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Exponential Dichotomy Of Ode's

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To my daughter

Angela Al-Hanna

with deepest love and adoration

EXPONENTIAL DICHOTOMY OF ODE'S

by

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Chapter 1

Introduction

In this work, we explore the notion of exponential dichotomy of ordinary differential equations on arbitrary Banach spaces. We refer the reader to the body of the dissertation for the main definitions and terminology. We pay particular attention to the stability of exponential dichotomy, more precisely, to the invariance of exponential dichotomy under small L^∞ perturbation of the underlying differential equation (See chapter 5). Our main goal is to demonstrate that under the assumption of exponential growth, exponential dichotomy of the ordinary differential equation

$$x^{'} = A(t)x, \tag{1.1}$$

is equavalent to the invertibility of the unbounded operator L on $C(\mathbb{R}, X)$. Where $C(\mathbb{R}, X)$ is the space of all bounded linear operators from $\mathbb{R} \to X$.

$$L : D = \left\{ x \in C(\mathbb{R}, X) : \frac{d}{dt}x - A(t)x \in C(\mathbb{R}, X) \right\},$$

defined by

$$Lx(t) = \frac{d}{dt}x(t) - A(t)x(t).$$

The body of my thesis will proceed as follows: In chapter two, I present the existence and uniqueness of ordinary differential equations both in the general and linear cases. Moreover, in chapter three, I introduce the Cauchy operator, and show that the Cauchy operator is linear, bounded, and invertible. Then, I define exponential dichotomy and introduce an example of a differential equation that is exponentially dichotomic versus another differential equation that is not exponentially dichotomic. Then, I move on to

chapter four to study the functional analytic characterization of exponential dichotomy. I introduce the main result and prove this result with the aid of some lemmas which will also be proved. Finally, in chapter five, I study the roughness of exponential dichotomy. Moreover, I show the invariance of exponential dichotomy under small L^∞ perturbation.

Chapter 2

Ordinary Differential Equations on a Banach Space

In this section, we present well known theorems on existence and uniqueness of solutions of differential equations on a Banach space X. To this end, we consider the initial value problem:

$$\begin{aligned} \frac{dx}{dt} &= f(t,x) \\ x(t_0) &= x_0, \end{aligned} \tag{2.1}$$

where $f(t,x)$ is a X valued function of variables $t \in \mathbb{R}$, $x \in X$ and we assume that $f : (a,b) \times X \to X$ is continuous.

2.1 Existence and Uniqueness of Ordinary Differential Equations

2.1.1 The General Case

Existence

The existence of solutions of equation (2.1), is a consequence of Banach's contraction principle. More precisely:

Theorem 2.1.1. *Suppose that there exists a neighborhood of a point* (t_0, x_0) *in which the*

function $f(t,x)$ is continuous and satisfies the Lipschitz condition:

$$\|f(t,x_2) - f(t,x_1)\| \leq M\|x_2 - x_1\|, \tag{2.2}$$

for some positive constant M. Then there exists a neighborhood of t_0 in which equation(2.1) has a unique solution $x = \phi(t)$ satisfying the initial condition $\phi(t_0) = x_0$.

Proof. We can see that there exists positive constants ε and η such that when $|t - t_0| \leq \varepsilon$ and $\|x - x_0\| \leq \eta$, the function $f(t,x)$ is continuous, satisfies (2.2), and bounded:

$$\|f(t,x)\| \leq M_1 < \infty. \tag{2.3}$$

let $\delta = min(\varepsilon, \frac{\eta}{M_1})$ and let $C_\delta(B)$ denote the Banach space of continuous functions $x(t)$ that are defined for $|t - t_0| \leq \delta$. Take their values in X, we will have the norm

$$|||x||| = \sup_{|t-t_0|\leq\delta} \|x(t)\|.$$

We consider in this space the closed ball $B_\eta(x_0) = \{x \in C_\delta / \|x - x_0\| \leq \eta\}$.

Equation (2.3) implies that the operator $(Tx)(t) = x_0 + \displaystyle\int_{t_0}^{t} f(\tau, x(\tau))d\tau$ maps B_η into itself, since $\|(Tx)(t) - x_0\| \leq \delta M_1 \leq \eta$, for $x(t) \in B_\eta$. When $x_1, x_2 \in B_\eta$, equation (2.2) implies the estimates:

$$\begin{aligned} \|(Tx_2)(t) - (Tx_1)(t)\| &\leq \int_{t_0}^{t} \|f(\tau, x_2(\tau)) - f(\tau, x_1(\tau))\| \, d\tau \\ &\leq M \int_{t_0}^{t} \|x_2(\tau) - x_1(\tau)\| \, d\tau \\ &\leq M(t - t_0) \, ||| \, x_2 - x_1 |||, \end{aligned}$$

which implies the estimates

$$\begin{aligned} \|(T^2x_2)(t) - (T^2x_1)(t)\| &\leq M \int_{t_0}^{t} \|Tx_2(\tau) - (Tx_1)(\tau)\| \, d\tau \\ &\leq M^2 \, ||| \, x_2 - x_1 \, ||| \int_{t_0}^{t} (\tau - t_0) \, d\tau \\ &= \frac{[M(t - t_0)]^2}{2!} \, ||| \, x_2 - x_1 \, ||| \, . \end{aligned}$$

