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Examples of hyperbolic knots with distance 3 toroidal surgeries in the 3-sphere

Cesar Garza
University of Texas at El Paso, cagarza@miners.utep.edu

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CÉSAR GARZA

Department of Mathematical Sciences

APPROVED:

Luis Valdez-Sánchez, Ph.D., Chair

Leslie Foged, Ph.D.

Jorge López, Ph.D.

Patricia D. Witherspoon, Ph.D.
Dean of the Graduate School

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CÉSAR GARZA

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Abstract

By the work of Thurston, any surgery on a hyperbolic knot in the 3-sphere produces a hyperbolic 3-manifold except in at most finitely many cases. So far, the figure-8 knot seems to be the best candidate for a hyperbolic knot with the most (8) non-trivial exceptional surgeries. In recent years, much progress has been made in the classification of hyperbolic knots admitting more than one exceptional toroidal surgery. In fact, such classification is known for toroidal surgeries with distance at least 4.

We give a classification of hyperbolic knots in S^3 admitting two toroidal surgeries at distance 3, whose slopes are represented by twice punctured essential separating tori. Such knots belong to a family $K(a, b, n)$, where a, b, n are integers and $\gcd(a, b) = 1$.

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Chapter 1

Introduction

Surfaces of non-negative Euler characteristic, i.e., spheres, disks, tori and annuli, play a special role in the theory of 3-dimensional manifolds. Following [20], we call a 3-manifold that contains no essential sphere, disk, torus or annulus *simple*. By [17] a 3-manifold M with non-empty boundary is simple if and only if M with its boundary tori removed has a hyperbolic structure of finite volume with totally geodesic boundary. On what follows, 3-manifolds are assumed to be compact, connected and orientable, while surfaces in a 3-manifold are assumed to be properly embedded and orientable, unless otherwise stated. Perelman's proof of Thurston's Geometrization Conjecture [12, 13] asserts that M is simple if and only if M is either hyperbolic or belongs to a certain class of small Seifert fiber spaces.

Let M be a 3-manifold with a torus boundary component T_0. A *slope* is an isotopy class of circles embedded in T_0. For any slope r on T_0, the *Dehn filling of M along r* is the manifold $M(r) = M \cup_{T_0} V_r$, where V_r is a solid torus glued to M via a homeomorphism between T_0 and ∂V_r which identifies r with the boundary of a meridian disk of V_r. The operation of Dehn filling is one of the central objects of study in 3-manifold topology; specifically, the study of non-negative Euler characteristic surfaces obtained by Dehn filling on 3-manifolds has seen great developments in the last twenty years.

By Thurston's Hyperbolic Dehn Surgery Theorem, there are only finitely many non-simple Dehn fillings on each torus boundary component of M. Though little can be said in general about a manifold M from a single non-simple Dehn filling, much information and even a complete classification can be obtained from two distinct non-simple Dehn fillings $M(r)$, $M(s)$. It turns out that in such cases the *distance* $\Delta = \Delta(r, s)$ between the slopes

r and s (i.e., their minimal geometric intersection number) is quite small. In fact, the least upper bound for Δ in non-simple Dehn fillings is already determined (see [9] and the references therein). If $M(r)$ contains a geometrically incompressible closed torus, we say that $M(r)$ is *toroidal* and r is a *toroidal slope.* If $M(r)$ is toroidal, then M contains an essential many-punctured torus T whose boundary components have slope r on T_0; we then say that T *represents* the slope r.

By [5], if r, s are toroidal slopes then $\Delta(r,s) \leq 8$; moreover there are exactly two manifolds M with $\Delta = 8$, one with $\Delta = 7$ and one with $\Delta = 6$. More recent work by Gordon and Wu [10] completes the classification of toroidal Dehn fillings at distance $\Delta \geq 4$. We will work with the case $\Delta = 3$ but in the following more restricted setting.

Two knots in S^3 are said to be equivalent if there is an automorphism in S^3 that maps one knot onto the other. Let K be a knot in S^3 and let $N(K)$ be a regular neighborhood of the knot. We denote by $X_K = S^3 \setminus \mathrm{int}N(K)$ the *exterior* of the knot in S^3. Thus X_K is a compact, connected, oriented 3-manifold with torus boundary T_0; we say that K is a *hyperbolic knot* if X_K is a hyperbolic 3-manifold. Let μ, λ be a standard meridian/longitude frame on T_0 (see [15]). Any slope on T_0 can be represented as a homology class, or as a fraction $\frac{a}{b} \in \mathbb{Q} \cup \{\frac{1}{0}\}$. Such a slope is said to be *integral* if $b = 1$, that is, if the slope is isotopic to K in $N(K)$. For r a boundary slope on T_0 we will denote by $K(r)$ the Dehn filled manifold $X_K(r)$. We also say that $K(r)$ has been obtained by Dehn surgery on K along r.

The classification of hyperbolic knots in S^3 admitting two toroidal surgeries at distance $\Delta \geq 4$ has been completed in recent years (cf. [10]). Gordon and Luecke proved in [7] that hyperbolic knots admit only integral toroidal slopes, except for the Eudave-Muñoz knots $k(l,m,n,p)$, which admit two toroidal Dehn surgeries if $\Delta \geq 5$ only; moreover, if one of the essential punctured tori S, T in X_K is nonseparating, K is a knot of genus one by the work of Gabai [4] and M. Teragaito classified this case in [16]. For these reasons we will assume S and T are separating tori with integral boundary slopes, so the number of boundary components $|\partial S|, |\partial T|$ are even. In this work we will only study the case $|\partial S| = 2 = |\partial T|$,

which is the one expecting to produce the largest families of examples of such knots in S^3. Let $K(a, b, n)$ be the family of knots introduced in Chapter 4. Our goal is to prove the following theorem:

Theorem 1.1. *Let K be a hyperbolic knot in S^3 admitting two toroidal surgeries $K(\sigma)$, $K(\tau)$ at distance $\Delta(\sigma, \tau) = 3$, represented by essential twice punctured separating tori S, T in X_K, respectively. Then K is equivalent to some knot of the form $K(a, b, n)$ where a, b, n are integers and* $\gcd(a, b) = 1$.

Here is a summary of the organization of this paper. Chapter 2 contains the definitions and fundamental results related to the graphs of intersection produced by two essential surfaces in M with transverse intersection. We also narrow the options for the graphs of S and T to only three cases. Due to its relevance with these families of graphs, in chapter 3 we present some basic properties of the genus two handlebody and its fundamental group, a free group on two generators. The cases obtained in chapter 2 are thoroughly studied in chapter 4, where we discard one of them. In the remaining two cases the corresponding pairs S, T are shown to be isotopic while giving rise to distinct graphs of intersection. Using the results in chapter 3 the geometric realization of these cases are shown to produce members of the family of knots $K(a, b, n)$ where a, b are relatively prime integers. Chapter 5 presents a direct way to represent each $K(a, b, n)$ by the expansion of the rational number $\frac{a}{b}$ in continued fractions. This chapter ends with the proof of Theorem 1.1 and presents some criteria to identify the elements in this family that represent the knots studied in this paper. Finally, in chapter 6 we give some directions for some applications of the technique used in Chapters 4 and 5 to another cases of non-simple surgeries on hyperbolic knots.

Chapter 2

Graphs of Intersection

2.1 Preliminaries

A 3-manifold M is said to be reducible if there exists a 2-sphere in M not bounding a 3-ball in M. Such sphere is called a *reducing sphere*. M is ∂-reducible if its boundary, denoted by ∂M, is compressible in M, in which case a compressing disk of ∂M is also called a *boundary reducing disk* of M. Let M be a hyperbolic 3-manifold with a torus T_0 as a boundary component. A surface of non-negative Euler characteristic in M is *essential* if it is incompressible, ∂-incompressible, and is not boundary parallel; a sphere (resp. disk) is essential if it is a reducing sphere (resp. boundary reducing disk). We use a, b to denote the numbers 1 or 2, with the convention that if they both appear in a statement then $\{a, b\} = \{1, 2\}$.

A slope on T_0 is a toroidal slope if $M(r_a)$ is toroidal. Let r_a be a toroidal slope on T_0. Denote by $\Delta = \Delta(r_1, r_2)$ the minimal geometric intersection number between r_1 and r_2. When $\Delta > 3$ the manifolds M have been determined in [5] and [10], so we will always assume $\Delta = 3$. Let $\widehat{F}_a$ be an essential torus in $M(r_a)$, and let $F_a = \widehat{F}_a \cap M$.

Let $n_a = |\partial F_a \cap T_0|$. Isotope each $\widehat{F}_a$ so that F_1, F_2 intersect transversely in arcs and circles which are essential on both F_a and n_a is minimal among all essential tori in $M(r_a)$. Denote by J_a the attached solid torus in $M(r_a)$, and by u_i ($i = 1, \ldots, n_a$) the components of $\widehat{F}_a \cap J_a$, which are all disks, labeled successively when traveling along J_a. Similarly let v_j be the disk components of $\widehat{F}_b \cap J_b$. Let Γ_a be the graph on F_a with the u_i's as (fat) vertices, and the arc components of $F_1 \cap F_2$ as edges. Similarly for Γ_b. The minimality of the number of components in $F_1 \cap F_2$ and the minimality of n_a imply that Γ_a has no trivial

loops, and that each disk face of Γ_a in $\widehat{F}_a$ has interior disjoint from F_b.

If e is an edge of Γ_a with an endpoint x on a fat vertex u_i, then x is labeled j if x is in $u_i \cap v_j$. In this case e is called a j-edge in Γ_a, and an i-edge in Γ_b. Labels in Γ_a are considered as mod n_b integers; in particular, $n_b + 1 = 1$. When going around ∂u_i, the labels of the endpoints of edges appear as $1, 2, \ldots, n_b$ repeated Δ times. Label the endpoints of edges in Γ_b similarly.

Each vertex of Γ_a is given a sign according to whether J_a passes $\widehat{F}_a$ from the positive side or negative side at this vertex. Two vertices of Γ_a are parallel if they have the same sign, otherwise they are antiparallel. Following the terminology of [19], if the vertices u_i of $\widehat{F}_a$ are all of the same parity we will say that F_a is *polarized*, and that it is *neutral* if there are the same number of vertices of either parity. Observe that if $\widehat{F}_a$ is a separating surface then n_a is even and u_i, u_j are parallel if and only if i, j have the same parity. For an arbitrary graph G, the valence of a vertex v of G, denoted by $val(v, G)$ is the number of edge endpoints of G that are incident to v. If G is clear from the context, we simply denote it by $val(v)$. Observe that for any vertex u of Γ_a, $val(u) = \Delta n_b$.

An edge of Γ_a is a positive edge if it connects parallel vertices. Otherwise it is a negative edge. A collection of edges in Γ_a whose union is a circle in $\widehat{F}_a$ (where the circle is constructed in the obvious way, by collapsing the vertices into points in $\widehat{F}_a$) is called a *cycle*. A cycle in Γ_a is nontrivial if it is not contained in a disk in $\widehat{F}_a$. We call a cycle in Γ_a consisting of a single edge a *loop edge*; notice that since F_a is orientable a loop edge in Γ_a is positive. If an edge e of Γ_a has the same label on its two endpoints we call it a *co-loop edge*. In other words, e is a loop on the other graph Γ_b.

When considering each family of parallel edges of Γ_a as a single edge $\widehat{e}$, we get the reduced graph $\widehat{\Gamma}_a$ on $\widehat{F}_a$. It has the same vertices as Γ_a. Each edge $\widehat{e}$ of $\widehat{\Gamma}_a$ represents a family of parallel edges in Γ_a. The number of such parallel edges is called the *weight* of $\widehat{e}$. We shall often refer to a family of parallel edges as simply a *family*.

A cycle in Γ_a consisting of two positive edges e, e' is a *Scharlemann bigon* if it bounds a disk with interior disjoint from the graph, and e, e' have the same pair of labels $\{j, j+1\}$

(with $j, j+1$ well defined mod n_b) at their two endpoints. A Scharlemann bigon with label pair, say, $\{1,2\}$ will also be called a (12)-Scharlemann bigon; this is a restricted version of the more general notion of a Scharlemann cycle, which we will not use here. If $\{e, e'\}$ is a Scharlemann bigon in Γ_a then the subgraph of Γ_b consisting of these edges and their vertices is called a *Scharlemann cocycle*

The following lemma summarizes several fundamental results regarding the properties of the graphs Γ_a. The proofs can be found in [10] and its references.

Lemma 2.1.1 ([10], §2). 1. (The Parity Rule) *An edge e is a positive edge in Γ_1 if and only if it is a negative edge in Γ_2.*

2. *A pair of edges cannot be parallel on both Γ_1 and Γ_2.*

3. *If Γ_a has a Scharlemann bigon, then $\widehat{F}_b$ is separating. In particular, Γ_b has the same number of positive and negative vertices, so n_b is even, and two vertices v_i, v_j of Γ_b are parallel if and only if i, j have the same parity.*

4. *If Γ_a has a Scharlemann bigon then the corresponding Scharlemann cocycle on Γ_b is essential.* □

Let u be a vertex of Γ_a, and P, Q two edge endpoints on ∂u. Let I be the interval on ∂u from P to Q along the direction induced by the orientation of u. The edge endpoints in Γ_a cut I into k subintervals for some k. We follow [10] and define the *distance* from P to Q on ∂u as $d_u(P, Q) = k$. Observe that in general $d_u(P, Q) \neq d_u(Q, P)$, so this is not a metric since symmetry fails. If P, Q are the only edge endpoints of e_1, e_2 on ∂u, respectively, then we define $d_u(e_1, e_2) = d_u(P, Q)$. Notice that for u on Γ_a, $d_u(P, Q) + d_u(Q, P) = \Delta n_b$. The following is proved in [5].

Lemma 2.1.2 ([5], Lemma 2.4). 1. *Suppose $P, Q \in \partial u_i \cap \partial v_k$ and $R, S \in \partial u_j \cap \partial v_l$. If $d_{u_i}(P, Q) = d_{u_j}(R, S)$ then $d_{v_k}(P, Q) = d_{v_l}(R, S)$.*

2. *Suppose that $P \in u_i \cap v_k$, $Q \in u_i \cap v_l$, $R \in u_j \cap v_k$ and $S \in u_j \cap v_l$. If $d_{u_i}(P, Q) = d_{u_j}(P, Q)$, then $d_{v_k}(R, S) = d_{v_l}(R, S)$.* □

Suppose two edges e_1, e_2 of Γ_a connect the same pair of vertices u_i, u_j (we allow $u_i = u_j$). Name p_k, q_k the endpoints of e_k on u_i, u_j, respectively, $k = 1, 2$. We say that e_1 and e_2 are *equidistant* if $d_{u_i}(p_1, p_2) = d_{u_j}(q_2, q_1)$ (note that the orders of the edge endpoints have been reversed). Thus for example a pair of parallel positive edges is always equidistant, but a pair of parallel negative edges is not unless their distance is exactly half of the valence of the vertices.

When $u_i \neq u_j$ the above equation can be written as $d_{u_i}(e_1, e_2) = d_{u_j}(e_2, e_1)$. However, when $u_i = u_j$, $d_{u_i}(e_1, e_2)$ is not well defined, and there are two choices for the pair p_k, q_k, but one can check that whether the equality $d_{u_i}(p_1, p_2) = d_{u_j}(q_2, q_1)$ holds is independent of the choice of p_k, q_k. We introduce the *Equidistance Lemma.* It follows from Lemma 2.1.2, and can also be found in [8].

Lemma 2.1.3 ([8], Lemma 2.8)**.** *Let e_1, e_2 be a pair of edges with $\partial e_1 = \partial e_2$ in both Γ_1 and Γ_2. Then e_1, e_2 are equidistant in Γ_1 if and only if they are equidistant in Γ_2.* □

We will need the concept of *jumping number* between the graphs Γ_1 and Γ_2 as introduced in [5]. Since $\Delta = 3$, the jumping number is 1, which means that if the Δ points of intersection between the vertices u_i and v_j are labeled consecutively as $x_1, x_2, \ldots, x_\Delta$ around u_i, then these points appear consecutively around v_j in the same order $x_1, x_2, \ldots, x_\Delta$ when read in some direction. This is the *jumping number one condition* or JN1, as referred in [19].

