms, this means that for some n, we have $A_n \cap T = \emptyset$.

In case of a coin: U is the set of all infinite sequences of results of flipping a coin; A_n is the set of all sequences that start with n heads but have some tail afterwards. Here, $\cap A_n = \emptyset$. Therefore, we can conclude that there exists an n for which all elements of A_n are abnormal. According to mechanics, the result of flipping a coin is uniquely determined by the initial conditions, i.e., on the initial positions and velocities of the atoms that form our muscles, atmosphere, etc. So, if we assume that in our world, only not-abnormal initial conditions can happen, we can conclude that for some n, the actual sequence of results of flipping a coin cannot belong to A_n. The set A_n consists of all elements that start with n heads and a tail after that. So, the fact that the actual sequence does not belong to A_n means that if an actual sequence has n heads, then it will consist of all heads. In plain words, if we have flipped a coin n times, and the results are n heads, then this coin is biased: it will always fall on head.

Let us describe this idea in mathematical terms [4, 11]. To make formal definitions, we must fix a formal theory: e.g., the set theory ZF (the definitions and results will not depend on what exactly theory we choose). A set S is called *definable* if there exists a formula $P(x)$ with one (free) variable x such that $P(x)$ if and only if $x \in S$.

Crudely speaking, a set is definable if we can *define* it in ZF. The set of all real numbers, the set of all solutions of a well-defined equations, every set that we can describe in mathematical terms is definable.

This does not means, however, that *every* set is definable: indeed, every definable set is uniquely determined by formula $P(x)$, i.e., by a text in the language of set theory. There are only denumerably many words and therefore, there are only denumerably many definable sets. Since, e.g., there are more than denumerably many set of integers, some of them are thus not definable.

Definition 1. A sequence of sets $A_1, \ldots, A_n, \ldots$ is called *definable* if there exists a formula $P(n, x)$ such that $x \in A_n$ if and only if $P(n, x)$.

Definition 2. Let U be a universal set.

- A non-empty set $T \subseteq U$ is called a *set of typical (not abnormal) elements* if for every definable sequence of sets A_n for which $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$, there exists an N for which $A_N \cap T = \emptyset$.
- If $u \in T$, we will say that u is *not abnormal.*
- For every property P, we say that *"normally, for all u, $P(u)$"* if $P(u)$ is true for all $u \in T$.

How are these definitions related to Kolmogorov complexity $K(x)$? Kolmogorov complexity – i.e., the shortest length of a program that generates x – enables us to define the notion of a *random sequence*, e.g., as a sequence s for which there exists a constant $c > 0$ for which, for every n, the (appropriate version of) Kolmogorov complexity $K(s_{|n})$ of its n-element subsequence $s_{|n}$ exceeds $n - c$. Crudely speaking, c is the amount of information that a random sequence has.

Random sequences in this sense do not satisfy the above definition, and are not in perfect accordance with common sense – because, e.g., a sequence that starts with 10^6 zeros and then ends in a truly random sequence is still random. Intuitively, for "truly random" sequences, c should be small, while for the above counter-example, $c \approx 10^6$. If we restrict ourselves to random sequences with fixed c, we satisfy the above definition.

There are many ways to define Kolmogorov complexity and random sequences [12]; it is therefore desirable to aim for results that are true in as general case as possible. In view of this desire, in the following text, we will not use any specific version of these definitions; instead, we will assume that Definition 2 is true.

It is possible to prove that abnormal elements do exist [4]; moreover, we can select T for which abnormal elements are as rare as we want: for every probability distribution p on the set U and for every ε, there exists a set T for which the probability $p(x \notin T)$ of an element to be abnormal is $\leq \varepsilon$:

Proposition 1. [11] *For every probability measure μ on a set U (in which all definable sets are measurable), and for every $\varepsilon > 0$, there exists a set T of typical elements for which $\mu(T) > 1 - \varepsilon$.*

What are the possible applications of these definitions? First, restriction to "not abnormal" solutions leads to regularization of ill-posed problems.

An ill-posed problem arises when we want to reconstruct the state s from the measurement results r. Usually, all physical dependencies are continuous, so, small changes of the state s result in small changes in r. In other words, a mapping $f : S \to R$ from the set of all states to the set of all observations is continuous (in some natural topology). We consider the case when the measurement results are (in principle) sufficient to reconstruct s, i.e., the case when the mapping f is 1-1. That the problem is ill-posed means that small changes in r can lead to huge changes in s, i.e., that the inverse mapping $f^{-1} : R \to S$ is *not* continuous.