From which we can obtain that

$$\|(T^n x_2)(t) - (T^n x_1)(t)\| \le \frac{[M(t-t_0)]^n}{n!} \; ||| \, x_2 - x_1 \, ||| \, .$$

Where n is a natural number.

Hence,

$$|||T^n x_2 - T^n x_1||| \le \frac{(\delta M)^n}{n!} \; ||| \, x_2 - x_1 \, ||| \, .$$

It follows that the operator T^n is a contraction in B_η for sufficiently large n. Therefore, by the contraction principle, there exists a solution of equation 2.1. See [1] □

Uniqueness

We see that by the contraction principle, there exists a solution and it is unique. Thus, $x(t) \in B_\eta$ is a unique solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau)) d\tau$$

Which is equivalent to equation(2.1). Therefore, for $x(t) \in B_\eta$ the theorem is proved.

2.2 The Linear Case

In the case of a linear differential equation, the existence theorem takes up a particularly interesting form. Consider the following linear differential equation on the interval $\mathcal{I}$

$$\frac{dx}{dt} = A(t)x + f(t) \tag{2.4}$$

where $A : \mathbb{R} \to C(X)$ is strongly continuous ($C(X)$ is the space of all bounded linear operators $A : X \to X$) and $f : \mathbb{R} \to X$. We present equations whose coefficients are strongly measurable and locally Bochner integrable. We recall some definitions and facts that are connected to these notions.

Definition 2.2.1. A vector function $x(t)$ on the interval $I = [a, b]$ with values in a Banach space X is said to be countably valued if it takes on $[a, b]$ no more than countable number of nonzero values $x_k, (k = 1, 2, ...)$, where the sets $E_k = \{t/\, x(t) = x_k, (k = 1, 2, ...)\}$ are Lebesgue integrable. See [1].

Definition 2.2.2. A countably valued function is Bochner integrable on $[a, b]$ if and only if the numerical function$\|x(t)\|$ is Lebesgue integrable on [a,b].See [1]

Definition 2.2.3. The Bochner integral of a countably valued function $x(t)$ is defined by:

$$\int_a^b x(t)dt = \sum_{k=1}^{\infty} x_k\, mE_k$$

Where mE_k is the Lebesgue measure of the set E_k. See [1].

Definition 2.2.4. A vector function $x(t)$ is said to be be strongly measurable on $[a, b]$, if for any almost everywhere convergent sequence of countably valued functions $x_n(t)$, we have

$$\lim_{t\in[a,b]} x_n(t) = x(t).$$

Moreover, if the function $x(t)$ is strongly measurable, the function $\|x(t)\|$ is Lebesgue measurable. If $\|x(t)\|$ is also integrable, $x(t)$ is said to be Bochner integrable (strongly integrable) and by definition

$$\int_a^b x(t)\, dt = \lim_{n\to\infty} \int_a^b x_n(t)\, dt.$$

We underline the equation relation

$$A \int_a^b x(t)\, dt = \int_a^b Ax(t)\, dt.$$

By considering in place of the space X, the space $C(X)$, we can transfer all the above definitions and notions to operator functions. In this connection the product of two integrable functions (operator functions or an operator function and a vector function) of which one is bounded is also an integrable function. If $x(t)$ is Bochner integrable, the equality

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} \|x(\tau) - x(t)\| d\tau = 0$$

is valid for almost all values $t \in [a, b]$ and, in particular, the function

$$y(t) = \int_a^t x(\tau) d\tau$$

is continuous and almost everywhere differentiable in [a,b]. See [1]

Existence

We consider the following differential equation on the interval $\mathcal{I}$

$$\frac{dx}{dt} = A(t)x + f(t), \quad t \in \mathcal{I} \tag{2.5}$$

in a Banach space X. We assume that the functions $f(t)$ and $A(t)$ with values in X and $C(X)$, respectively, are strongly measurable and Bochner integrable on subintervals of $\mathcal{I}$.

Definition 2.2.5. A solution of the equation (2.5) is a continuous function $x(t)$ which is differentiable and satisfies (2.5) almost everywhere.