2.2 Possible cases for Γ_a

Next we restrict to the case $n_1 = n_2 = 2$ and, since $\widehat{F}_a$ is separating, the two vertices of Γ_a have opposite sign. Two graphs in a surface are equivalent if there is a homeomorphism of the surface taking one to the other. We can explicitly identify the reduced graph $\widehat{\Gamma}_a$ and establish some additional properties of Γ_a. We start with a well-known result from [5]

Lemma 2.2.1 ([5], Lemma 5.2)**.** *The reduced graph $\widehat{\Gamma}_a$ is a subgraph of the graph illustrated*

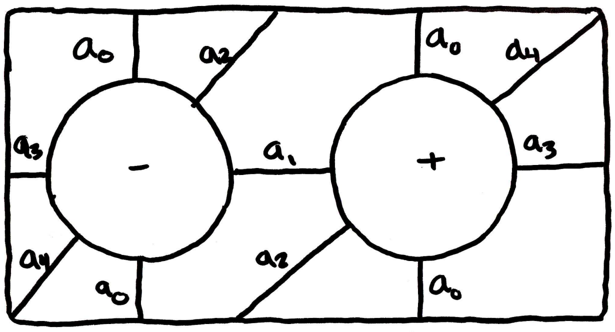

Figure 2.1: Parallelism classes for Γ_a

in Fig. 2.1. □

Thus Γ_a is determined by a quintuple $(a_0; a_1, a_2, a_3, a_4)$ of non-negative integers, as shown in Fig. 2.1. We write $\Gamma_a = (a_0; a_1, a_2, a_3, a_4)$, a graph on a twice-punctured torus with two families of loops of weight a_0, and four families of edges $\widehat{e}_i$ with weight sequence $a_1, \ldots, a_4$; and we abbreviate $(0; a_1, \ldots, a_4)$ to $(a_1, \ldots, a_4)$. The weight sequence is defined up to cyclic rotation and reversal of order. Hence:

$$\begin{aligned}(a_0; a_1, a_2, a_3, a_4) &\cong (a_0; a_3, a_4, a_1, a_2) \cong (a_0; a_4, a_3, a_2, a_1) \\ &\cong (a_0; a_2, a_1, a_4, a_3),\end{aligned} \tag{2.1}$$

and if $a_0 = 0$:

$$(a_1, a_2, a_3, a_4) \cong (a_2, a_3, a_4, a_1). \tag{2.2}$$

We are studying the case that each F_a is separating, $\Delta = 3$ and $n_a = 2$, so the two boundary components have opposite sign. Call the vertices of Γ_a $+$ and $-$, $a = 1, 2$. Let $\Gamma_1 = (a_0; a_1, a_2, a_3, a_4), \Gamma_2 = (b_0; b_1, b_2, b_3, b_4)$. Without loss of generality we may assume that $a_0 \geq b_0$. The following lemma summarizes the properties of Γ_1 and Γ_2.

Lemma 2.2.2 ([5], §6). *1.* $2a_0 + \sum_{i=1}^4 a_i = 2b_0 + \sum_{i=1}^4 b_i = 2\Delta = 6$*;*

2. $2a_0 = \sum_{i=1}^4 b_i, \quad 2b_0 = \sum_{i=1}^4 a_i$*;*

3. $2 \leq a_0 \leq 3$

4. $a_i, b_i \leq 2, \quad i = 1, 2, 3, 4.$

5. $a_i \equiv a_j \pmod 2, \quad i, j = 1, 2, 3, 4$ *and similarly for the b's.* □

Our next result narrows the options for the graphs Γ_1, Γ_2.

Lemma 2.2.3. *Suppose F_1, F_2 are neutral punctured tori with $n_1 = n_2 = 2$ and $\Delta = 3$. Then one of the following holds.*

1. $\Gamma_1 = (3; 0, 0, 0, 0)$ *and* $\Gamma_2 = (2, 2, 2, 0)$ *with edge correspondence as in Fig. 2.2.*

2. $\Gamma_1 = (2; 2, 0, 0, 0)$ *and* $\Gamma_2 = (1; 2, 2, 0, 0)$ *with edge correspondence as in Fig. 2.4.*

3. $\Gamma_1 = (2; 2, 0, 0, 0)$ *and* $\Gamma_2 = (1; 1, 1, 1, 1)$ *with edge correspondence as in Fig. 2.5.*

Proof. By Lemma 2.2.2, a_0 is either 2 or 3. Assume first $a_0 = 3$. Then $\Gamma_1 = (3; 0, 0, 0, 0)$ and $b_0 = 0$. Since $b_i \equiv b_j \pmod 2$, the only possibility for Γ_2 is $(2, 2, 2, 0)$, by the symmetries of (2.1), (2.2). The graphs are as shown in Fig. 2.2.

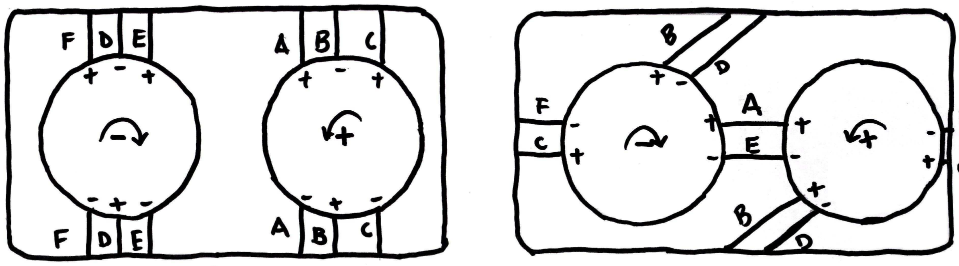

Figure 2.2: $\Gamma_1 = (3; 0, 0, 0, 0)$ and $\Gamma_2 = (2, 2, 2, 0)$

The $+$ edges in vertex $+$ of Γ_1 are in the order A, B, C. By the JN1 condition, these edges appear in the same order around vertex $+$ of Γ_2. The order of the edges with label $-$ at vertex $-$ of Γ_1 is D, E, F, so this is the same order of the $-$ edges around vertex $-$ in Γ_2. Consider the loops A, E in Γ_1. Denote by $A^{\pm}, E^{\pm}$ the endpoints of A and E with label $\pm$. Since $d_+(A^+, A^-) = d_-(E^+, E^-)$, A and E must be parallel in Γ_2 by Lemma 2.1.2.

This determines completely the identification between the edges in Γ_1 and Γ_2.

If $a_0 = 2$, up to symmetry the only possibility for Γ_1 is $(2; 2, 0, 0, 0)$, where clearly $b_0 = 1$. By Lemma 2.2.2, $\Gamma_2 = (1; 2, 2, 0, 0)$, $\Gamma_2 = (1; 1, 1, 1, 1)$ or $\Gamma_2 = (1; 2, 0, 2, 0)$. We show that the latter case is impossible.

Suppose $\Gamma_2 = (1; 2, 0, 2, 0)$. Label the edges of Γ_1 as in Fig. 2.3. By the parity rule, B and E represent loops in the vertices $+$ and $-$, respectively, of Γ_2. The rest is determined by the JN1 condition as above. In Γ_1, A and C are parallel positive edges, hence they are equidistant. However, in Γ_2, $4 = d_-(A, C) \neq d_+(C, A) = 2$, violating the Equidistance Lemma. This rules out the possibility $\Gamma_2 = (1; 2, 0, 2, 0)$.

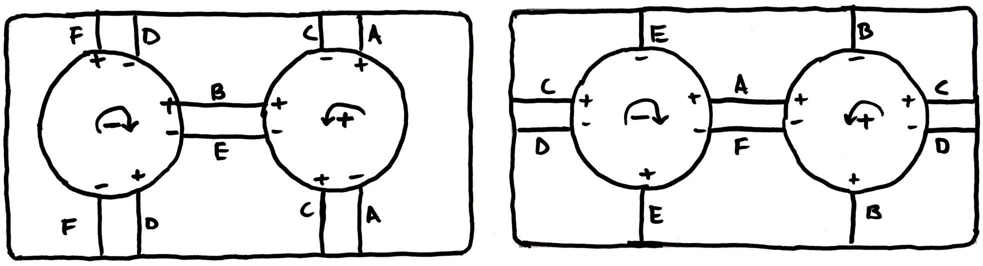

Figure 2.3: $\Gamma_1 = (2; 2, 0, 0, 0), \Gamma_2 = (1; 2, 0, 2, 0)$ is impossible

If $\Gamma_2 = (1; 2, 2, 0, 0)$ or $\Gamma_2 = (1; 1, 1, 1, 1)$, we label the edges of Γ_1 as before. Thus edges B and E are still loops in Γ_2 and so, the JN1 condition determines the rest of the identification between the edges in both cases, as shown in Figs. 2.4 and 2.5. $\square$

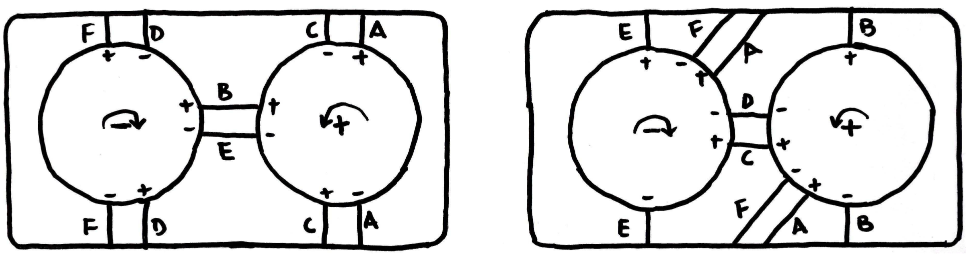

Figure 2.4: $\Gamma_1 = (2; 2, 0, 0, 0)$ and $\Gamma_2 = (1; 2, 2, 0, 0)$

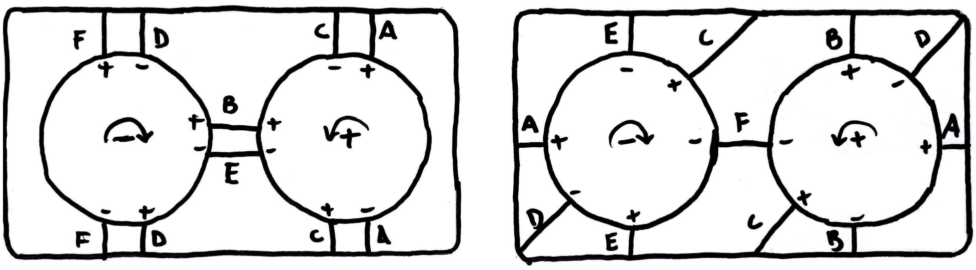

Figure 2.5: $\Gamma_1 = (2; 2, 0, 0, 0)$ and $\Gamma_2 = (1; 1, 1, 1, 1)$

Observe that the conclusion of Lemma 2.2.3 holds for exteriors of hyperbolic knots in arbitrary 3-manifolds, not only in S^3.

In chapter 4 we will determine exactly which of these graphs are *geometrically realizable* in S^3, in the sense that they are generated as graphs of intersection between twice punctured tori in nontrivial knot exteriors in S^3 with boundary slopes at distance 3.

Chapter 3

Topology of genus two handlebodies

3.1 Basic definitions

In the previous chapter we obtained only three possible minimal graphs of intersection between essential twice punctured separating tori in the exterior of a hyperbolic knot K. The topological analysis of these cases relies heavily on the properties of *genus two handlebodies*. We introduce such properties in this chapter by means of *compression bodies*.

A compression body W is a cobordism rel ∂ between surfaces $\partial_+ W$ and $\partial_- W$ such that $W \cong \partial_+ W \times I \cup$ 2-handles $\cup$ 3-handles and $\partial_- W$ has no 2-spheres components. W is a *handlebody* if $\partial_+ W$ is closed and connected and $\partial_- W = \emptyset$ and W is *trivial* if $W \cong \partial_+ W \times I$. When F is a closed surface in a 3-manifold, a compression body for F is, by definition, a compression body W in the 3-manifold for which $\partial_+ W = F$.

Let M be an irreducible 3-manifold. By [2], there exist a compression body $W \subset M$ for ∂M, unique up to isotopy, with the following properties.

1. $\overline{M \setminus W}$ is irreducible and ∂-irreducible.

2. $\partial_- W$ is incompressible in M.

3. Every surface in M that consists of disks can be isotoped inside of W.

4. Every compression body $W' \subset M$ for ∂M can be isotoped inside of W.

Such a compression body W is called a *maximal compression body* for ∂M.

A *complete disk system* $\mathbf{D}$ for a handlebody W is a disjoint union of disks $(\mathbf{D}, \partial \mathbf{D}) \subset$

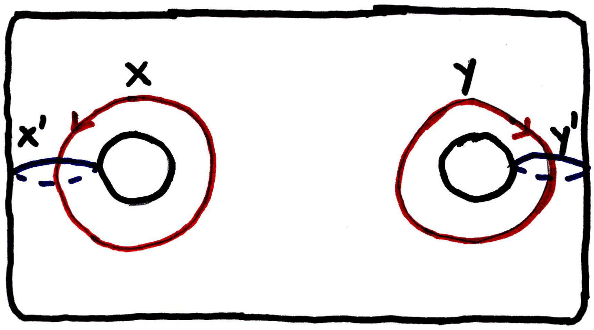

Figure 3.1: A genus two handlebody and its complete disk system

$(W, \partial_+ W)$ such that W cut along $\mathbf{D}$ is homeomorphic to B^3. A genus two handlebody H along with a complete disk system $\mathbf{D} = \{x', y'\}$ is depicted in Fig. 3.1.

Note that the *duals* x, y of the disks x', y' constitute a basis for $F_2 = \pi_1(H)$ (rel some base point), a *free group on two generators.* We now introduce some general properties about this group.

We write $F_2 = \langle x, y | - \rangle$. Elements of F_2 are called *words* in x and y. For convenience, we will denote the inverse u^{-1} of u by $\bar{u}$ and by $[u, v]$ the commutator $uv\bar{u}\bar{v}$ of any two words u, v in F_2. We say that u is *primitive* if, for some $v \in F_2$, $\{u, v\}$ represents a basis; and u is a *power* if there is a nontrivial element $w \in F_2$ and an integer $k \geq 2$ such that $u = w^k$. We write $u \equiv v$ if u is in the conjugacy class of either v or $\bar{v}$. The word $u \in F_2$ is *cyclically reduced* if, for $\varepsilon = \pm 1$ and $z \in \{x, y\}$, $z^{\varepsilon}, z^{-\varepsilon}$ do not occur consecutively nor at the beginning and end in u. Note that $u \equiv v$ whenever v is a cyclic reduction of u or $\bar{u}$.

The following is a characterization of primitive or power elements in $F_2 = \pi_1(H)$. It can be found in [3].

Lemma 3.1.1 ([3])**.** *If an element u in $F_2 = \langle x, y | - \rangle$ is primitive or a power of a primitive then there is a basis $\{a, b\} \subset \{x, \bar{x}, y, \bar{y}\}$ of F_2 and a positive integer n such that either $u \equiv ab^{n-1}$, $u \equiv a^{n+1}$, or $u \equiv ab^{m_1} \cdots ab^{m_k}, k \geq 2$, where each m_i is either n or $n+1$.* $\square$

3.2 Companion annuli, roots

In this section we follow closely the discussion in [14, §2.4]. Let M be a compact, orientable, irreducible and atoroidal 3-manifold with connected boundary, and let γ be a nontrivial circle in ∂M. Let A be an annular regular neighborhood of γ in ∂M. A properly embedded separating annulus A' in M is a *companion annulus* for γ in M if A' is not boundary parallel. By [18], the region cobounded in M by A and A' is a solid torus. We call this the *companion solid torus* of γ in M. Such companion annulus and solid torus for γ in M are unique up to isotopy. Define the *multiplicity* $\mu(\gamma)$ of γ in M as 1 if γ has no companion annuli, and as the number of times γ wraps around its companion solid torus if γ has a companion annulus. Note that $\mu(\gamma) \geq 2$ iff γ has a companion annulus.

Now we present several fundamental facts about circles in ∂H which may represent primitive or power elements in $\pi_1(H)$. Let γ be a loop in H. Denote by $[\gamma]$ the element of $\pi_1(H)$ representing γ rel some base point. A properly embedded disk in H, separating the manifold into two solid torus components is a *waist disk* of H.

Lemma 3.2.1 ([14], Lemma 2.3)**.** *Let H be a handlebody of genus two and γ be a nontrivial circle in ∂H.*

1. *$\partial H \setminus \gamma$ compresses in H iff $[\gamma]$ is primitive or a power in $\pi_1(H)$, in which case there is a waist disk of H disjoint from γ. In particular, if $[\gamma]$ is a power in $\pi_1(H)$ then it is a power of a primitive element.*

2. *γ has multiplicity $n \geq 2$ in H iff $[\gamma] = u^n$ for some primitive element $u \in \pi_1(H)$.* □

If a circle γ satisfies condition (1) in the previous lemma, we say that γ is primitive in H.

By attaching a 2-handle to a genus two handlebody we may obtain the manifold $\mathcal{T} \times I$, $\mathcal{T}$ a torus. The following result gives a necessary and sufficient algebraic condition for this. The result is well known, see for example [14].

Lemma 3.2.2 ([14], Lemma 2.5). *Let H be a genus two handlebody and $\mathcal{T}$ be a closed torus. Let γ be a circle embedded in ∂H and $M = H \cup_\gamma N(D)$ be the manifold obtained by attaching a 2-handle $N(D)$ to H along γ. Then M is homeomorphic to $\mathcal{T} \times I$ iff $[\gamma] \equiv [a, b]$ for some (hence any) basis $\{a, b\}$ of $\pi_1(H)$.* $\square$

The previous lemma can be generalized to the case of attaching a 2-handle to a genus-two sub-handlebody of H.

Lemma 3.2.3 ([14], Lemma 2.6). *Let H be a genus two handlebody and $\gamma_0, \gamma_1, \gamma_2$ be disjoint nontrivial circles embedded in ∂H. Let $\mathcal{T}$ denote a closed torus.*

1. *If $A_0 \subset H$ is a companion annulus for γ_0 with core α_0 and corresponding companion solid torus $V_0 \subset H$, then $H' = \overline{H \setminus V_0}$ is a genus two handlebody, and there is a common waist disk D of H and H' such that*

   (a) $H' = W_0 \cup_D W_1$ for some solid tori W_0, W_1 in H' with $D = \partial W_0 \cap \partial W_1 = W_0 \cap W_1$ and $A_0 \subset \partial W_0 \setminus D$.