We will show that if we restrict ourselves to states S that are not abnormal, then the restriction of f^{-1} will be continuous, and the problem will become well-posed.

Definition 3. A definable metric space (X, d) is called *definably separable* if there exists a definable everywhere dense sequence $x_n \in X$.

Proposition 2. [8, 9, 11] *Let S be a definably separable definable metric space, T be a set of all not abnormal elements of S, and $f : S \to R$ be a continuous 1-1 function. Then, the inverse mapping $f^{-1} : R \to S$ is continuous for every $r \in f(T)$.*

In other words, if we know that we have observed a not abnormal state (i.e., that $r = f(s)$ for some $s \in T$), then the reconstruction problem becomes well-posed. So, if the observations are accurate enough, we get as small guaranteed intervals for the reconstructed state s as we want.

Another application is justification of physical induction. From the viewpoint of an experimenter, a physical theory can be viewed as a statement about the results of physical experiments. If we had an infinite sequence of experimental results $r_1, \ldots, r_n, \ldots$, then we will be able to tell whether the theory is correct or not. So, a theory can be defined as a set of sequences $r_1, r_2, \ldots$ that are consistent with its equations, inequalities, etc. In real life, we only have finitely many results $r_1, \ldots, r_n$, so, we can only tell whether the theory is *consistent* with these results or not, i.e., whether there is an infinite sequence $r_1, r_2, \ldots$ that starts with the given results that satisfies the theory.

It is natural to require that the theory be *physically meaningful* in the following sense: if all experiments confirm the theory, then this theory should be correct. An example of a theory that is not physically meaningful is easy to give: assume that a theory describes the results of tossing a coin, and it predicts that at least once, there should be a tail. In other words, this theory consists of all sequences that contain at least one tail. Let us assume that actually, the coin is so biased that we always have heads. Then, this infinite sequence does not satisfy the given theory. However, for every n, the sequence of the first n results (i.e., the sequence of n heads) is perfectly consistent with the theory, because we can add a tail to it and get an infinite sequence that belongs to the set $\mathcal{T}$. Let us describe this idea in formal terms.

Definition 4. Let a definable set R be given. Its elements will be called *possible results of experiments.* By S, we will denote the set of all possible sequences $r_1, r_n, \ldots$, where $r_i \in R$. By a *theory*, we mean a definable subset $\mathcal{T}$ of the set of all infinite sequences S. If $r \in \mathcal{T}$, we say that a sequence r *satisfies* the theory $\mathcal{T}$, or, that for this sequence r, *the theory $\mathcal{T}$ is correct.*

Comment. A theory is usually described by its axioms and deduction rules. The theory itself consists of all the statements that can be deduced from the axioms by using deduction rules. In most usual definitions, the resulting set is recursively enumerable (r.e.) – hence definable. We therefore define a theory as a definable set.

Definition 5. We say that a finite sequence $(r_1, \ldots, r_n)$ is *consistent* with the theory $\mathcal{T}$ if there exists an infinite sequence $r \in \mathcal{T}$ that starts with $r_1, \ldots, r_n$ and

that satisfies the theory. In this case, we will also say that the *first n experiments confirm the theory.*

Definition 6. We say that a theory $\mathcal{T}$ is *physically meaningful* if the following is true for every sequence $r \in S$:

> If for every n, the results of first n experiments from r confirm the theory $\mathcal{T}$, then, the theory $\mathcal{T}$ is correct for r.

In this case, the universal set consists of all possible infinite sequence of experimental results, i.e., $U = S$. Let $T \subseteq S$ be the set of all typical (not abnormal) sequences.

Proposition 3. [7] *For every physically meaningful theory $\mathcal{T}$, there exists an integer N such that if a sequence $r \in S$ is not abnormal and the first N experiment confirm the theory $\mathcal{T}$, then this theory $\mathcal{T}$ is correct.*

Idea of the proof: as A_n, we take the set of all the sequences r for which either the first n experiments confirm $\mathcal{T}$ or $\mathcal{T}$ is not correct for r.

This result shows that we can *confirm* the theory based on finitely many observations. The derivation of a general theory from finitely many experiments is called *physical induction* (as opposed to *mathematical induction*). The general physical induction is difficult to justify, to the extent that a prominent philosopher C. D. Broad has called the unsolved problems concerning induction a *scandal of philosophy* [3]. We can say that the notion of "not abnormal" justifies physical induction by making it a provable theorem (and thus resolves the corresponding scandal).

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