Thus, a solution of the integral equation

$$x(t) = x_0 + \int_{t_o}^t A(\tau)x(\tau) d\tau + \int_{t_0}^t f(\tau) d\tau,$$

where $x_0 = x(t_0)$, is by definition a solution of equation (2.5). If $f(t)$ is continuous and $A(t)$ is strongly continuous, a solution of equation (2.2) is continuously differentiable at each point $t \in \mathcal{I}$ and equation (2.5) is satisfied for all $t \in \mathcal{I}$. We consider instead of (1.1), the more general equation

$$x(t) = g(t) + \int_{t_0}^t A(\tau) d\tau$$

with a continuous vector function $g(t)$ on $\mathcal{I}$ and we need to show that it has a continuous solution on any interval $[a, b] \subset \mathcal{I}$. Let $C([a, b], X)$ be the Banach space of continuous functions on $[a, b]$ with values in the Banach space X and a norm defined as follows:

$$|||x||| = \max_{t \in [a,b]} \|x(t)\| \tag{2.6}$$

Now we are going to consider the operator:

$$(Sx)(t) = g(t) + \int_{t_0}^{t} A(\tau)x(\tau)d\tau, \tag{2.7}$$

notice that $S : C(X) \to C(X)$ is continuous.

For $n \in N$, we can use induction to verify the formula:

$$\begin{aligned}(S^n x)(t) = \quad & g(t) + \int_{t_0}^{t} A(t_1)g(t_1)dt_1 \\ & + \int_{t_0}^{t}\int_{t_0}^{t_2} A(t_2)A(t_1)g(t_1)dt_1dt_2 + ... \\ & + \int_{t_0}^{t}\int_{t_0}^{t_{n-1}} ... \int_{t_0}^{t_2} A(t_{n-1})...A(t_1)g(t_1)dt_1...dt_{n-1} + .. \\ & + \int_{t_0}^{t}\int_{t_0}^{t_n} ... \int_{t_0}^{t_2} A(t_n)A(t_{n-1})...A(t_1)x(t_1)dt_1..dt_n.\end{aligned}$$

Which, in turn, implies the equation:

$$(S^n x_2)(t) - (S^n x_1)(t) = \int_{t_0}^{t}\int_{t_0}^{t_n} ... \int_{t_0}^{t_2} A(t_n)A(t_{n-1})...A(t_1)[x_2(t) - x_1(t)dt_1..dt_n.$$

We also get the estimate,

$$\begin{aligned}&\|(S^n x_2)(t) - (S^n x_1)(t)\| \\ &\qquad \leq ||| \, x_2 - x_1 \, ||| \int_{t_0}^{t}\int_{t_0}^{t_n} ... \int_{t_0}^{t_2} \|A(t_n)\|\|A(t_{n-1}).....\|A(t_1)\|dt_1....dt_n\end{aligned}$$

Hence, we get the equality:

$$\begin{aligned}&\int_{t_0}^{t}\int_{t_0}^{t_n} ... \int_{t_0}^{t_2} \|A(t_n)\|\|A(t_{n-1}).....\|A(t_1)\|dt_1....dt_n \\ &\quad = \frac{1}{n!}\int_{t_0}^{t}\int_{t_0}^{t} ... \int_{t_0}^{t} \|A(t_n)\|\|A(t_{n-1}).....\|A(t_1)\|dt_1....dt_n \\ &\quad = \frac{1}{n!}\left[\int_{t_0}^{t} \|A(\tau)\|d\tau\right]^n .\end{aligned}$$

We finally obtain the estimate :

$$|||S^n x_2 - S^n x_1||| \leq \frac{1}{n!}\left[\int_{a}^{b} \|A(\tau)\|d\tau\right]^n ||| \, x_2 - x_1 \, ||| \, .$$

This show that when n is sufficiently large , S^n is a contraction operator in $C(X)$. So,equation

$$x(t) = g(t) + \int_{t_0}^{t} A(\tau)d\tau,$$

has a unique continuous solution. This solution can be obtained from the relation

$$x(t) = \lim_{n\to\infty} S^n x_0(t),$$

for any $x_0(t) \in C(X)$ and it can be represented by the series

$$\begin{aligned} x(t) = \quad & g(t) + \int_{t_0}^{t} A(t_1)g(t_1)dt_1 \\ & + \sum_{n=2}^{\infty} \int_{t_0}^{t} \int_{t_0}^{t_n} ... \int_{t_0}^{t_2} A(t_n)A(t_{n-1}).....A(t_1)g(t_1)dt_1....dt_n \\ = \quad & g(t) + \sum_{k=1}^{\infty} g_k(t), \end{aligned}$$

where

$$g_k(t) = \int_{t_0}^{t} A(\tau)g_{k-1}(\tau)d\tau \quad g(t_0) = g(t).$$

This series is majorized in norm by the series:

$$|||g\ ||| \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\int_{t_0}^{t} \|A(\tau)\|d\tau \right]^n \right\},$$

which implies

$$|||x||| \leq |||g\ ||| \exp\left\{ \int_{a}^{b} \|A(\tau)\|d\tau \right\}. \tag{2.8}$$

We consider in particular the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} A(\tau)x(\tau)d\tau,$$

which is equivalent to the equation (1.1), (See [1]).