   (b) If β_1 is a core of W_1 then $\{w_0 = [\alpha_0], w_1 = [\beta_1]\}$ is a basis for $\pi_1(H')$.

   (c) If $H' \cup_{\gamma_2} N(D)$ is the manifold obtained by attaching a 2-handle $N(D)$ along γ_2, then $H' \cup_{\gamma_2} N(D) \approx \mathcal{T} \times I$ iff $[\gamma_2] \equiv [a^{\mu(\gamma_0)}, b]$ for some (hence any) basis $\{a, b\}$ of $\pi_1(H)$ such that $[\gamma_0] \equiv a^{\mu(\gamma_0)}$.

   *We say that H' was obtained by taking the $\mu(\gamma_0)$-*root *of γ_0 in H.*

2. *Suppose $A_0, A_1 \subset H$ are disjoint companion annuli for γ_0, γ_1, respectively, with corresponding cores α_0, α_1 and companion solid tori $V_0, V_1 \subset H$. Then $H' = \overline{H \setminus (V_0 \sqcup V_1)}$ is a genus two handlebody, and there is a common waist disk D of H and H' such that*

   (a) $H' = W_0 \cup_D W_1$ for some solid tori W_0, W_1 in H' with $D = \partial W_0 \cap \partial W_1 = W_0 \cap W_1$, $A_0 \subset \partial W_0 \setminus D$, and $A_1 \subset \partial W_1 \setminus D$.

   (b) $\{w_0 = [\alpha_0], w_1 = [\alpha_1]\}$ is a basis for $\pi_1(H')$.

(c) If $H' \cup_{\gamma_2} N(D)$ *is the manifold obtained by attaching a 2-handle* $N(D)$ *along* γ_2*, then* $H' \cup_{\gamma_2} N(D) \approx \mathcal{T} \times I$ *iff* $[\gamma_2] \equiv [a^{\mu(\gamma_0)}, b^{\mu(\gamma_1)}]$ *for some (hence any) basis* $\{a, b\}$ *of* $\pi_1(H)$ *such that* $[\gamma_0] \equiv a^{\mu(\gamma_0)}$ *and* $[\gamma_1] \equiv b^{\mu(\gamma_1)}$*.*

As before, we say that H' *is the result of taking the roots of* γ_0 *and* γ_1 *in* H*.* □

The process of taking roots in handlebodies will be useful in the next chapter, where the need to study knots embedded in a genus two handlebody will arise.

Chapter 4

Geometrically realizable graph pairs

4.1 Introduction

In this chapter we study the geometric realization of the graphs described in Lemma 2.2.3 and discard the cases that do not occur inside S^3. From now on, we assume that both essential tori F_1, F_2, represented here by S, T, are separating and $M = X_K$ for some hyperbolic knot K.

Let T_B, T_W denote the closure of the components of $X_K \setminus T$, the *black* and *white* sides of T, respectively. Define S_B and S_W similarly. A face of Γ_S (resp. Γ_T) contained in the black (white) side of T (S) is called a *black* (*white*) face. Recall that $N(K)$ is the solid torus in S^3 obtained by taking a regular neighborhood of K and T_0 is the torus boundary of the exterior of the knot X_K. For $Z \in \{S, T\}$ and $* \in \{B, W\}$, define the annulus $A^Z_* = Z_* \cap T_0$; since Z has integral boundary slope, the core K^Z_* of A^Z_* is isotopic to K in S^3. For simplicity, all such cores will be denoted by K.

The boundary ∂Z_* is the surface $Z \cup A^Z_*$, a genus two surface. The following lemma gives a sufficient condition for Z_* to be a genus two handlebody.

Lemma 4.1.1. *If the graph Γ_S (resp. Γ_T) contains a black (white) disk face D then T_B (T_W, S_B, S_W) is a genus two handlebody.*

Proof. Since X_K is hyperbolic and T is essential, T_B is irreducible and atoroidal. Assume Γ_S contains a black disk face D; the other cases can be similarly handled. Let W be the maximal compression body for ∂T_B in T_B, so that, by Section 3.1, $\partial_- W$ is incompressible in T_B. As D is a compression disk for ∂T_B, $\partial_- W$ is either empty or consists of one or two

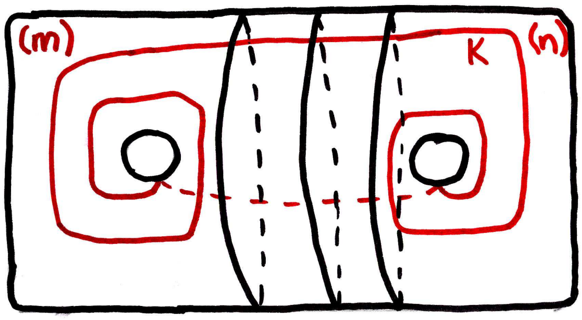

Figure 4.1: K in S_B

tori in T_B. The fact that T_B is atoroidal now implies $\partial_- W = \emptyset$ and hence that $W = T_B$ is a genus two handlebody. $\square$

4.2 The case $\Gamma_S = (3; 0, 0, 0, 0), \Gamma_T = (2, 2, 2, 0)$

Since Γ_T has 3 black disk faces, S_B is a genus two handlebody. Moreover, the three disks are parallel in S_B and since S is separating, the disks are separating in S_B, that is, they are waist disks. Since these disks are bigons, the knot K passes two times trough each disk. Modulo a homeomorphism, Fig. 4.1 shows S_B, its waist disks and K, winding m times around the left solid torus and n times around the right one. Notice we must have $m, n \geq 2$, as otherwise K would be primitive in S_B and hence S would compress in S_B by Lemma 3.2.1. Decompose K as the connected sum of circles ζ_0, ζ_1 disjoint from K, in the obvious way, where $[\zeta_0] \equiv y^n, [\zeta_1] \equiv x^m$, where the circles x and y are like in Fig. 3.1. Let $A'_0, A'_1; V_0, V_1$ be the respective companion annuli and solid tori. Then by Lemma 3.2.3 we can take roots of ζ_0 and ζ_1 in S_B to obtain $S'_B = \overline{S_B \setminus (V_0 \sqcup V_1)}$, which is a genus two handlebody, and K now winds one time around each solid tori of S'_B. See Fig. 4.2.

Since the slopes are integral, each component of $\partial S, \partial T$ may be assumed to lie in T_0, up to homeomorphism, as shown in Fig. 4.3; which gives the models corresponding to the possible signs of the oriented intersection number $\partial S \cdot \partial T$.

Consider the white hexagon Q disk face in Γ_T. Let $A'_S = T_0 \setminus \text{int} A^S_B$ be the complementary

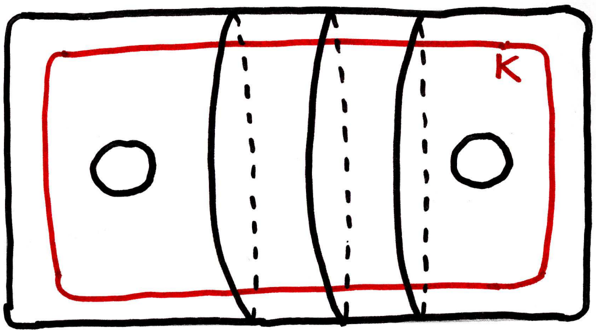

Figure 4.2: K in S'_B after taking the roots of ζ_0 and ζ_1

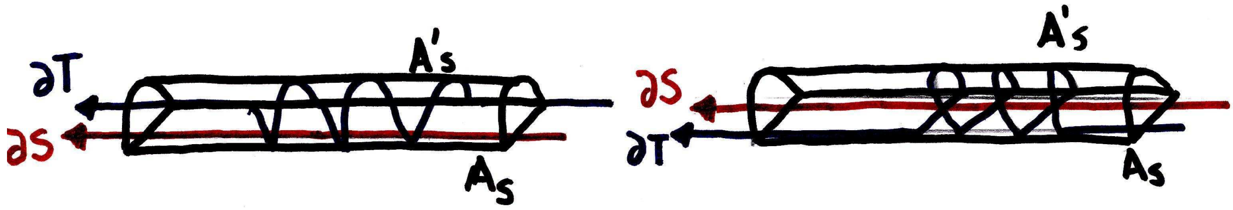

Figure 4.3: The circles $\partial S, \partial T$ in T_0

annulus of A^S_B in T_0. Observe that $\partial Q \subset S'_B \cup_{A^S_B} N(K)$. Therefore, since A^S_B and A'_S are parallel in $N(K)$, we can extend Q via meridian disks of $N(K)$ to a disk Q^* properly embedded in $S^3 \setminus \mathrm{int}S'_B$ with $\partial Q^* \subset S'_B$.

In general, if the graphs of intersection Γ_S, Γ_T yield (say) S_B as a genus two handlebody, we can properly embed all the white faces F_W of Γ_T in $S^3 \setminus \mathrm{int}S_B$. The collection of circles ∂F_W in ∂S_B is called the *white spectrum* of Γ_T in S_B and it is obtained as follows. Draw A^S_B in ∂S_B with core $K^S_B \approx K$. Label the intersections of K with the black faces of Γ_T as $+, -$ alternatively. Assign an orientation to $\partial S \cdot \partial T$. Because of the JN1 condition, the loop ∂F_W follows K until it intersects one of the black faces. Then it continues along the boundary of the black face according to $\partial S \cdot \partial T$ until reaching another intersection with K and moving along K until reaching another label with the same sign. The process continues until the final closed curve ∂F_W is complete.

Fig. 4.4 shows the white spectrum in S'_B for the only white face in Γ_T, the hexagon Q^*,

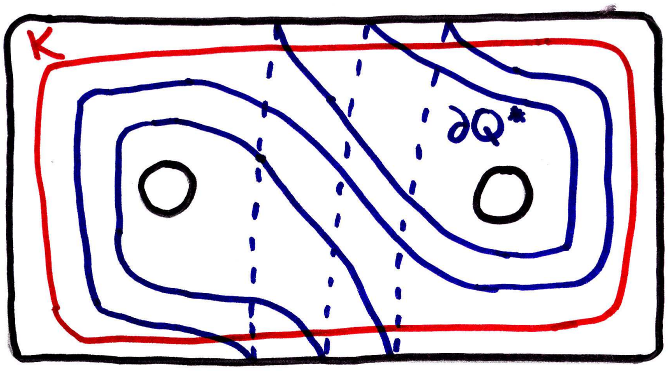

Figure 4.4: ∂Q^* in S'_B

with respect to $\partial S \cdot \partial T = -1$. Since the other case can be treated along the same lines and yields the same conclusion, we will only concentrate in this case.

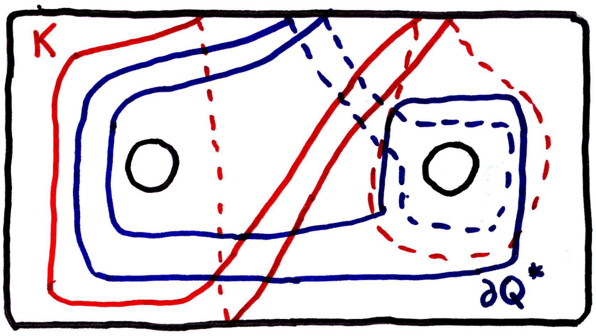

Figure 4.5: K and ∂Q^* in S'_B after the twist

Fig. 4.5 shows K and ∂Q^* after performing one full negative twist along the waist disk of S'_B. Observe that ∂Q^* represents the commutator $[x, y]$ of $\pi_1(S'_B)$, where S'_B is identified with the handlebody in Fig. 3.1. By Lemma 3.2.2, $S'_B \cup_{\partial Q^*} N(D) \approx \mathcal{T} \times I$, where $N(D)$ is a 2-handle; equivalently $S'_B \approx \widetilde{\mathcal{T}} \times I$, where $\widetilde{\mathcal{T}}$ is a once punctured torus and $\widetilde{\mathcal{T}} \times 0 = \widetilde{\mathcal{T}}_0$ and $\widetilde{\mathcal{T}} \times 1 = \widetilde{\mathcal{T}}_1$ are the two once punctured tori separated by ∂Q^* in $\partial S'_B$. We call these surfaces the *lids* of S'_B.

Let $\{i, j\} = \{0, 1\}$. By capping off $\partial \widetilde{\mathcal{T}}_i$ with a disk we obtain a torus $\mathcal{T}_i$. Observe that each annulus A'_i (relabeling it if necessary) lies in $\mathcal{T}_i$. Consider the oriented circles $\alpha_i, \beta_i \subset \widetilde{\mathcal{T}}_i$

in Fig. 4.6. Each pair $\{\alpha_0, \alpha_1\}$ and $\{\beta_0, \beta_1\}$ cobound an annulus in S'_B. Moreover, the pair of circles $\{\alpha_i, \beta_i\}$ intersect at one point and represent a basis for both $\pi_1(S'_B)$ and $\pi_1(\mathcal{T}_i) = \mathbb{Z} \oplus \mathbb{Z}$.

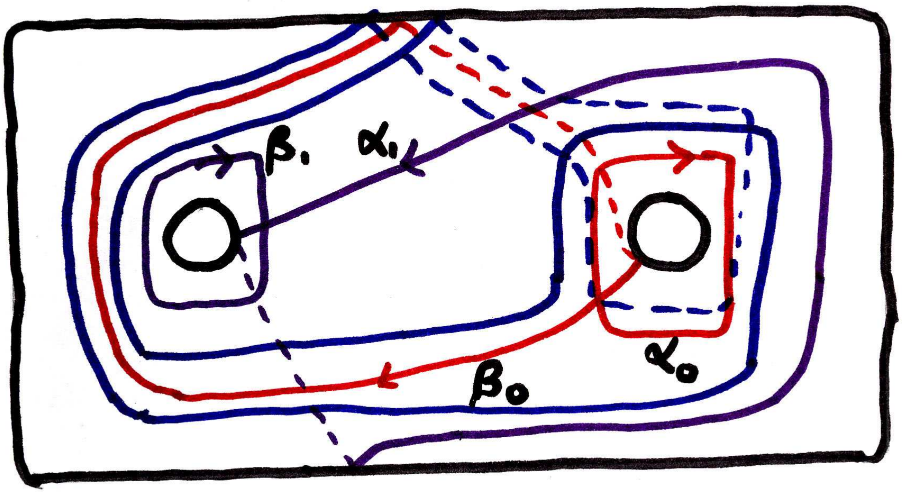

Figure 4.6: $\alpha_0, \beta_0, \alpha_1, \beta_1$ in S'_B

Since $S'_B = \widetilde{\mathcal{T}} \times I$, any loop $\gamma_i = a\alpha_i + b\beta_i$, $a, b \in \mathbb{Z}, \gcd(a, b) = 1$ in $\pi_1(\widetilde{\mathcal{T}}_i)$ represents a primitive circle $\gamma_i(\alpha_i, \beta_i)$ in $\pi_1(S'_B)$. By reversing the orientation of γ_i if necessary, we can assume $b \geq 0$. Since $\{\alpha_i, \alpha_j\}$ and $\{\beta_i, \beta_j\}$ cobound annuli, $\gamma_j = a\alpha_j + b\beta_j$ is isotopic to γ_i in S'_B. Isotope K so that it lies in $\mathrm{int}S'_B$ and then attach a 2-handle to S'_B along γ_i. By Lemma 3.2.1 the result is a solid torus V_{γ_i} containing the knot K in its interior; that is:

$$K \subset V_{\gamma_i} = S'_B \cup (\text{2-handle along } \gamma_i). \tag{4.1}$$

We now compute the winding number $\omega(K)$ of K in V_{γ_i} in terms of a and b. Consider the intersections of K with α_i, β_i, according to the outward normal unit vector rule. Clearly $K \cdot \alpha_0 = 0, K \cdot \alpha_1 = 1, K \cdot \beta_0 = -1, K \cdot \beta_1 = 0$. Because of the product structure, γ_i can be isotoped to γ_j in $\widetilde{\mathcal{T}} \times I$. The winding number $\omega(K)$ is the oriented intersection number between K and the disk bounded by γ_i in V_{γ_i}, which is the sum of the oriented intersection number between K and γ_i in the two lids, i.e. $|a(0)+b(-1)+a(1)+b(0)| = |a-b|$. Observe that this is also the winding number of K with respect to V_{γ_j}.

We will show that each torus $\mathcal{T}_i$ is necessarily *unknotted* in S^3, i.e. that each component of $S^3 \setminus \mathcal{T}_i$ is a solid torus. We will need the following lemma.

Lemma 4.2.1. *The knot K is not a core of V_{γ_i} for any $\gamma_i = a\alpha_i + b\beta_i$, $\frac{a}{b} \notin \{\frac{1}{0}, \frac{0}{1}\}$.*

Proof. Since $\omega(K) = |a-b|$, if $|a-b| \neq 1$, K is not a core in V_{γ_i}. We show that K is also not a core if $|a-b| = 1$, $\frac{a}{b} \notin \{\frac{1}{0}, \frac{0}{1}\}$ by showing that some surgery on K in V_{γ_i} does not yield a solid torus. Since K is primitive in S'_B, then up to homeomorphism K lies in S'_B as shown in Fig. 4.7. In the figure, the loops u and v are the duals of the *disk of primitivity* D_x and the *disk of compressibility* D_y for K, respectively, and these loops constitute a basis for $\pi_1(S'_B)$. We take the point of intersection between α_i and β_i as the base point for $\pi_1(S'_B)$. Note that the circles $\alpha_0, \beta_0, \alpha_1, \beta_1$ represent respectively $v, \bar{u}\bar{v}, \bar{v}, u\bar{v}$. Since K is a core of the left one handle of S'_B, any surgery on K in V_{γ_i} can be obtained by first performing surgery on $K \subset \mathrm{int}S'_B$ and then attaching a 2-handle along γ_i. By Lemma 3.1.1 and since $|a-b| = 1$,

$$\gamma_i(\alpha_i, \beta_i) = \underbrace{\alpha_i\beta_i \ldots \alpha_i\beta_i}_{k \geq 0 \text{ times}} \alpha_i$$

or

$$\gamma_i(\alpha_i, \beta_i) = \underbrace{\alpha_i\beta_i \ldots \alpha_i\beta_i}_{k \geq 0 \text{ times}} \beta_i$$

Now perform $\frac{0}{1}$-surgery on K. This yields a new genus two handlebody S''_B with basis $\{w, v\}$ where u, D_y form a complete disk system for S''_B and w is represented by D_x. By looking at the oriented intersections of α_i, β_i with u, we see that $\alpha_0 = v, \beta_0 = v\bar{w}\bar{v}w\bar{v}\bar{w}$ and $\alpha_1 = vw\bar{v}\bar{w}\bar{v}w, \beta_1 = v$. This gives:

$$\gamma_0(w, v) = \underbrace{vv\bar{w}\bar{v}w\bar{v}\bar{w} \ldots vv\bar{w}\bar{v}w\bar{v}\bar{w}}_{k \geq 0 \text{ times}} v = (\bar{w}\bar{v}w\bar{v}\bar{w}v^2)^k v$$

or

$$\gamma_0(w, v) = \underbrace{vv\bar{w}\bar{v}w\bar{v}\bar{w} \ldots vv\bar{w}\bar{v}w\bar{v}\bar{w}}_{k \geq 0 \text{ times}} \bar{w}\bar{v}w\bar{v}\bar{w} = (v^2\bar{w}\bar{v}w\bar{v}\bar{w})^k v\bar{w}\bar{v}w\bar{v}\bar{w}$$

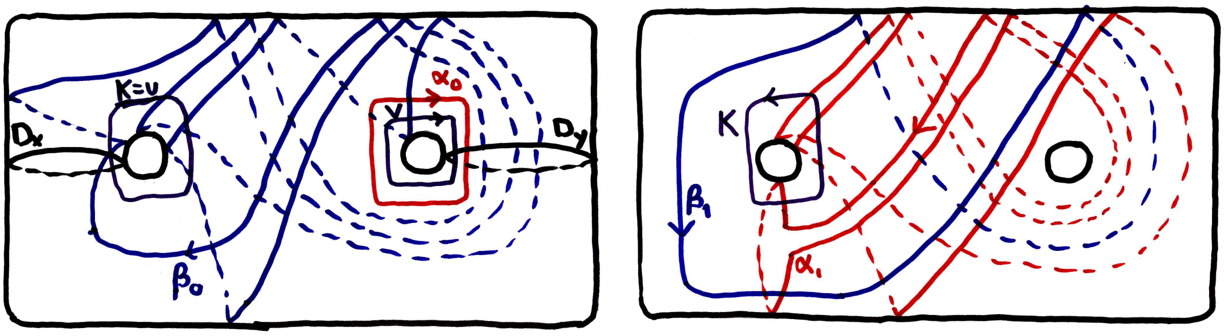

Figure 4.7: K and the circles α_i, β_i in S'_B after the homeomorphism

Similarly,

$$\gamma_1(w,v) = \underbrace{vw\bar{v}\bar{w}\bar{v}wv\ldots vw\bar{v}\bar{w}\bar{v}wv}_{k\geq 0\ \text{times}} vw\bar{v}\bar{w}\bar{v}w = (v^2w\bar{v}\bar{w}\bar{v}w)^k vw\bar{v}\bar{w}\bar{v}w$$

or

$$\gamma_1(w,v) = \underbrace{vw\bar{v}\bar{w}\bar{v}wv\ldots vw\bar{v}\bar{w}\bar{v}wv}_{k\geq 0\ \text{times}} v = (w\bar{v}\bar{w}\bar{v}wv^2)^k v$$

The word represented by $\gamma_i(w,v)$ fails to be primitive except for the cases $\gamma_0 = \alpha_0$ and $\gamma_1 = \beta_1$. For the rest, $S''_B \cup_{\gamma_i} N(D)$ is not a solid torus and K is not trivial in V_{γ_i}.

If $\gamma_0 = \alpha_0$, $S'_B \cup_{\gamma_0} N(D)$ is obtained by capping off the right solid torus of S'_B with a 2-handle, and clearly K is the core of V_{γ_0}. A similar argument shows that when $\gamma_1 = \beta_1$, K is the core of V_{γ_1}. $\square$

It is well known that any closed torus in S^3 bounds a solid torus on one side and that any solid torus in S^3 nontrivially containing a hyperbolic knot must be unknotted. Without loss of generality, we may assume that $\mathcal{T}_0$ bounds a solid torus V_{γ_0} away from $\mathcal{T} \times I$ whose meridian disk has boundary the circle $\gamma_0 = a\alpha_0 + b\beta_0$

Corollary 4.2.2. *$\mathcal{T}_0$ bounds a solid torus in both sides if $\frac{a}{b} \notin \{\frac{0}{1}, \frac{1}{0}, \frac{1}{1}\}$.*

Proof. If a and b are nonzero, the previous lemma shows that K is not a core of V_{γ_0}. If K is contained in a 3-ball in V_{γ_0}, then $\omega(K) = |a - b| = 0$, in which case $a = b = 1$, since $\gcd(a, b) = 1$. Thus K is a nontrivial knot in V_{γ_0} if $\frac{a}{b} \notin \{\frac{0}{1}, \frac{1}{0}, \frac{1}{1}\}$ and since K is hyperbolic $\mathcal{T}_0$ must be unknotted in S^3. $\square$

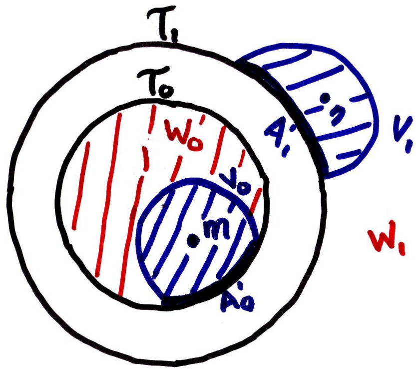

Figure 4.8: The annuli A_0', A_1'

Remark 4.2.3. *In light of the above corollary, it follows that any knot K with the aforementioned properties can be obtained from an arbitrary unknotted embedding of the solid torus V_{γ_0} in S^3 under suitable conditions on γ_0 and the embedding. Such suitable conditions will be discussed in each particular case.*

Recall that the solid torus V_0 whose boundary ∂V_0 is the union of the annuli A_0, A_0', is embedded in V_{γ_0} with $A_0' = V_0 \cap V_{\gamma_0}$ and A_0 properly embedded in V_{γ_0}. Therefore A_0 is parallel to ∂V_{γ_0} in V_{γ_0} and hence the meridian disk of V_0 extends to a meridian disk of V_{γ_0} bounded by the circle γ_0 (see Fig. 4.8, where for simplicity $V_{\gamma_i} = W_i$). Since n is the number of times the meridian disk of V_0 intersects the core of A_0', while $b > 0$ is the number of times the meridian disk of V_{γ_0}, or equivalently, the circle γ_0, intersects the core of A_0', it follows that $b = n \geq 2$, so the cases $\frac{a}{b} = \frac{0}{1}, \frac{1}{0}$ or $\frac{1}{1}$ do not occur.

Since the circle $\gamma_0 = a\alpha_0 + b\beta_0, \gcd(a, b) = 1, b \geq 2$ represents a primitive element in $\pi_1(S_B')$, there exists a homeomorphism mapping γ_0 to the circle shown in Fig. 4.9. Observe

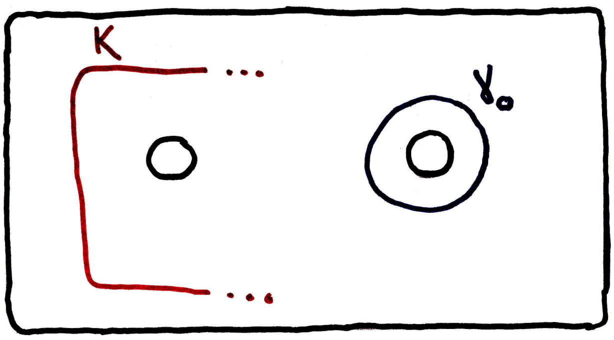

Figure 4.9: γ_0 in S'_B after the homeomorphism

that in the figure, the pair (V_{γ_0}, K) is obtained by simply attaching a 2-handle around γ_0. Originally, K is represented by the circles α_0, β_0 with an addition of a suitable positive full twist around their point of intersection (Fig. 4.10). Therefore, after the homeomorphism, K is also obtained by adding a suitable positive full twist around the intersection of the image of the circles α_0, β_0.

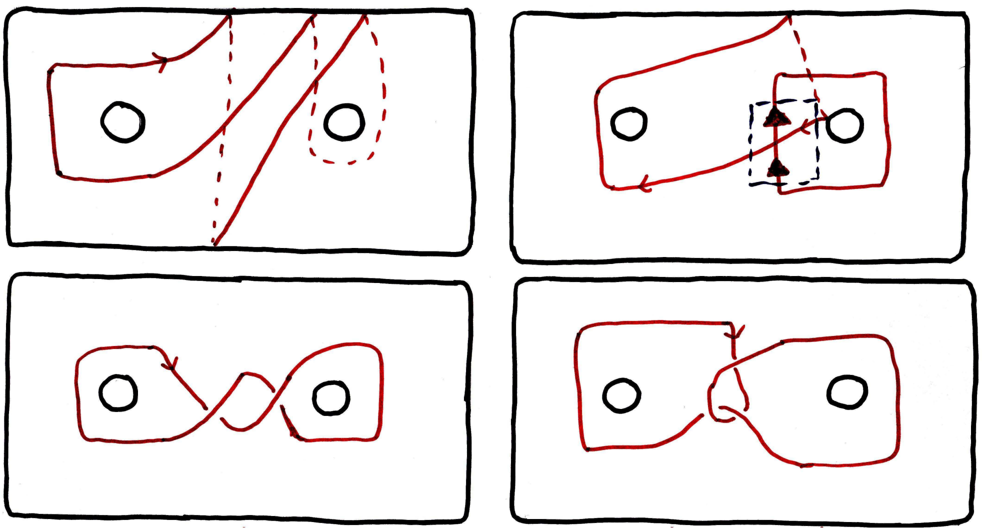

Figure 4.10: At the left, the knot K; at the right, α_0, β_0 in with a twist

Note that, in general, such a homeomorphism is not unique. In chapter 5 we introduce a *canonical embedding* for each γ_0, depending only on the parameters a, b. The pair (V_{γ_0}, K) obtained by by this canonical embedding will be denoted by $(V_{\gamma_0}, K(a, b, 0))$. Further

embeddings of V_{γ_0} in S^3 may then be obtained by full twisting $n \in \mathbb{Z}$ times the canonical embedding of V_{γ_0}, giving rise to the family of pairs $(V_{\gamma_0}, K(a,b,n))$. A classification of the family of knots $K(a,b,n)$ will be discussed in Chapter 5.

4.3 The case $\Gamma_S = (2;2,0,0,0), \Gamma_T = (1;2,2,0,0)$

In this case the black faces of Γ_S consist of two black Scharlemann bigons D_1, D_2 and one more black bigon D_3, so T_B is a genus two handlebody by Lemma 4.1.1. Since the knot K intersects each Scharlemann bigon two times in the same direction, D_1, D_2 are nonseparating in T_B and can be either parallel or nonparallel in T_B. The latter case will be seen to produce the graph $\Gamma_T = (1;1,1,1,1)$ in the following section; therefore D_1 and D_2 must be parallel in T_B. If D_3 is nonseparating, then we can draw a circle in T_B intersecting at only one point the black disks, which implies T is nonseparating, contrary to our assumption. Thus, up to homeomorphism, the scenario is the one shown in Fig. 4.11.

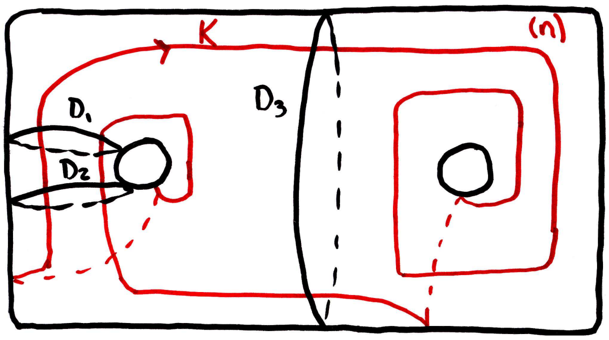

Figure 4.11: K in T_B

Let A_B^T be the annulus in T_B with core $K_B^T \approx K$. Then $\partial T_B \setminus \text{int} A_B^T = T$, with $\partial A_T^B = \partial T$. The circle K intersects exactly two different classes of parallelism between negative edges in Γ_T, confirming that $\Gamma_T = (1;2,2,0,0)$.

Since K is nonprimitive in T_B, it winds $n \geq 2$ times around the solid torus on the right of Fig. 4.11. As in Section 4.2, we decompose K as the connected sum of two circles ζ_0, ζ_1, where $[\zeta_0] \equiv y^n, [\zeta_1] \equiv x^2$ (see Fig. 4.11). Let A_0', V_0 be the companion annulus and torus in

T_B, respectively, for ζ_0, and let $T_B' = \overline{T_B \setminus V_0}$ be the resulting genus two handlebody after taking the root of ζ_0 in T_B. The pair is now equivalent to the pair (T_B, K) in Fig. 4.11 with $n = 1$.

We now construct the white spectrum of Γ_S in T_B'. The white faces of Γ_T consist of a 4-gon Q and an annulus A_W, which can be properly embedded in $S^3 \setminus \mathrm{int}T_B'$ with $\partial Q, \partial A_W$ as in Fig. 4.12. Analogously to the case of ∂Q^* in Section 4.2, ∂Q is the commutator of $\pi_1(T_B')$, and hence $T_B' \cup_{\partial Q} N(D) \approx \mathcal{T} \times I$ and $T_B' \approx \widetilde{\mathcal{T}} \times I$. Let $\widetilde{\mathcal{T}}_0, \widetilde{\mathcal{T}}_1$ be the lids of T_B' and define the pair of circles $\alpha_i, \beta_i \subset \widetilde{\mathcal{T}}_i$ as in the previous section. Note that both components of ∂A_W lie in $\widetilde{\mathcal{T}}_1$ and are parallel to β_1; in particular, ∂A_W is disjoint from α_0, β_0.

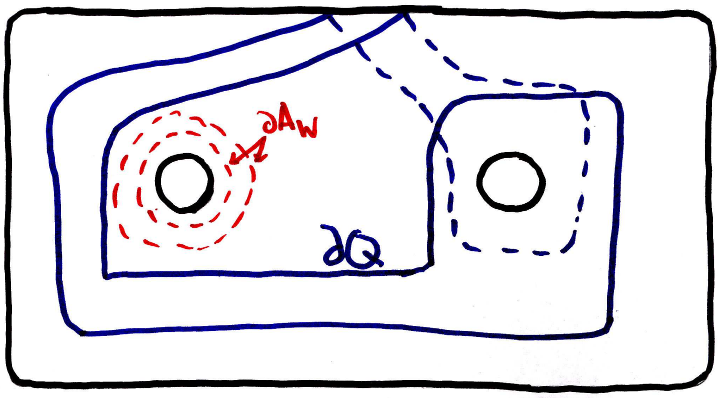

Figure 4.12: ∂Q and ∂A_W in T_B'

As in section 4.2, we define $\gamma_i = a\alpha_i + b\beta_i$ with $\gcd(a, b) = 1$ and the solid torus

$$K \subset V_{\gamma_i} = T_B' \cup (\text{2-handle along } \gamma_i). \tag{4.2}$$

Reversing the orientation of γ_i if necessary, we may assume $b \geq 0$. To compute $\omega(K)$ in V_{γ_i}, first observe that $K \cdot \alpha_0 = 0, K \cdot \beta_0 = 2, K \cdot \alpha_1 = -2, K \cdot \beta_1 = -1$. We proceed as in the previous section and obtain $\omega(K) = |a(0) + b(2) + a(-2) + b(-1)| = |2a - b|$. The following lemma is analogous to Lemma 4.2.1.