Uniqueness

Since we showed that S is a contraction in $C(X)$, then by Banach contraction principle we conclude that equation (2.5) has a unique solution. The material presented in this chapter is essentially from [1].

Chapter 3

Exponential Dichotomy

In this chapter, we present the Cauchy opreator of equation(1.1) along with the definition of exponential dichotomy, introduce an example of an ordinary differential equation which is exponentially dichotomic and an example of an ordinary differential equation which is not exponentially dichotomic.

3.1 Cauchy operator

We define the Cauchy operator $U(t)$ associated to the problem

$$\frac{dx}{dt} = A(t)x \tag{3.1}$$

as follows:

$$U(t) : X \to X;$$

$$U(t)(x) = x(t),$$

where $x(t)$ is the unique solution to the IVP :

$$\begin{aligned} \frac{dx}{dt} &= A(t)x \\ x(0) &= x_0. \end{aligned} \tag{3.2}$$

We next prove the following proposition:

Proposition 3.1.1. *The Cauchy operator $U(t)$ is linear, invertible, and bounded.*

Proof. To prove that $U(t)$ is linear, take $x \in X$, $y \in X$ then

$$U(t)(x+y)$$

is the unique solution of (3.2). Consider z such that $z(0) = x+y$ and let $x(t)$ be the unique solution to (3.2) with $x(0) = x$, and $y(t)$ be the solution to (3.2) with $y(0) = y$. Then,

$$x(t) + y(t) = z(t)$$

is a solution of (3.2) and

$$z(0) = x(0) + y(0) = x + y.$$

Hence,

$$U(t)(x+y) = z(t) = x(t) + y(t) = U(t)(x) + U(t)(y)$$

Now, we need to show that

$$U(t)(rx) = rU(t)x,$$

for any $r \in \mathbb{R}$ and for any $x \in X$. $U(t)(x)$ is the unique solution of (3.2). Consider z such that

$$z(0) = rx$$

and let $x(t)$ be the unique solution of (3.2) with $x(0) = x$. Then, $z(t) = rx(t)$ is a solution of (3.2) and $z(0) = rx(0) = rx(t)$. So,

$$U(t)(rx) = z(t) = rx(t) = rU(t)x.$$

Thus, $U(t)$ is linear. Moreover, to show that $U(t)$ is invertible, we should show that $U(t)$ is both surjective and injective. First, to show that $U(t)$ is injective, let $U(t)(x) = 0$ then the unique solution to (3.2) such that $x(0) = x$ is 0 at t. (Since $x(t) = 0$). Now, $y(s) = x(s+t)$, $y(0) = x(t) = 0$

$$\frac{dy}{dt}(s) = A(s+t)y(s)$$
$$y(0) = 0.$$

So, $y(s) = 0$ for all s and $y(-t) = x(0) = x = 0$. Thus, $U(t)$ is injective.

Second, to show that $U(t)$ is surjective, take $x \in X$ and look for $y \in X$ such that $U(t)(y) = x$. So, we have

$$\frac{dy}{dt} = A(t)y$$
$$y(0) = y,$$

and

$$\frac{dz}{dt} = A(t+s)z$$
$$z(0) = x.$$

Let $y(s) = z(s-t)$. then,

$$y^{'}(s) = z^{'}(s-t) = A(s)z(s-t) = A(s)y(s)$$
$$y(t) = z(0) = x.$$

So, $U(t)(y(0)) = y(t) = x$, which implies that $U(t)$ is surjective. Therefore, $U(t)$ is invertible. Now, we need to show that $U(t)$ is bounded. Boundedness follows from the existence of the solution of the linear differential equation (3.2). We have $x(t) = U(t)x_0$. So, from equation (2.8), we can conclude that:

$$\|U(t)\| \leq e^{\int_{t_0}^{t} \|A(\tau)\| d\tau},$$

which implies that $U(t)$ is bounded. This concludes the proof of the proposition. □

Definition 3.1.1. Let X be a Banach space and let I be an interval on $\mathbb{R}$. Consider $C(X)$ and $A : I \to C(X)$ be a strongly continuous function. Consider the differential equation

$$\frac{dx}{dt} = A(t)x. \tag{3.3}$$

Let $U(t)$ be the Cauchy operator corresponding to equation (3.3) such that $U(t)(x) = x(t)$. Where $x(t_0)$ is the unique solution for equation (3.2). We say that the equation (3.2) has

exponential dichotomy iff there exist two bounded projections P and Q with $P+Q=I$ and positive constants N_i,υ_i, $i=1,2$ for which the following estimates hold for any $t,s\in I$

$$\|U(t)PU^{-1}(s)\| \leq N_1 e^{-\upsilon_1(t-s)} \qquad \text{if} \quad t \geq s \tag{3.4}$$

$$\|U(t)QU^{-1}(s)\| \leq N_2 e^{-\upsilon_2(s-t)} \qquad \text{if} \quad t \leq s. \tag{3.5}$$

Let us assume that a differential equation has exponential dichotomy, then for $t\geq s$ one has:

$$\begin{aligned} U(t)Px_0 &= U(t)PU^{-1}(s)U(s)Px_0 \\ \|U(t)Px_0\| &\leq \|U(t)PU^{-1}(s)\|\|U(s)Px_0\|. \end{aligned}$$

In particular for $s=0$, we have:

$$\|U(t)Px_0\| \leq ce^{-\alpha t}\left\|Px_0\right\|.$$

Hence, $\lim_{t\to\infty}\|U(t)Px_0\|=0$.

Also for $s\leq t$, one has:

$$\begin{aligned} U(t)Px_0 &= U(t)PU^{-1}(s)U(s)Px_0 \\ \|U(t)Px_0\| &\leq \|U(t)PU^{-1}(s)\|\|U(s)Px_0\| \\ \|U(t)Px_0\| &\leq ce^{-\alpha(t-s)}\left\|U(s)Px_0\right\|. \end{aligned}$$

For $t=0, s\leq t$, we have:

$$\|Px_0\|\frac{1}{c}e^{(-\alpha s)} \leq \left\|U(s)Px_0\right\|,$$

so that $\lim_{s\to-\infty}\|U(s)Px_0\| = \infty$. We also have for $t \le s$

$$\begin{aligned}
U(t)(I-P)x_0 &= U(t)(I-P)U^{-1}(s)U(s)(I-P)x_0 \\
\|U(t)(I-P)x_0\| &\le \|U(t)(I-P)U^{-1}(s)\|\|U(s)(I-P)x_0\| \\
\|U(0)(I-P)x_0\| &\le ce^{-\alpha(s-0)}\|U(s)(I-P)x_0\| \\
\frac{1}{c}\|(I-P)x_0\|e^{\alpha s} &\le \|U(s)(I-P)x_0\|.
\end{aligned}$$

Hence,

$$\lim_{s\to\infty}\|U(s)(I-P)x_0\| = \infty.$$

Also for $t \le s$

$$\begin{aligned}
U(t)(I-P)x_0 &= U(t)(I-P)U^{-1}(s)U(s)(I-P)x_0 \\
\|U(t)(I-P)x_0\| &\le \|U(t)(I-P)U^{-1}(s)\|\|U(s)(I-P)x_0\| \\
\|U(t)(I-P)x_0\| &\le ce^{\alpha(t)}\|(I-P)x_0\| \\
\frac{1}{c}\|U(t)(I-P)x_0\|e^{-\alpha t} &\le \|(I-P)x_0\|.
\end{aligned}$$

$\lim_{t\to-\infty}\|U(t)(I-P)x_0\| = 0$.

To see what dichotomy means in the general case, it is convenient to rewrite (3.4) and (3.5) in the equivalent form:

$$\begin{aligned}
&\|U(t)P\xi\| &&\le ke^{-\alpha(t-s)}\|U(s)P\xi\| &&\text{for} \quad t \ge s \\
&\|U(t)(I-P)\xi\| &&\le Le^{-\beta(s-t)}\|U(s)(I-P)\xi\| &&\text{for} \quad s \ge t \\
&\|U(t)Px^{-1}(t)\| &&\le M &&\text{for all} \quad t,
\end{aligned}$$

where k, l, m are positive constants and ξ is an arbitrary constant vector. Let us show how

we obtain these equations from the definition of exponential dichotomy.

$$\begin{aligned}\|U(t)P\xi\| &= \|U(t)PU(s)U^{-1}(s)P\xi\| \\ &\leq \|U(t)PU^{-1}(s)\|\|U(s)P\xi\| \\ &\leq ke^{-\alpha(t-s)}\|U(s)P\xi\|.\end{aligned}$$

3.2 Examples

We now pause to present some examples of systems with and without exponential dichotomy. We will consider elementary differential equations on $X = \mathbb{R}$.