Lemma 4.3.1. *The knot K is not the core of V_{γ_i} for any $\gamma_i = a\alpha_i + b\beta_i, \frac{a}{b} \notin \{\frac{0}{1}, \frac{1}{1}\}$.*

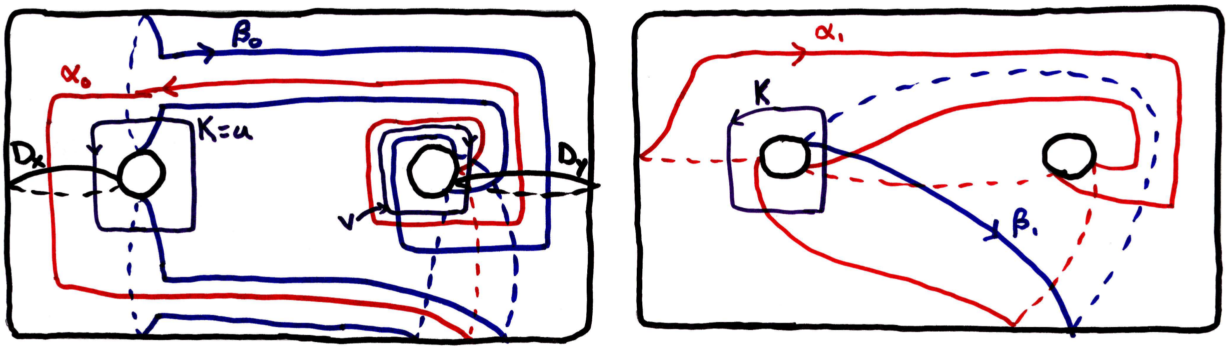

Figure 4.13: K and the circles α_i, β_i in T'_B after the homeomorphism

Proof. As in Lemma 4.2.1, we just need to analyze the case $|2a - b| = 1$. If $\frac{a}{b} \notin \{\frac{0}{1}, \frac{1}{1}\}$ we will find a surgery of K on V_{γ_i} not yielding a solid torus. K is primitive on S'_B, so we take a homeomorphism of this manifold such that K is now like in Fig. 4.13. Let u, v, D_x and D_y be as in the proof of Lemma 4.2.1. We take the point of intersection between α_i and β_i as the base point for $\pi_1(S'_B)$. The circles $\alpha_0, \beta_0, \alpha_1, \beta_1$ represent, in the same order, $u\bar{v}^2, v, \bar{u}v^2, \bar{v}$. Again, K is a core in a solid torus of S'_B, so we first perform surgery on $K \subset \mathrm{int}T'_B$ and then attach the 2-handle along γ_i. By Lemma 3.1.1 and since $|2a - b| = 1$,

$$\gamma_i(\alpha_i, \beta_i) = \underbrace{\alpha_i{\beta_i}^2 \ldots \alpha_i{\beta_i}^2}_{k \geq 0 \text{ times}} \alpha_i\beta_i$$

or

$$\gamma_i(\alpha_i, \beta_i) = \underbrace{\alpha_i{\beta_i}^2 \ldots \alpha_i{\beta_i}^2}_{k \geq 0 \text{ times}} \beta_i$$

Now perform $\frac{1}{1}$-surgery on K (see Fig. 4.14). This yields a new genus two handlebody T''_B with basis $\{w, v\}$. By looking at the oriented intersections of α_i, β_i with μ', we see that $\alpha_0 = \bar{v}w\bar{v}, \beta_0 = vw\bar{v}wv$ and $\alpha_1 = \bar{w}v^2, \beta_1 = \bar{w}\bar{v}$. This gives:

$$\gamma_0(w, v) = \underbrace{\bar{v}w\bar{v}(vw\bar{v}wv)^2 \ldots \bar{v}w\bar{v}(vw\bar{v}wv)^2}_{k \geq 0 \text{ times}} \bar{v}w\bar{v}vw\bar{v}wv = (\bar{v}wv^2w\bar{v}w^3)^k\bar{v}w^3$$

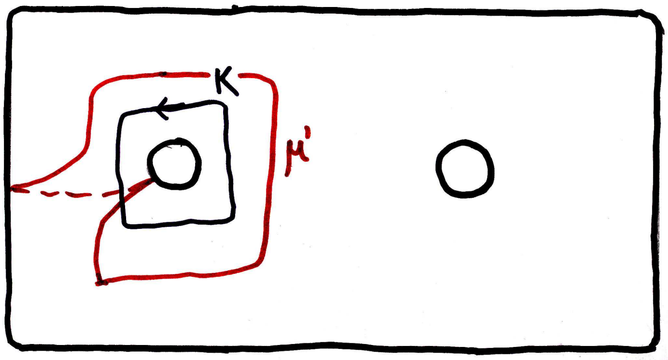

Figure 4.14: The loop μ' in T'_B is the new meridian after the surgery

or

$$\gamma_0(w,v) = \underbrace{\bar{v}w\bar{v}(vw\bar{v}wv)^2 \dots \bar{v}w\bar{v}(vw\bar{v}wv)^2}_{k\geq 0\,\text{times}} vw\bar{v}wv = (\bar{v}wv^2w\bar{v}w^3)^k\bar{v}wv^2w$$

Similarly,

$$\gamma_1(w,v) = \underbrace{\bar{w}v^2(\bar{w}\bar{v})^2 \dots \bar{w}v^2(\bar{w}\bar{v})^2}_{k\geq 0\,\text{times}} \bar{w}v^2\bar{w}\bar{v} = (\bar{w}v^2(\bar{w}\bar{v})^2)^k\bar{w}v^2\bar{w}\bar{v}$$

or

$$\gamma_1(w,v) = \underbrace{\bar{w}v^2(\bar{w}\bar{v})^2 \dots \bar{w}v^2(\bar{w}\bar{v})^2}_{k\geq 0\,\text{times}} \bar{w}\bar{v} = (\bar{w}v^2(\bar{w}\bar{v})^2)^k\bar{w}\bar{v}$$

Clearly $\gamma_i(w,v)$ is not primitive if $k \geq 1$, that is, if $\frac{a}{b} \notin \{\frac{0}{1}, \frac{1}{1}\}$. This completes the proof. $\square$

If K is trivial then $\omega(K) = |2a-b| = 0$, so $a=1, b=2$, since $\gcd(a,b)=1$. Using the same argument as in section 4.2, we have the following corollary.

Corollary 4.3.2. *T_i bounds a solid torus in both sides if $\frac{a}{b} \notin \{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}\}$*

Since $n \geq 2$ is the number of times the meridian disk of V_0 intersects the core of A'_0, while $b \geq 0$ is the number of times the meridian disk of V_{γ_0}, or equivalently, the circle γ_0,

intersects the core of A'_0, it follows that $b = n \geq 2$, so the cases $\frac{a}{b} = \frac{0}{1}$ or $\frac{1}{1}$ do not occur. In section 4.5 we show that this graph pair is *equivalent*, in some sense, to the one in section 4.2, so the case $\frac{a}{b} = \frac{1}{2}$ was actually studied in the previous section.

4.4 The case $\Gamma_S = (2; 2, 0, 0, 0), \Gamma_T = (1; 1, 1, 1, 1)$ is impossible

Assume now that $\Gamma_S = (2; 2, 0, 0, 0), \Gamma_T = (1; 1, 1, 1, 1)$. Let D_1, D_2, D_3 be as in the previous section. Then T_B is a genus two handlebody. Here the two Scharlemann bigons D_1, D_2 of Γ_S are nonparallel in T_B, so they constitute a complete disk system for T_B, and since T is separating, the remaining disk D_3 is also nonseparating. Fig. 4.15 shows T_B, K, D_1, D_2 and D_3 up to homeomorphism.

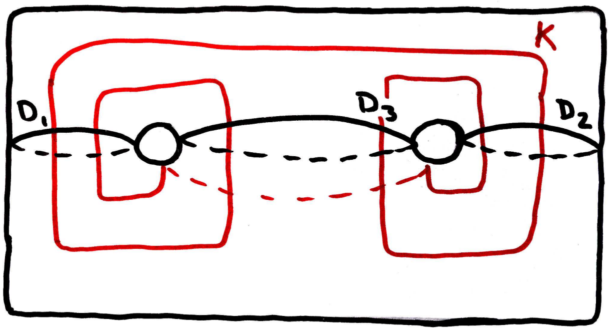

Figure 4.15: K, D_1, D_2 and D_3 in T_B

The only two options for the word in $\pi_1(T_B)$ represented by K are x^2y^2 and $xyxy = (xy)^2$. The latter is a power of a primitive element, so T_B compresses along K and this implies that $\widehat{T}$ is compressible, a contradiction. Thus K represents x^2y^2. Observe that K never passes two consecutive times through the same black disk, so there are no parallelisms in Γ_T and $\Gamma_T = (1; 1, 1, 1, 1)$. We construct the white spectrum of Γ_S in T_B as before, focusing on the boundary ∂Q of the white 4-gon Q in Γ_S. Fig. 4.16 shows the circle ∂Q after a slight

isotopy in ∂T_B, while Fig. 4.17 shows both ∂Q and K after performing a full negative twist around the left one-handle of T_B.

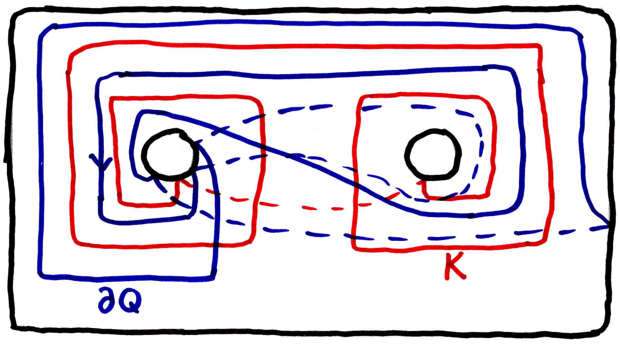

Figure 4.16: K and ∂Q in T_B

Since ∂Q represents the word $xyx\bar{y}\bar{x}\bar{y}$ in $\pi_1(T_B)$ with respect to the standard complete disk system D_x, D_y (see Fig. 3.1), it follows that the fundamental group of the manifold $T_B \cup_{\partial Q} N(Q)$ is

$$\langle x, y|\ xyx\bar{y}\bar{x}\bar{y} = 1\rangle \approx \left\langle X, Y|\ X^2 = Y^3\right\rangle,$$

where $X = xy$ and $Y = yxy$. Therefore the manifold $T_B \cup_{\partial Q} N(Q)$ is homeomorphic to the exterior of the trefoil knot, which by [6] admits a unique (up to orientation) embedding in S^3. Since, in the embedding of $T_B, \partial Q$ shown in Fig. 4.16, the circle ∂Q bounds a disk in $S^3 \setminus \mathrm{int} T_B$, it follows that the embedding of K in the figure, is the unique possible embedding of K in S^3. But then K is the trivial knot, which is not hyperbolic. This shows that the given graph pair (Γ_S, Γ_T) cannot be geometrically realizable in S^3.

4.5 Isotopies between essential tori

Two pairs of graphs of intersection produced by twice punctured tori are said to be *equivalent* if their geometric realization in S^3 produce equivalent knot exteriors.

In this section we prove that the two possible cases for graphs of intersection $\Gamma_S = (3; 0, 0, 0, 0), \Gamma_T = (2, 2, 2, 0)$ and $\Gamma_S = (2; 2, 0, 0, 0), \Gamma_T = (1; 2, 2, 0, 0)$ for hyperbolic knots

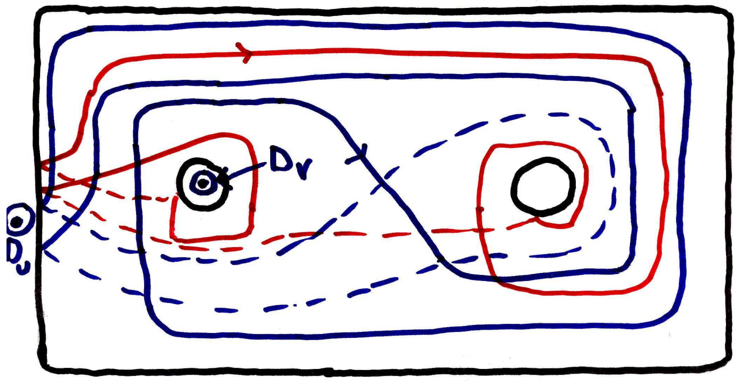

Figure 4.17: K and ∂Q in T_B after the twist

with two toroidal surgeries at distance 3 are, in fact, equivalent. Thus, we can move freely between the two realizable cases of graphs of intersection.

Let h be a proper arc on a disk face D of, say, Γ_S with boundary on the edges of Γ_S. Replacing two small arcs of Γ_S centered at ∂h with two parallel copies of h we obtain a new graph Γ'_S obtained from Γ_S by *surgery along* h.

Let $X(S,T)$ be obtained in X_K from $N(S,T,T_0)$ by capping off its 2-sphere boundary components with 3-balls. Let F be a face in Γ_S and choose an arbitrary boundary component $\partial_0 F$ in F, consisting of edges $\{e_i\}$ and corners of vertices $\{c_{i,j}\}$. Two edges e_i, e_j are connected by a corner $c_{i,j}$ whose labels at its endpoints are $+,-$ or $-,+$, according to the orientation of ∂F. Then $\partial_0 F \subset T \cup T_0$ consists of edges e_i, e_j connected by *spanning arcs* in T_0 going from vertex $+$ to vertex $-$ or viceversa, according to $c_{i,j}$.

Lemma 4.5.1. *$\partial_0 F$ is essential in ∂T_B or ∂T_W, for any face F and any boundary component $\partial_0 F$ in Γ_S.*

Proof. Proceed by contradiction and assume $\partial_0 F$ bounds a disk D in, say, ∂T_B. Isotope D so as to intersect transversely with the two vertices $+$ and $-$. The result is a graph Γ_D in D, with edges in $\{+,-\}$ and endpoints in $\partial_0 F$ (see Fig. 4.18). Choose an outermost edge e in Γ_D and cut D through e. We obtain a new disk D_1 with boundary an edge e_i and part of a vertex $+$ or $-$, but that is impossible, since all edges are essential in T. $\square$

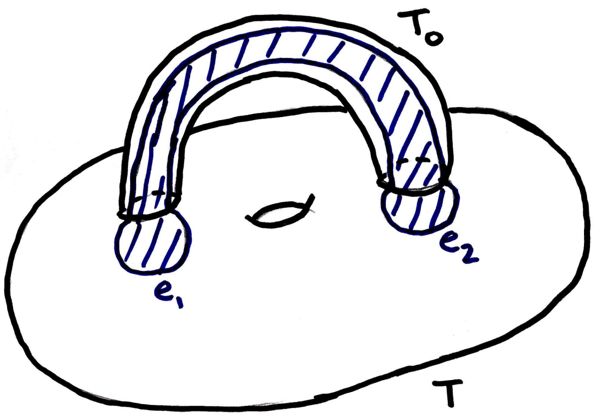

Figure 4.18: The edges e_1, e_2 are not essential in T

Observe that the previous assertion may be generalized to graphs of intersection between arbitrary surfaces F_a with an arbitrary number of boundary components n_a.

Lemma 4.5.2. *Let K be a hyperbolic knot admitting two essential, separating twice punctured tori S, T with boundary slopes at distance 3. Then $\partial X(S,T)$ is a union of tori.*

Proof. By the previous sections, we see that either $\Gamma_S = (3; 0, 0, 0, 0), \Gamma_T = (2, 2, 2, 0)$ or $\Gamma_S = (2; 2, 0, 0, 0), \Gamma_T = (1; 2, 2, 0, 0)$. $X(S,T)$ can be constructed by adding thickened faes of Γ_S to $N(T \cup T_0)$, which has two boundary components of genus 2. In both graphs of intersection, S has a disk face in each side of T. The boundary of such a disk is an essential loop in $T \cup T_0$ by the previous lemma. The addition of a 2-handle along this curve compresses a genus two boundary component to one or two tori. It follows that $\partial X(S,T)$ is a union of tori. □

Consider one of the components of $\partial X(S,T)$ containing the proper arc h on the white 4-gon face Q of S. We will find a proper arc k in the white side of T such that $h \cup k$ cobound a disk D in T_W (Fig. 4.19). Therefore we can isotope S through this disk D to obtain a new surface S'. The new intersection graph Γ'_S is obtained by surgery along h and then $\Gamma'_S = (3; 0, 0, 0, 0)$, like in section 4.2. In section 4.3 we proved that the circle ∂Q

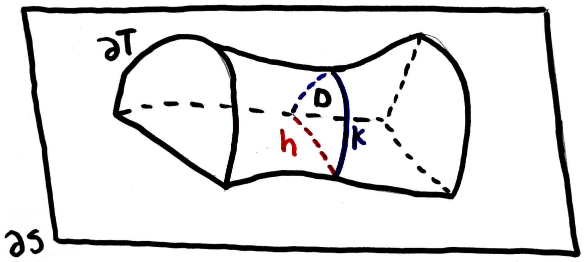

Figure 4.19: A $\partial-$compression disk in $X(S,T)$

represents a commutator, so it is a separating circle in ∂T_B. Since this is also ∂T_W, ∂Q separates ∂T_W.

Lemma 4.5.3. *Let D be a proper surface in T_W. If ∂D separates ∂T_W then D separates T_W.*

Proof. Every surface in intT_W is nonseparating, so $H_2(T_W) = 0$. The inclusion $i : (D, \partial D) \to (T_W, \partial T_W)$ induces a map between two long exact sequences of homology groups. Since ∂D separates ∂T_W the induced map $i_* : H_1(\partial D) \to H_1(\partial T_W)$ is the zero map. Thus the map between long exact sequences is as follows:

$$\begin{array}{ccccccc}
\ldots \to & H_2(T_W) = 0 & \to & H_2(T_W, \partial T_W) & \to & H_1(\partial T_W) & \to \ldots \\
& \uparrow & & \uparrow i_* & & \uparrow 0 & \\
\ldots \to & H_2(D) = 0 & \to & H_2(D, \partial D) & \overset{\approx}{\to} & H_1(\partial D) & \to \ldots
\end{array}$$

By exactness of the upper sequence, the map i_* is also the zero map. Therefore D separates T_W. $\square$

The same exactness of sequences can extend this lemma, but we will not need that. This gives that Q separates T_W into two once-punctured tori and, up to homeomorphism, Q is a waist disk of T_W. One of the punctured tori contains the white annulus A_W of the graph Γ_S. Since any two disjoint loops in a once-punctured torus are parallel, the boundary components of the annulus are parallel in T_W. In section 4.3 we proved this using the white spectrum in T_B'. Furthermore, we concluded that ∂A_W is in the lid disjoint from α_0. Since

the latter circle is in the right side of T_W separated by Q, we can draw A_W in the left side. Each boundary component consist of edges and one corner of a vertex in Γ_S. Thus K passes one time through each boundary component in T_W. These loops represent a (p, q) torus knot in the punctured torus, and since K is not primitive, $|q| \geq 2$. Since Q is a 4-gon, $|K \cdot Q| = 4$. In the other punctured torus, the two arcs must be parallel, either because of Γ_T or simply because the arcs in the other side are nonparallel. Moreover, each arc must wind at least one time along the right solid torus, since otherwise the number of intersection points between K and Q is not minimal in T_W.

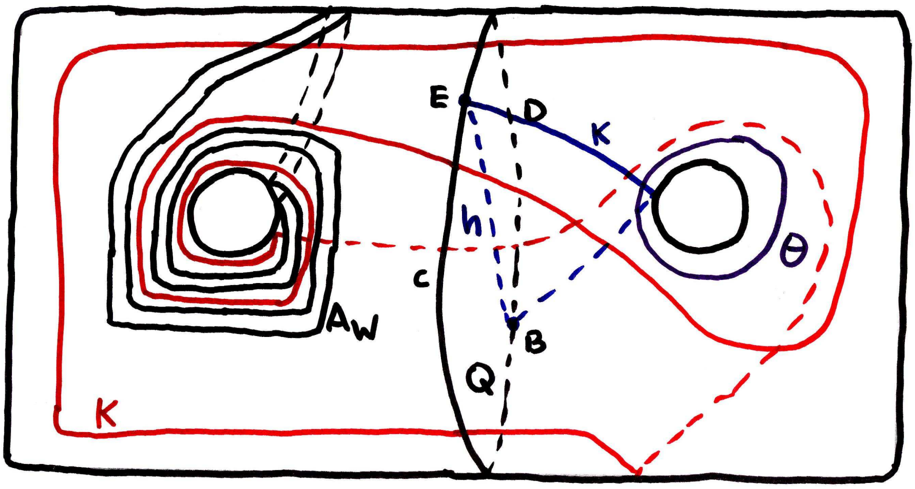

Figure 4.20: K in T_W

Gathering these results, T_W is like in Fig. 4.20, where for simplicity, K only winds one time in the right solid torus. We need to label the edges according to Γ_T. Observe that the black annulus A_B in Γ_T is in the right side of T_W and has the circle θ as its core. Clearly θ is either a primitive or a power of a primitive loop in T_W. Each component of A_B consists of the edges C, D, B and E and C, D must be parallel. With this, we label the edges as in Fig. 4.20 and draw an arc h from B to E in Q and another arc k from B to E in A_B. It is easy to see that $h \cup k$ bound a disk in T_W iff θ is a primitive element; we just need to slide the primitivity disk of θ through parallel copies of θ (see Fig. 4.21).

Since T_W and T_B have common boundary, after removing a companion solid torus V_1 in

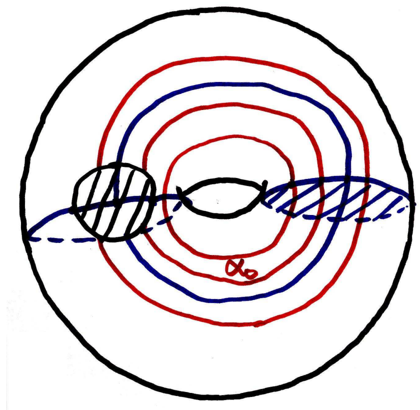

Figure 4.21: A $\partial-$compression disk is equivalent to a primitiviy disk for α_0

T_B to obtain T'_B, θ represents the word α_0 in this manifold (see Section 4.3). To return to the original manifold T_B, we need to reattach the companion torus V_1 in T'_B. In the white side, this represents the invert process: remove a solid torus which converts $\theta = \alpha_0$ to a primitive element in T_W (see Fig. 4.22). Therefore there exists a $\partial-$compressible disk in $X(S,T)$ between h and k and we can isotope S by performing a surgery on Γ_S along h (see Fig. 4.23). This proves:

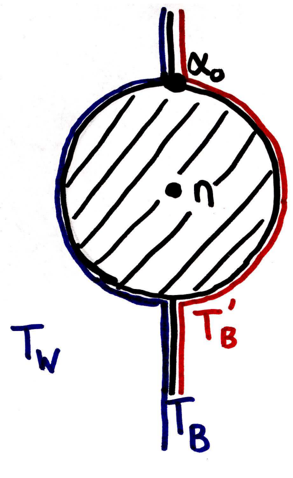

Figure 4.22: α_0 as part of T_B, T'_B and T_W

Lemma 4.5.4. *Suppose S, T are essential twice punctured tori in a hyperbolic knot exterior*

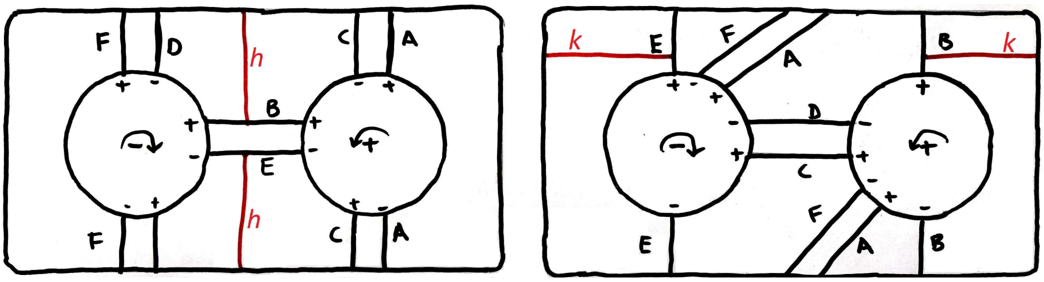

Figure 4.23: The arcs h and k in Γ_S, Γ_T respectively

in S^3 *which produce the graph pair* $\Gamma_S = (2; 2, 0, 0, 0), \Gamma_T = (1; 2, 2, 0, 0)$. *Then* S *and* T *may be isotoped so as to produce the graph pair* $\Gamma_S = (3; 0, 0, 0, 0), \Gamma_T = (2, 2, 2, 0)$.

Now assume $\Gamma_S = (3; 0, 0, 0, 0), \Gamma_T(2, 2, 2, 0)$. Using a similar argument we prove that, in fact the process can be reversed and this reduces to $\Gamma_S = (2; 2, 0, 0, 0), \Gamma_T = (1; 2, 2, 0, 0)$ by isotopies of S and T. By Lemma 4.5.2 $\partial X(S, T)$ is a union of tori. According to Fig. 2.2, Γ_T only has a white hexagon Q. Since ∂Q represents a commutator in ∂S_W, it separates ∂S_W. By the same argument of Lemma 4.5.3, Q separates S_W and is thus a waist disk. Since Γ_S has only two families of parallel edges, each with weight three, we must have two families of three parallel arcs each in $\partial Q \subset S_W$. To construct K in ∂S_W, we draw the parallelism between three edges as in Fig. 4.24.

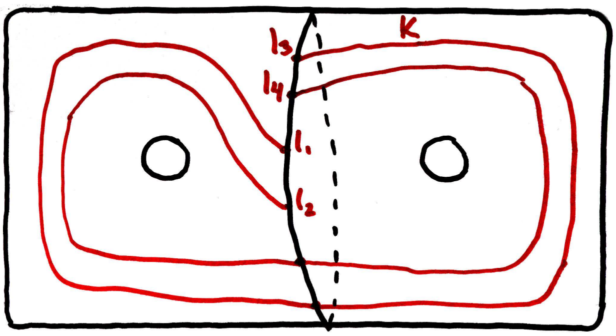

Figure 4.24: Parallelism in S_W between three arcs in Q

Observe that by cutting S_W by Q and these arcs we obtain two annuli. The endpoints l_1, l_2 and l_3, l_4 lie in different annuli and in each case, they lie in one boundary component.

Since Q separates S_W, we need to join the endpoints l_1, l_2 and l_3, l_4 and since each pair lie in one boundary component of an annulus, such arc must be parallel to the rest. Fig. 4.25 illustrate this, up to homeomorphism. We label the edges in S_W as follows. Choose any family of three parallel arcs in ∂Q and set them to $\{A, B, C\}$. Only one of these arcs forms part of two bigons in S_W. Label this arc as B, according to Γ_S. The other family of arcs are the edges $\{D, E, F\}$, and the edge D is uniquely determined by a similar argument. Orient ∂Q in S_W and label the rest of the edges according to Γ_T. Analyzing ∂Q in Γ_T, the pairs $\{A, F\}, \{C, E\}$ appear consecutively, regardless of the orientation of ∂Q, thus for the two annuli A_0, A_1 in Γ_S we can construct arcs h_0, h_1 from A to E and from C to F respectively. We can apply the same argument as in Lemma 4.5.4 since the cores of A_0, A_1 represent powers of α_0 or β_0 and in this case we are reattaching companion tori in both circles in S'_B. Performing surgery on Γ_S in either of these arcs gives $\Gamma_S = (2; 2, 0, 0, 0)$, due to the symmetry given by formula (2.1). Fig. 4.26 shows the arcs h_1, k_1 in Γ_T, Γ_S, respectively. We have proven:

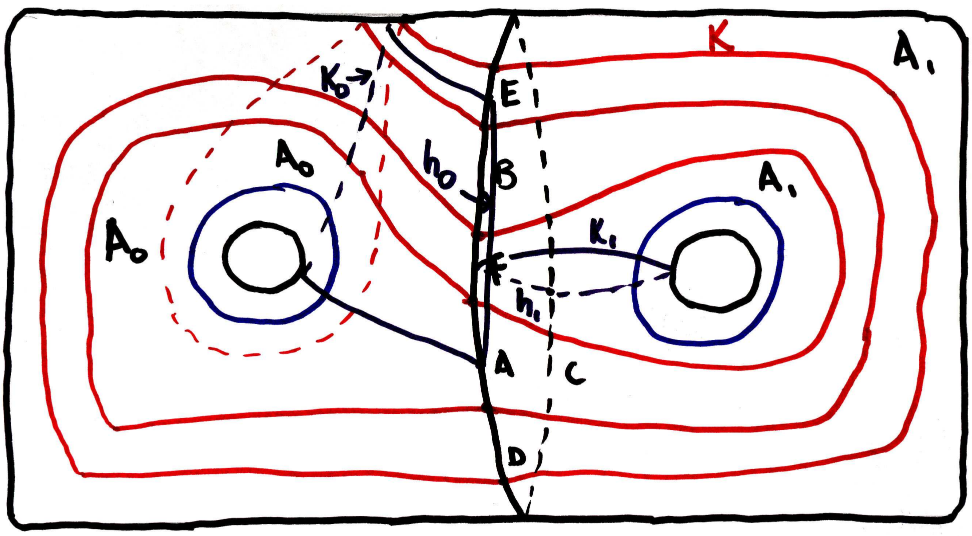

Figure 4.25: The $\partial-$compression disks $h_0 \cup k_0$ and $h_1 \cup k_1$

Lemma 4.5.5. *Suppose S, T are essential twice punctured tori in a hyperbolic knot exterior in S^3 which produce the graph pair $\Gamma_S = (3; 0, 0, 0, 0), \Gamma_T = (2, 2, 2, 0)$. Then S and T may*

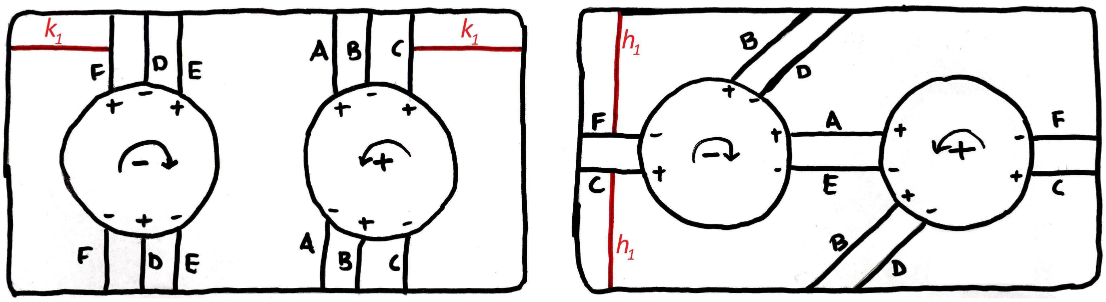

Figure 4.26: The arcs h_1, k_1 in Γ_T, Γ_S respectively

be isotoped so as to produce the graph pair $\Gamma_S = (2; 2, 0, 0, 0)$, $\Gamma_T = (1; 2, 2, 0, 0)$.

By the two previous lemmas we thus obtain:

Corollary 4.5.6. *The graphs* $\Gamma_S = (3; 0, 0, 0, 0), \Gamma_T = (2, 2, 2, 0)$ *and* $\Gamma_S = (2; 2, 0, 0, 0)$, $\Gamma_T = (1; 2, 2, 0, 0)$ *are equivalent.*

We will focus on $\Gamma_S = (3; 0, 0, 0, 0), \Gamma_T = (2, 2, 2, 0)$ only because the classification of the knots is simpler in this case. By the final remarks in section 4.2, we conclude that if a hyperbolic knot K admits two essential twice punctured separating tori S, T in X_K with boundary slopes at distance 3 then K is equivalent to some $K(a, b, n)$. To finish the proof of Theorem 1.1 we need to define $K(a, b, 0)$. We do this on the next chapter.

Chapter 5

The knots $K(a, b, n)$

Here we give a classification of the knots $K(a, b, n)$ introduced in section 4.2. In chapter 4 we proved that all hyperbolic knots K with two essential, separating twice punctured tori with boundary slopes at distance 3 must be in $K(a, b, n)$. Let $\gamma_0 = a\alpha_0 + b\beta_0, \gcd(a, b) = 1$ be as in section 4.2. By the remarks in that section we can assume $b \geq 2$. We need to choose a canonical homeomorphism of S'_B to define $K(a, b, 0)$. Each γ_0 determines a fraction $\frac{a}{b} \in \mathbb{Q}$; as we will see, it is possible to use the theory of *continued fractions* to produce the desired canonical homeomorphism of S'_B. Below we give a brief exposition of continued fractions; for a detailed exposition and the proofs of the following lemmas, see [11].

5.1 Continued fractions

Lemma 5.1.1. *Any rational number $\frac{a}{b}, \gcd(a, b) = 1$ can be represented by a finite simple continued fraction*

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_N}}}},$$

where each a_i is an integer ($i = 0, \ldots, N$) and, for $i \geq 1$, a_i is positive. The integers a_i are obtained by Euclid's algorithm:

$$\begin{array}{rcll} a &=& a_0 b + k_1 & (0 < k_1 < b), \\ b &=& a_1 k_1 + k_2 & (0 < k_2 < k_1), \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ k_{N-2} &=& a_{N-1} k_{N-1} + k_N & (0 < k_N < k_{N-1}), \\ k_{N-1} &=& a_N k_N, & \\ k_N &=& 1 \quad \Box & \end{array}$$

To make the notation shorter, we write $\frac{a}{b} = [a_0, a_1, \ldots, a_N]$. We call $a_0, a_1, \ldots, a_N$ the *quotients* of the continued fraction and

$$[a_0, a_1, \ldots, a_n] \quad (0 \leq n \leq N)$$

the nth *convergent* to $[a_0, a_1, \ldots, a_N]$. These convergents are calculated easily by means of the following lemma.

Lemma 5.1.2. *Define p_n, q_n by*

$$p_{-2} = 0, \;\; q_{-2} = 1, \;\; p_{-1} = 1, \;\; q_{-1} = 0, \tag{5.1}$$

and, for $0 \leq n \leq N$,

$$p_n = a_n p_{n-1} + p_{n-2}, \;\; q_n = a_n q_{n-1} + q_{n-2}. \tag{5.2}$$

Thus

$$[a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}. \quad \Box$$

Observe that if $\frac{a}{b} = [a_0, a_1, \ldots, a_N]$ then $a = p_N, b = p_N$.