Example 3.2.1. Consider the differential equation on $\mathbb{R}$

$$\frac{dx}{dt} = x. \tag{3.6}$$

We will show that equation (3.6) has an exponential dichotomy with projection $P = 0$. The general solution of this equation is

$$\begin{aligned}x(t) &= ce^t \\ x(0) &= c,\end{aligned}$$

with its Cauchy operator:

$$\begin{aligned}U(t)(x_0) &= x_0e^t \\ U^{-1}(t)(x_0) &= x_0e^{-t}.\end{aligned}$$

Now, we will show that both of the inequalities (3.4) and (3.5) hold for $N_1 = 0, \upsilon_1 = 1, N_2 = 1, \upsilon_2 = -1$ To this effect, observe that for $x \in B$, one has

$$\begin{aligned}(U(t)PU^{-1}(s))(x) &= U(t)P(xe^{-s}) \\ &= U(t)(0) \\ &= 0.\end{aligned}$$

Therefore, inequality (3.4) is true for $N_1 = 0, \upsilon_1 = 1$. To see that inequality (3.5) holds, we set

$$\begin{aligned}(U(t)(I-P)U^{-1}(s))(x) &= U(t)(I-P)(xe^{-s}) \\ &= U(t)(xe^{-s}) \\ &= xe^{-s}e^{t} \\ &= xe^{t-S},\end{aligned} \tag{3.7}$$

which implies,

$$\begin{aligned}\|xe^{t-s}\| &= |e^{t-s}|\|x\| \\ &\leq e^{t-s},\end{aligned} \tag{3.8}$$

since $\|x\| \leq 1$. So, equation (3.5) is satisfied. Thus, the differential equation (3.6) is exponentially dichotomic.

Example 3.2.2. Consider the differential equation on $\mathbb{R}$

$$\frac{dx}{dt} = ix. \tag{3.9}$$

The general solution to this equation is $x(t) = ce^{it}$. We will show that equation (3.9) does not have an exponential dichotomy. Let $x_0 \in Px$ then $x_0 = Px_0$. If $x_0 \neq 0$

$$U(t)(x_0) = x(t) = x_0e^{it}.$$

Then,

$$|x(t)| = |x_0||\cos t + i\sin t| = |x_0|,$$

so, we have: $|x(t)| \nrightarrow 0$ as $t \to \infty$ and $|x(t)| \nrightarrow \infty$ as $t \to -\infty$. Thus, equation (3.9) does not have exponential dichotomy.

Chapter 4

Functional Analytic Characterization of Exponential Dichotomy

In this chapter, we explore some functional-analytic aspects of exponential dichotomies. We recall that given a closed unbounded linear operator T with domain $D(T)$ on a Banach space X, $D(T)$ becomes a Banach space when furnished with the norm $\|x\| = \|x\| + \|T(x)\|$ and $T : D(T) \to T$ defines a bounded linear operator. We now introduce the Banach space $C(\mathbb{R}, X)$ consisting of all bounded, continuous functions $x : \mathbb{R} \to X$ with the supremum norm $\|x\| = \max_{t\in\mathbb{R}} \|x(t)\|$ and we consider the (unbounded) operator

$$L : D = \left\{ x \in C(\mathbb{R}, X) : \frac{d}{dt}x - A(t)x \in C(\mathbb{R}, X) \right\},$$

defined by

$$Lx(t) = \frac{d}{dt}x(t) - A(t)x(t).$$

Our main result establishes that under the assumption of exponential growth, exponential dichotomy of the differential equation (1.1) is equivalent to the invertibiity of L. More precisely, we have the following theorem:

Theorem 4.1. *Let X and $A(t)$ be as specified in chapter(1). Assume there exists a positive constant c such that for any solution x of (1.1) the estimate*

$$\left\| U(t)U^{-1}(t_0) \right\| \leq \alpha e^{\beta|t-t_0|}, \tag{4.1}$$

holds for $t, t_0 \in \mathbb{R}$. Then the differential equation (1.1) has exponential dichotomy if and only if the operator L is invertible.

We start by presenting two technical lemmas:

Lemma 4.2. *If*

$$L : \mathcal{D}(L) \to C(\mathbb{R}, X) \tag{4.2}$$

has a bounded inverse, then there exists positive constants C_i, α_i $i = 1, 2$ depending only on the operator norm of L^{-1} and α, β in (4.1) such any solution u of (1.1) for which

$$\|u\|_{L^\infty([0,\infty)),X)} < \infty \tag{4.3}$$

satisfies the estimate

$$\|u(t)\| \leq C_1 e^{-\alpha_1(t-s)} \|u(s)\| \text{ for } t \geq s \geq 0 \tag{4.4}$$

and that any solution u of (1.1) for which

$$\|u\|_{L^\infty((-\infty,0],X)} < \infty \tag{4.5}$$

satisfies the estimate

$$\|u(t)\| \leq C_2 e^{-\alpha_2(s-t)} \|u(s)\| \text{ for } t \leq s \leq 0. \tag{4.6}$$

Proof. Let $s \leq 0$, consider a cut-off function $\psi \in C_0^\infty(\mathbb{R})$ supported on $(-\infty, s]$ for which $\psi \equiv 1$ on $(-\infty, -s-1]$, $0 \leq \psi \leq 1$ and $\|\psi'\|_{L^\infty} \leq 2$. If u is a solution of (1.1) then $\psi u \in \mathcal{D}(L)$ and $L(\psi u) = \psi' u$ if in addition u is bounded on the left semi-axis (i.e, satisfying the condition (4.5)), then by virtue of the invertibility of L, one has

$$\begin{aligned}
\|u\|_{L^\infty((-\infty,s-1])} &\leq \|\psi u\|_{L^\infty((-\infty,s-1])} \\
&\leq \|\psi u\|_{L^\infty(\mathbb{R})} \\
&\leq \|L^{-1}\| \|\psi' u\|_{L^\infty(\mathbb{R})} \\
&\leq 2\|L^{-1}\| \|U(t)U^{-1}(s)u(s)\|_{L^\infty(\mathbb{R})} \\
&\leq 2\alpha e^\beta \|L^{-1}\| \|u(s)\|_X,
\end{aligned}$$

which in conjunction with the estimate

$$\sup_{t\in[s-1,s]} \|u(t)\| = \sup_{t\in[s-1,s]} \|U(t)U^{-1}(s)u(s)\| \leq \alpha e^{\beta}\|u(s)\|$$

yields the inequality

$$\|u(t)\| \leq 2\alpha e^{\beta} \max\{1, \|L^{-1}\|\}\|u(s)\| \quad \text{for} \quad t \leq s \leq 0. \tag{4.7}$$

In identical fashion it can be established that any solution of (1.1) which is bounded on $[0,\infty)$ satisfies the inequality

$$\|u(t)\| \leq 2\alpha e^{\beta} \max\{1, \|L^{-1}\|\}\|u(s)\| \quad \text{for} \quad t \geq s \geq 0. \tag{4.8}$$

Next, we consider an arbitrary interval $[a,b] \subset (-\infty,0]$, an arbitrary solution of (1.1) satisfying (4.7) subject to $\|w(a)\| \geq \frac{1}{2}$ and $\|w(b)\| \leq 1$. Then a-fortiori, for $t \in [a,b]$ and $C = 2\alpha e^{\beta} \max\{1, \|L^{-1}\|\}$, one has the inequality

$$\frac{1}{2C} \leq \|w(t)\| \leq C. \tag{4.9}$$

Take $\epsilon > 0$ and a cut-off function ψ supported on $[a,b]$, equal to 1 on $[a+\epsilon, b-\epsilon]$, $|\psi| \leq 1$; set

$$g(t) = \psi(t)w(t)\|w(t)\|^{-1}\ ,\ u(t) = w(t)\int_{-\infty}^{t} \psi(s)\|w(s)\|^{-1}\,ds.$$

Elementary calculations show that $Lu = g$ and it follows from the assumption on L that

$$\frac{1}{2c^2}(b-a-2\epsilon) \leq \|u\|_{L^\infty(\mathbb{R})} \leq \|L^{-1}\|\|g\|_{L^\infty(\mathbb{R})} \leq \left\|L^{-1}\right\|. \tag{4.10}$$

We conclude that if u is any solution of (1.1) which is bounded on the left semi-axis and $t > N > 2C^2\|L^{-1}\|$, then the inequality

$$\|u(s-t)\| \leq \frac{1}{2}\|u(s)\| \tag{4.11}$$

holds for any $s \leq 0$. Next, let $t \leq s \leq 0$, set

$$-n-1 = \sup\{i \in \mathbb{Z} : iN < t-s\};$$

it follows then from (4.11) that for some constant $0 < \epsilon < T$ depending only on $\|L^{-1}\|$ and the exponential growth, one has

$$\|u(t)\| = \|u(t-s+s)\| \leq \frac{1}{2^n}\|u(s)\| = e^{\ln 2(-n)}\|u(s)\| \tag{4.12}$$

$$= e^{\ln 2(\frac{t-s}{N}+\frac{\epsilon}{N})}\|u(s)\| \tag{4.13}$$

which proves (4.6). The proof of (4.4) is identical and will be omitted. Lemma (4.2) is proved. Next, we set

$$X_1 = \{x \in X : \exists v : v^{'} = A(t)v\,,\, v(0) = x \;\text{ and }\; \|v\|_{L^\infty(([0,\infty))} < \infty\}$$

and

$$X_2 = \{x \in X : \exists v : v^{'} = A(t)v\,,\, v(0) = x \;\text{ and }\; \|v\|_{L^\infty((-\infty,0])} < \infty\}.$$