Lemma 5.1.3. *The convergents to a simple continued fraction $[a_0, a_1, \ldots, a_N]$ are in their lowest terms; i.e., $\gcd(p_n, q_n) = 1$ for $0 \leq n \leq N$. The integers p_n and q_n satisfy*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}. \quad \Box$$

We call

$$a'_n = [a_n, a_{n+1}, \ldots, a_N] \quad (0 \leq n \leq N)$$

the nth *complete quotient* of the continued fraction $[a_0, a_1, \ldots, a_N]$. Hence

$$\frac{a}{b} = a'_0 = a_0 + \frac{1}{a'_1}$$

or, more generally,

$$\frac{a}{b} = \frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}} \quad (0 \leq n \leq N).$$

Lemma 5.1.4. *If $\frac{a}{b}$ is representable by a simple continued fraction with an odd (even) number of convergents, it is also representable by one with an even (odd) number.*

Proof. Indeed, if $a_N \neq 1$, it is easy to see that

$$[a_0, a_1, \ldots, a_N] = [a_0, a_1, \ldots, a_N - 1, 1],$$

and if $a_N = 1$, $[a_0, a_1, \ldots, a_{N-1}, 1] = [a_0, a_1, \ldots, a_{N-2}, a_{N-1} + 1]$. □

Apart from this ambiguity, the representation of a rational number with a continued fraction is unique:

Lemma 5.1.5 ([11]). *A rational number can be expressed as a finite simple continued fraction in just two ways, one with an even and the other with an odd number of convergents. In one form, the last partial quotient is 1, in the other it is greater than 1.* □

5.2 An algorithm for $K(a, b, n)$

We return to our analysis of the hyperbolic knots K in S^3 with exterior X_K admitting two essential twice punctured separating tori with boundary slopes at distance 3. In chapter 4 we proved that such a knot K can be obtained from a genus two handlebody S'_B by attaching a 2-handle along a circle $\gamma_0 = a\alpha_0 + b\beta_0$, a and b are integers with $\gcd(a, b) = 1, b \geq 2$.

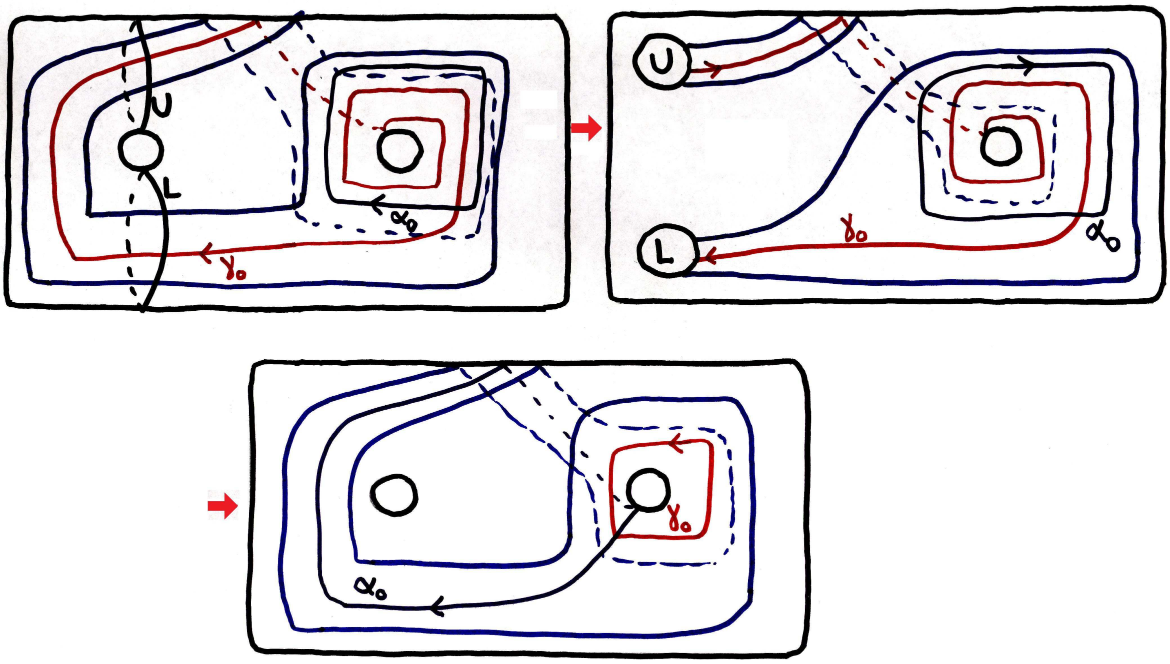

Figure 5.1: A homeomorphism for $\gamma_0 = 2\alpha_0 + \beta_0$ in S'_B

Consider the circle $\gamma_0 = m\alpha_0 + \beta_0$. Fig. 5.1 shows the case $m = 2$. This loop is a primitive element in $\pi_1(S'_B)$. We can "lift up" 2 of the feet of the handlebody and start a homeomorphism of S'_B, by sliding one of the feet along γ_0, and modifying the circles α_0, β_0 accordingly. At the end of the homeomorphism, each lid $\widetilde{\mathcal{T}_i}$ is mapped into itself, $m\alpha_0 + \beta_0$ reduces to $-\alpha_0$, α_0 and β_0 are now β_0 and $-\alpha_0 - m\beta_0$, considered as elements in $\pi_1(\mathcal{T}_0) = \mathbb{Z} \oplus \mathbb{Z}$ with basis $\{\alpha_0, \beta_0\}$. Thus the homeomorphism yields an automorphism $A_m : \pi_1(\mathcal{T}_0) \to \pi_1(\mathcal{T}_0)$ represented by the matrix

$$A_m = \begin{bmatrix} 0 & -1 \\ 1 & -m \end{bmatrix}, m \in \mathbb{Z}$$

By abuse of notation, we will sometimes refer to the homeomorphism also as A_m, if there is no room for confusion.

Similarly, using the loop $\gamma_0 = \alpha_0 + n\beta_0$ we produce a homeomorphism of S'_B mapping each lid onto itself, and the circles $\alpha_0 + n\beta_0, \alpha_0, \beta_0$ to $\alpha_0, \alpha_0 - n\beta_0, \beta_0$ respectively. Again,

this induces an automorphism $B_n : \pi_1(\mathcal{T}_0) \to \pi_1(\mathcal{T}_0)$ with matrix representation

$$B_n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}, n \in \mathbb{Z}$$

Here we will also refer to the homeomorphism itself as B_m, if the context is clear enough to distinguish it from the actual automorphism.

Thus the image of an arbitrary loop $\zeta = a\alpha_0 + b\beta_0$ under any of the homeomorphisms above is given by matrix multiplication $A_m\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & -m \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a-mb \end{bmatrix}$, whereas $B_n\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b-na \end{bmatrix}$.

Our goal is to systematically apply these homeomorphisms on S'_B so that the loop $\gamma_0 = a\alpha_0 + b\beta_0$ is reduced to either $\pm\alpha_0$ or $\pm\beta_0$. After attaching the 2-handle along γ_0 we have K in the solid torus V_{γ_0}. In chapter 4 we showed that K is obtained by finding the final images of the circles α_0, β_0 under this homeomorphism and adding a positive full twist. Fig. 5.2 illustrate this when these images are the circles $-2\alpha_0 - \beta_0$ and β_0, respectively.

By the above remarks, the concatenation of homeomorphisms translates into a composition of automorphisms at the $\pi_1(S'_B)$ level, which in turn become matrix multiplications. Let $\gamma_0 = a\alpha_0 + b\beta_0, \gcd(a,b) = 1$ be the attaching curve for the 2-handle. Recall that we can impose $b \geq 2$.

Let $\lfloor x \rfloor$ be the greatest lower integer function and let $[a_0, a_1, \ldots, a_N]$ be the continued

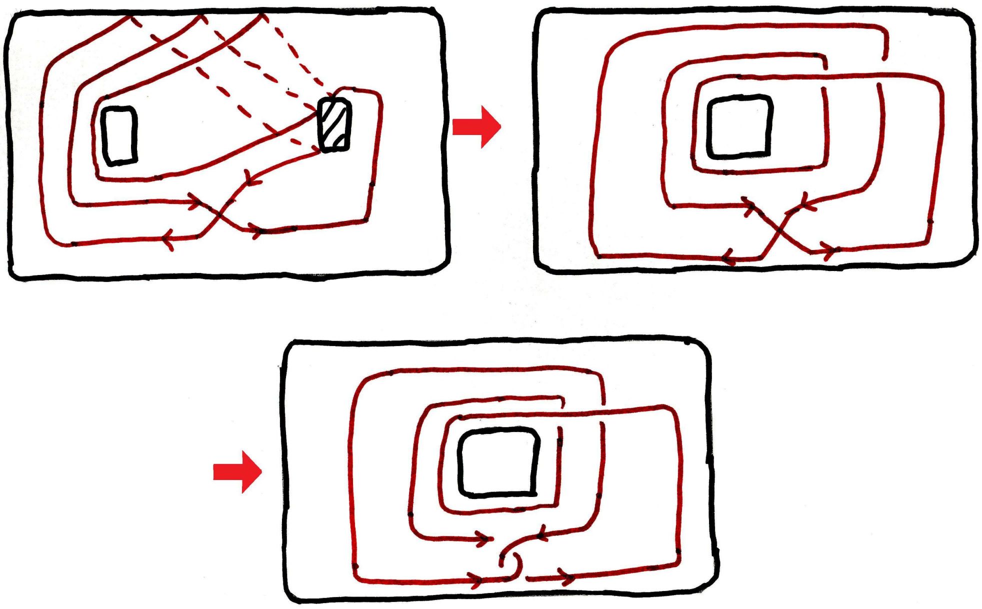

Figure 5.2: The process of drawing the images of α_0, β_0 and adding a positive full twist to obtain $K(a, b, 0)$ in S'_B

fraction expansion for $\frac{a}{b}$, with the quotients a_i obtained from Euclid's algorithm:

$$\begin{aligned}
a &= a_0 b + k_1 && (0 < k_1 < b), \\
b &= a_1 k_1 + k_2 && (0 < k_2 < k_1), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
k_{N-2} &= a_{N-1} k_{N-1} + k_N && (0 < k_N < k_{N-1}), \\
k_{N-1} &= a_N k_N, \\
k_N &= 1, \\
k_{N+1} &= 0,
\end{aligned}$$

where we can define $k_0 = b$.

Lemma 5.2.1. *If $a_0 \neq 0$, then*

$$A_{(-1)^n a_n} \ldots A_{-a_1} A_{a_0} \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} (-1)^{\lfloor \frac{n+2}{2} \rfloor} k_n \\ (-1)^{\lfloor \frac{n+1}{2} \rfloor} k_{n+1} \end{bmatrix}$$

for $0 \le n \le N$.

In particular,

$$A_{(-1)^N a_N} \dots A_{-a_1} A_{a_0} \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} (-1)^{\lfloor \frac{N+2}{2} \rfloor} \\ 0 \end{bmatrix}$$

Proof. If $n = 0$ then

$$A_{a_0} \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} -b \\ a - a_0 b \end{bmatrix} = \begin{bmatrix} -k_0 \\ k_1 \end{bmatrix} = \begin{bmatrix} (-1)^{\lfloor \frac{0+2}{2} \rfloor} k_0 \\ (-1)^{\lfloor \frac{0+1}{2} \rfloor} k_1 \end{bmatrix}$$

Assume now that

$$A_{(-1)^{n-1} a_{n-1}} \dots A_{-a_1} A_{a_0} \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} (-1)^{\lfloor \frac{(n-1)+2}{2} \rfloor} k_{n-1} \\ (-1)^{\lfloor \frac{(n-1)+1}{2} \rfloor} k_n \end{bmatrix},$$

for $0 \le n < N$. Then

$$A_{(-1)^n a_n} \dots A_{-a_1} A_{a_0} \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = A_{(-1)^n a_n} \left(\begin{bmatrix} (-1)^{\lfloor \frac{n+1}{2} \rfloor} k_{n-1} \\ (-1)^{\lfloor \frac{n}{2} \rfloor} k_n \end{bmatrix} \right)$$

$$= \begin{bmatrix} (-1)^{1+\lfloor \frac{n}{2} \rfloor} k_n \\ (-1)^{\lfloor \frac{n+1}{2} \rfloor} k_{n-1} - (-1)^{n+\lfloor \frac{n}{2} \rfloor} a_n k_n \end{bmatrix} = \begin{bmatrix} (-1)^{\lfloor \frac{n+2}{2} \rfloor} k_n \\ (-1)^{\lfloor \frac{n+1}{2} \rfloor} k_{n+1} \end{bmatrix}.$$

The induction is complete. □

Lemma 5.2.2. *If* $a_0 \neq 0$, *then*

$$A_{(-1)^n a_n} \dots A_{-a_1} A_{a_0} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} (-1)^{\lfloor \frac{n+3}{2} \rfloor} p_{n-1} \\ (-1)^{\lfloor \frac{n+2}{2} \rfloor} p_n \end{bmatrix}$$

for $0 \le n \le N$.

In particular,

$$A_{(-1)^N a_N} \dots A_{-a_1} A_{a_0} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} (-1)^{\lfloor \frac{N+3}{2} \rfloor} p_{N-1} \\ (-1)^{\lfloor \frac{N+2}{2} \rfloor} a \end{bmatrix}$$

Proof. We check the case $n = 0$ first.

$$A_{a_0}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -a_0 \end{bmatrix} = \begin{bmatrix} (-1)^{\lfloor \frac{0+3}{2} \rfloor} p_{-1} \\ (-1)^{\lfloor \frac{0+2}{2} \rfloor} p_0 \end{bmatrix}$$

Assume now that

$$A_{(-1)^{n-1}a_{n-1}} \dots A_{-a_1} A_{a_0} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} (-1)^{\lfloor \frac{(n-1)+3}{2} \rfloor} p_{(n-1)-1} \\ (-1)^{\lfloor \frac{(n-1)+2}{2} \rfloor} p_{n-1} \end{bmatrix},$$

for $0 \leq n < N$. Then

$$A_{(-1)^n a_n} \dots A_{-a_1} A_{a_0} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = A_{(-1)^n a_n} \left(\begin{bmatrix} (-1)^{\lfloor \frac{n+2}{2} \rfloor} p_{n-2} \\ (-1)^{\lfloor \frac{n+1}{2} \rfloor} p_{n-1} \end{bmatrix}\right)$$

$$= \begin{bmatrix} (-1)^{1+\lfloor \frac{n+1}{2} \rfloor} p_{n-1} \\ (-1)^{\lfloor \frac{n+2}{2} \rfloor} p_{n-2} - (-1)^{n+\lfloor \frac{n+1}{2} \rfloor} a_n p_{n-1} \end{bmatrix} = \begin{bmatrix} (-1)^{\lfloor \frac{n+3}{2} \rfloor} p_{n-1} \\ (-1)^{\lfloor \frac{n+2}{2} \rfloor} p_n \end{bmatrix}.$$

□

Lemma 5.2.3. *If $a_0 \neq 0$, then*

$$A_{(-1)^n a_n} \dots A_{-a_1} A_{a_0} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} (-1)^{\lfloor \frac{n+1}{2} \rfloor} q_{n-1} \\ (-1)^{\lfloor \frac{n}{2} \rfloor} q_n \end{bmatrix}$$

for $0 \leq n \leq N$.

In particular,

$$A_{(-1)^N a_N} \dots A_{-a_1} A_{a_0} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} (-1)^{\lfloor \frac{N+1}{2} \rfloor} q_{N-1} \\ (-1)^{\lfloor \frac{N}{2} \rfloor} b \end{bmatrix}$$

Proof. Proceed by induction as usual. For $n = 0$ we have

$$A_{a_0}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1)^{\lfloor \frac{0+1}{2} \rfloor} q_{-1} \\ (-1)^{\lfloor \frac{0}{2} \rfloor} q_0 \end{bmatrix}$$

Suppose now that

$$A_{(-1)^{n-1}a_{n-1}} \dots A_{-a_1} A_{a_0} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} (-1)^{\lfloor \frac{(n-1)+1}{2} \rfloor} q_{(n-1)-1} \\ (-1)^{\lfloor \frac{n-1}{2} \rfloor} q_{n-1} \end{bmatrix},$$

for $0 \le n < N$. Then

$$A_{(-1)^n a_n} \dots A_{-a_1} A_{a_0}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = A_{(-1)^n a_n}\left(\begin{bmatrix}(-1)^{\lfloor\frac{n}{2}\rfloor} q_{n-2}\\(-1)^{\lfloor\frac{n+3}{2}\rfloor} q_{n-1}\end{bmatrix}\right)$$

$$= \begin{bmatrix}(-1)^{1+\lfloor\frac{n+3}{2}\rfloor} q_{n-1}\\(-1)^{\lfloor\frac{n}{2}\rfloor} q_{n-2} - (-1)^{n+\lfloor\frac{n+3}{2}\rfloor} a_n q_{n-1}\end{bmatrix} = \begin{bmatrix}(-1)^{\lfloor\frac{n+1}{2}\rfloor} q_{n-1}\\(-1)^{\lfloor\frac{n}{2}\rfloor} q_n\end{bmatrix}.$$

□

For any two matrices $\begin{bmatrix}a\\b\end{bmatrix}, \begin{bmatrix}c\\d\end{bmatrix}$, $a, b, c, d \in \mathbb{Z}$, let

$$A_m\left(\begin{bmatrix}a & c\\b & d\end{bmatrix}\right)$$

denote the 2×2 matrix with columns the images in A_m of $\begin{bmatrix}a\\b\end{bmatrix}, \begin{bmatrix}c\\d\end{bmatrix}$ in this order. By Lemma 5.1.4, we can impose the condition N odd or N even. Let I denote the 2×2 identity matrix. Combining these results and the properties of the greatest lower integer function, we have the result

Lemma 5.2.4.

$$A_{(-1)^N a_N} \dots A_{-a_1} A_{a_0}(I) = \begin{cases} (-1)^{\frac{N+2}{2}} \begin{bmatrix}-q_{N-1} & p_{N-1}\\-b & a\end{bmatrix} & N \text{ even, } a_0 \neq 0\\[2ex] (-1)^{\frac{N+1}{2}} \begin{bmatrix}q_{N-1} & -p_{N-1}\\-b & a\end{bmatrix} & N \text{ odd, } a_0 \neq 0 \quad \square \end{cases}$$

We now classify the case $a_0 = 0$. It is no surprise that the results and proofs are very similar to the previous case.

Lemma 5.2.5. *If* $a_0 = 0$ *then*

$$A_{(-1)^N a_N} \dots A_{a_2} B_{a_1} \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} (-1)^{\lfloor \frac{N}{2} \rfloor} \\ 0 \end{bmatrix}$$

$$A_{(-1)^N a_N} \dots A_{a_2} B_{a_1} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} (-1)^{\lfloor \frac{N+3}{2} \rfloor} q_{N-1} \\ (-1)^{\lfloor \frac{N+2}{2} \rfloor} b \end{bmatrix}$$

$$A_{(-1)^N a_N} \dots A_{a_2} B_{a_1} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} (-1)^{\lfloor \frac{N+1}{2} \rfloor} p_{N-1} \\ (-1)^{\lfloor \frac{N}{2} \rfloor} a \end{bmatrix}$$

Proof. All the statement are proved by induction and using the same argument as the previous lemmas. □

Again, by properties of the greatest lower integer function, we have the following

Lemma 5.2.6.

$$A_{(-1)^N a_N} \dots A_{a_2} B_{a_1}(I) = \begin{cases} (-1)^{\frac{N}{2}} \begin{bmatrix} -q_{N-1} & p_{N-1} \\ -b & a \end{bmatrix} & N \textit{ even, } a_0 = 0 \\ \\ (-1)^{\frac{N+3}{2}} \begin{bmatrix} q_{N-1} & -p_{N-1} \\ -b & a \end{bmatrix} & N \textit{ odd, } a_0 = 0 \quad \square \end{cases}$$

To make the notation shorter, let $C_{\frac{a}{b}}$ denote either $A_{(-1)^N a_N} \dots A_{-a_1} A_{a_0}$ or $A_{(-1)^N a_N} \dots A_{a_2} B_{a_1}$, according to the continued fraction of $\frac{a}{b}$. Then the following corollary summarizes our previous results.

Corollary 5.2.7. *Let* $\gamma_0 = a\alpha_0 + b\beta_0$. *Let* $\frac{a}{b} = [a_0, \dots, a_N]$. *Then the homeomorphism* $C_{\frac{a}{b}}$

maps γ_0 to $\pm\alpha_0$ and $C_{\frac{a}{b}}(I)$ is one of the following

$$(-1)^{\frac{N+2}{2}}\begin{bmatrix}-q_{N-1} & p_{N-1}\\ -b & a\end{bmatrix} \quad N \text{ even, } a_0 \neq 0; \quad (-1)^{\frac{N+1}{2}}\begin{bmatrix}q_{N-1} & -p_{N-1}\\ -b & a\end{bmatrix} \quad N \text{ odd, } a_0 \neq 0$$

$$(-1)^{\frac{N}{2}}\begin{bmatrix}-q_{N-1} & p_{N-1}\\ -b & a\end{bmatrix} \quad N \text{ even, } a_0 = 0; \quad (-1)^{\frac{N+3}{2}}\begin{bmatrix}q_{N-1} & -p_{N-1}\\ -b & a\end{bmatrix} \quad N \text{ odd, } a_0 = 0 \quad \square$$

By Lemma 5.1.3, all these matrices lie in $\mathrm{SL}_2(\mathbb{Z})$. Note that, since any fraction has an odd and an even continued fraction, for any loop $\gamma_0 = a\alpha_0 + b\beta_0, \gcd(a,b) = 1$ there are two possible definitions for $C_{\frac{a}{b}}$. Since $b \geq 2$ only one of this expansions has a last quotient a_N different from 1. We set this $C_{\frac{a}{b}}$ as the *canonical homeomorphism* for the pair (S'_B, γ_0) and its associated knot K as $K(a,b,0)$. Thus the knot $K = K(a,b,0)$ in V_{γ_0} is obtained by taking the images of the circles $\alpha_0 = \begin{bmatrix}1\\0\end{bmatrix}, \beta_0 = \begin{bmatrix}0\\1\end{bmatrix}$ under the appropriate $C_{\frac{a}{b}}$ and adding a full positive twist in the intersection point of the circles. Corollay 5.2.7 shows the four possibilities of these images according to the continued fraction expansion $\frac{a}{b} = [a_0, \ldots, a_N]$. Moreover, any two different homeomorphisms involved to transform γ_0 to $\pm\alpha_0$ will give a knot $K(a,b,n)$, obtained from $K(a,b,0)$ by n full twists in V_{γ_0}. Hence the previous continued fraction algorithm gives the same family of knots if the expansion is even or odd. We have proven that K is equivalent to some $K(a,b,n)$. This completes the proof of Theorem 1.1.

Consider a matrix $C_{\frac{a}{b}}(I)$ in any of the 4 cases in Corollary 5.2.7. To this matrix we can associate a knot $K(a,b,n)$ as in the previous paragraph. It is easy to see that the matrix $-C_{\frac{a}{b}}(I)$ produces the same knot $K(a,b,n)$ but with its orientation reversed. By our definition of equivalent knots, the two knots are equivalent. Thus we can disregard orientations and work in the projective special linear group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. We assume this in what follows. For any $M \in \mathrm{SL}_2(\mathbb{Z})$, let $\overline{M} = \{M, -M\}$ denote its image in $\mathrm{PSL}_2(\mathbb{Z})$. The following corollary is then obvious.

Corollary 5.2.8. *Let* $\gamma_0 = a\alpha_0 + b\beta_0$. *Let* $\frac{a}{b} = [a_0, \ldots, a_N]$. *Then the homeomorphisms* $C_{\frac{a}{b}}$ *maps* γ_0 *to* $\pm\alpha_0$ *and* $\overline{C_{\frac{a}{b}}(I)}$ *is one of the following*

$$\overline{\begin{bmatrix} -q_{N-1} & p_{N-1} \\ -b & a \end{bmatrix}}, N \text{ even} \; ; \; \overline{\begin{bmatrix} q_{N-1} & -p_{N-1} \\ -b & a \end{bmatrix}}, N \text{ odd.} \qquad \square$$

Given two loops $\zeta_1 = c\alpha_0 + d\beta_0, \zeta_2 = e\alpha_0 + f\beta_0$ in $\widetilde{\mathcal{T}}_0$ such that

$$M = \begin{bmatrix} c & e \\ d & f \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

we can determine if these circles (modulo orientation) represent an element $\overline{C_{\frac{a}{b}}(I)}$ yielding a valid knot K with the aforementioned properties, for some fraction $\frac{a}{b}$. We will see that there is a symmetry in the constructions for $\frac{a}{b}$ and $-\frac{a}{b}$, but first we need some preparatory lemmas. Recall that one of the topological constraints is that $b \geq 2$. This implies that the continued fraction expansion for $\frac{a}{b}$ has $N \geq 1$ quotients.

Lemma 5.2.9. *Let* $1 \leq \frac{a}{b} = [a_0, a_1, \ldots, a_N]$. *Then* $0 < \frac{b}{a} \leq 1$, *with continued fraction expansion* $[0, a_0, a_1, \ldots, a_N]$

Proof. Since $a_0 \geq 1$

$$\frac{b}{a} = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_N}}}} = [0, a_0, a_1, \ldots, a_N]$$

$\square$

Lemma 5.2.10. *Let* $\frac{a}{b} = [a_0, a_1, \ldots, a_N], N \geq 1$ *and* $a_1 \geq 2$. *Then* $-\frac{a}{b} = [-a_0 - 1, 1, a_1 - 1, a_2, \ldots a_N]$.

Proof. Let $-\frac{a}{b} = [b_0, b_1, \ldots, b_M]$. Let $a_1' = [a_1, a_2, \ldots, a_N]$ be the first complete quotient of the continued fraction. Since $a_1 \geq 2, a_1' \geq 2$. Thus $a_1' - 1 = [a_1 - 1, a_2, \ldots, a_N] \geq 1$ and

$0 < \frac{1}{a_1'-1} \leq 1$. By the previous lemma, $\frac{1}{a_1'-1} = [0, a_1 - 1, a_2, \ldots, a_N]$. Since

$$-\frac{a}{b} = b_0 + \frac{1}{b_1'} = -a_0 - \frac{1}{a_1'} = -a_0 - 1 + 1 - \frac{1}{a_1'}$$

and $0 < 1 - \frac{1}{a_1'} < 1$, $b_0 = -a_0 - 1$ (note that this equality does not require $a_1 \geq 2$), $b_1' = 1 + \frac{1}{a_1'-1} = [1, a_1 - 1.a_2, \ldots, a_N]$. Therefore, $-\frac{a}{b} = [-a_0 - 1, 1, a_1 - 1, a_2, \ldots a_N]$. $\square$

Corollary 5.2.11. *Let* $\frac{a}{b} = [a_0, a_1, \ldots, a_N], N \geq 1$ *and* $-\frac{a}{b} = [b_0, b_1, \ldots, b_M], M \geq 1$. *Then either* $a_1 \geq 2$ *or* $b_1 \geq 2$.

Proof. By the previous lemma, $b_0 = -a_0 - 1$, regardless of the value of a_1. Write $\frac{a}{b} = a_0 + \frac{1}{a_1'}$ and $-\frac{a}{b} = -a_0 - 1 + \frac{1}{b_1'}$. Then, after adding both terms and simplifying we get

$$1 = \frac{1}{a_1'} + \frac{1}{b_1'}.$$

Thus either $a_1' \geq 2$ or $a_1' \geq 2$, which gives $a_1 \geq 2$ or $b_1 \geq 2$. $\square$

Lemma 5.2.12. *Let* $\frac{a}{b} = [a_0, a_1, \ldots, a_N], -\frac{a}{b} = [-a_0 - 1, 1, a_1 - 1, a_2, \ldots, a_N]$ *and let*

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n], \quad 0 \leq n \leq N;$$
$$\frac{r_m}{s_m} = [-a_0 - 1, 1, a_1 - 1, a_2, \ldots, a_{m-1}], \quad 2 \leq m \leq N + 1.$$

Then

$$s_n = q_{n-1}, r_n = -p_{n-1}, \quad 1 \leq n \leq N + 1$$

Proof. For $n = 1$ we have $\frac{r_1}{s_1} = [-a_0 - 1, 1] = -a_0$, so $r_1 = -a_0 = -p_0$ and $s_1 = 1 = q_0$. If $n = 2$ then $\frac{r_2}{s_2} = [-a_0 - 1, 1, a_1 - 1] = \frac{-a_0 a_1 - 1}{a_1}$, so $r_1 = -a_0 a_1 - 1 = -p_1$ and $s_2 = a_1 = q_1$. For $3 \leq n \leq N + 1$ the nth quotient of $-\frac{a}{b}$ is the $(n-1)$th quotient of $\frac{a}{b}$. By the recursive formula (5.2), the statement follows. $\square$

Corollary 5.2.13. *By relabeling if necessary, we have*

$$\overline{C_{\frac{a}{b}}(I)} = \overline{\begin{bmatrix} -q_{N-1} & p_{N-1} \\ -b & a \end{bmatrix}}, \quad \overline{C_{-\frac{a}{b}}(I)} = \overline{\begin{bmatrix} q_{N-1} & p_{N-1} \\ -b & -a \end{bmatrix}} \quad N \text{ even}$$

$$\overline{C_{\frac{a}{b}}(I)} = \overline{\begin{bmatrix} q_{N-1} & -p_{N-1} \\ -b & a \end{bmatrix}}, \quad \overline{C_{-\frac{a}{b}}(I)} = \overline{\begin{bmatrix} q_{N-1} & p_{N-1} \\ b & a \end{bmatrix}} \quad N \text{ odd}$$

In all cases, $\overline{C_{\frac{a}{b}}(I)}$ *and* $\overline{C_{-\frac{a}{b}}(I)}$ *differ by a factor of* -1 *in the diagonal.*

Proof. By the topological constraint $b \geq 2$, the number of quotients N in any of these continued fractions is at least one. Up to relabeling, the previous results show that

$$\frac{a}{b} = [a_0, a_1, \ldots, a_N], -\frac{a}{b} = [-a_0 - 1, 1, a_1 - 1, a_2, \ldots, a_N].$$

Let p_n, q_n, r_n, s_n be as in Lemma 5.2.12, so $s_n = q_{n-1}, r_n = -p_{n-1}$, $1 \leq n \leq N+1$. If N is even then $N+1$ is odd and viceversa. Assume first that N is even. Corollary 5.2.8 gives

$$\overline{C_{\frac{a}{b}}(I)} = \overline{\begin{bmatrix} -q_{N-1} & p_{N-1} \\ -b & a \end{bmatrix}}, \quad \overline{C_{-\frac{a}{b}}(I)} = \overline{\begin{bmatrix} q_{N-1} & p_{N-1} \\ -b & -a \end{bmatrix}}.$$

And if N is odd,

$$\overline{C_{\frac{a}{b}}(I)} = \overline{\begin{bmatrix} q_{N-1} & -p_{N-1} \\ -b & a \end{bmatrix}}, \quad \overline{C_{-\frac{a}{b}}(I)} = \overline{\begin{bmatrix} -q_{N-1} & -p_{N-1} \\ -b & -a \end{bmatrix}} = \overline{\begin{bmatrix} q_{N-1} & p_{N-1} \\ b & a \end{bmatrix}}. \quad \square$$

Summarizing, let $\overline{M} = \overline{\begin{bmatrix} c & e \\ d & f \end{bmatrix}} \overline{M^*} = \overline{\begin{bmatrix} -c & e \\ d & -f \end{bmatrix}} \in \mathrm{PSL}_2(\mathbb{Z})$. By Corollary 5.2.8, if $\overline{M}$ represents some knot $K(a, b, n)$ then $a = \pm d, b = \mp f$. Since b must be positive, the fraction $\frac{a}{b}$ is completely determined, so we just need to check the even and odd cases of $\overline{C_{\frac{a}{b}}(I)}$ and compare them with $\overline{M}$. By the previous corollary, $\overline{M}$ represents the knot $K(a, b, n)$ iff $\overline{M^*}$ represents the knot $K(-a, b, n)$.

Chapter 6

Future Work

The final part of the previous section showed us that only certain elements in $\mathrm{PSL}_2(\mathbb{Z})$ yield knots $K(a, b, n)$ with two twice-punctured tori with boundary slopes at distance 3 in their exterior. It remains to determine in which cases such tori are essential in both the white and black sides and finally, which knots in the family $K(a, b, n)$ are hyperbolic. If the knot $K(a, b, 0)$ is trivial in S^3 then the work of Motegi et al. [1] shows that $K(a, b, n)$, obtained by twisting the solid torus V_{γ_0} $|n| \geq 2$ times has positive Gromov volume so $K(a, b, n)$ is either hyperbolic or a satellite knot. Further work must be done to discard that $K(a, b, n)$ is satellite. If $K(a, b, 0)$ is nontrivial then a topological analysis may show that for a, b and n large enough the tori S, T are essential in X_K and $K(a, b, n)$ is hyperbolic, as desired.

The hypothesis of the knot embedded in S^3 was only used in two steps: when we showed that the torus $\mathcal{T}_i$ bounds a solid torus on one side or the other in Section 4.2 and when we embedded V_{γ_0} in S^3 in a unique fashion (modulo full twists). It is possible to extend our results to knots in arbitrary hyperbolic manifolds, but this task may turn to be quite intricated.

The construction developed in chapters 4 and 5 are so general that they may be applied to the case $\Delta < 3$, whenever a spectrum gives a proper disk in one of the sides of an essential torus such that its boundary represents a commutator after, if necessary, removing suitable companion tori (see sections 4.2, 4.3). Finally, the case that one of the essential torus has more than two boundary components may be analized in a similar fashion, as long as the other essential torus has still two boundary components.

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Curriculum Vitae

César Garza was born on August 9, 1983 in Delicias, Mexico. The son of César Villalobos and María Guadalupe Garza, he graduated from the Instituto Tecnológico y de Estudios Superiores de Monterrey in the spring of 2001 and entered to College at the same University. While pursuing a bachelor's degree in mechatronics, he worked with Delphi Automotive Systems in 2005 and later worked with the company full-time as a software engineer after receiving his bachelor's of science degree from the Instituto Tecnológico y de Estudios Superiores de Monterrey in 2005.

In the spring of 2007, he entered the Graduate School of The University of Texas at El Paso. While pursuing a master's degree in Mathematical Sciences he worked as a Teaching Assistant and, in the final semester, as a Lecturer. He was a guest speaker during the 17th Annual Meeting of the South Texas Mathematics Consortium in 2009 in Edinburg, Texas. He is a member of the AMS and UTEP's College of Science Graduate Students Council. In March 2009 he received a fellowship from The University of Texas at Austin to enter in their Ph.D. program in Math.

Permanent address: Órix Norte, 6619, Fracc. Santa Teresa, 32606.
Ciudad Juárez, Chihuahua. Mexico.