□

Lemma 4.3. *Let X_1 and X_2 be the spaces defined above. Then*

$$X = X_1 \oplus X_2$$

algebraically and topologically. Moreover, the differential equation (1.1) is exponentially dichotomic with the projection on X_1

Proof. By virtue of (4.7) and (4.8), X_i, $i = 1, 2$ is a closed subspace of X. Let $\alpha \in C^\infty(\mathbb{R})$ with $\alpha \equiv 0$ on $(-\infty, 0]$, $\alpha \equiv 1$ on $[1,\infty)$ and $\|\alpha^{'}\|_{L^\infty(\mathbb{R})} < \infty$; for an arbitrary $x \in X$ let u be the unique solution of (1.1) with $u(0) = x$. Set $g = \alpha^{'}u$. Invoking again the invertibility of L we can find a unique $v \in \mathcal{D}(L)$ such that $L(v) = g$.
Writing

$$w = (1-\alpha)u + v,$$

it is immediate that

$$Lw = -\alpha^{'}u + (1-\alpha)u^{'} - (1-\alpha)Au + Lv = 0;$$

in addition,

$$\|w\|_{L^\infty([0,\infty)} \leq \|v\|_{L^\infty(\mathbb{R})} + \|u\|_{L^\infty([0,1])} < \infty.$$

We have proved that $w(0) \in X_1$ and since $v(0) \in X_2$, that $X = X_1 \oplus X_2$. Finally, due to the invertibility of L, if z is bounded on the real line and $Lz = 0$, then necessarily z is identically 0 on $\mathbb{R}$, so that the sum is direct as claimed. □

Lemma 4.4. *Under the assumption of exponential growth, we set*

$$P(t) = U(t)PU^{-1}(t)\,, \quad t \in \mathbb{R}$$

and show that

$$\sup_{t\in\mathbb{R}} \|P(t)\| = \sup_{t\in\mathbb{R}} \left\|U(t)PU^{-1}(t)\right\| < \infty. \tag{4.14}$$

Proof. For $x \in X$, denote the solutions of the initial value problems ($P(t) \neq 0$ and $Q(t) \neq 0$)

$$\begin{aligned} \frac{d}{ds}u &= A(s)u \\ u(t) &= \frac{P(t)x}{\|P(t)x\|} \end{aligned}$$

and

$$\begin{aligned} \frac{d}{ds}v &= A(s)v \\ v(t) &= \frac{Q(t)x}{\|Q(t)x\|} \\ x &= P(t)x + Q(t)x \end{aligned}$$

by u and v respectively and write

$$w(\xi) = \|P(t)x\|\, u(\xi) + \|Q(t)x\|\, v(\xi).$$

Hence, for $\xi \geq t$ and some $0 \leq \tilde{\alpha} < \alpha$, one has

$$\begin{aligned} C\,\|x\|\, e^{\tilde{\alpha}(\xi-t)} &\geq \|w(\xi)\| \\ C\,\|x\|\, e^{\tilde{\alpha}(\xi-t)} &\geq \|Q(t)x\|\,\frac{1}{C}e^{\alpha(\xi-t)} - \|P(t)\|\,Ce^{-\alpha(\xi-t)} \\ \|x\| &\geq \|Q(t)x\|\,C_1e^{(\alpha-\tilde{\alpha})(\xi-t)} - \|P(t)\|\,C_2e^{-(\alpha+\tilde{\alpha})(\xi-t)}. \end{aligned} \tag{4.15}$$

Recall that there exists a positive constant c such that for each $\xi \geq t$, the inequalities

$$\|u(\xi)\| \leq C \|u(t)\| e^{-\alpha(\xi-t)}$$
$$\|v(\xi)\| \geq \frac{1}{c} \|v(t)\| e^{\alpha(\xi-t)}$$

hold. Choose $\xi \geq \max \left\{ \frac{1}{\alpha} \ln(c) + t; t - \frac{1}{\alpha} \ln \left(\frac{\|Q(t)x\|}{2c \|P(t)x\|} \right) \right\}$
Then, (4.15) yields

$$\|Q(t)x\| \frac{1}{2} \leq c \|x\| e^{\alpha(\xi-t)},$$

from which (4.14) follows automatically. $\square$

Proof. For the necessity, take $f \in C(\mathbb{R}, X)$ and set $u(t) = \int_{-\infty}^{\infty} G(t,s)f(s)ds$, where G is the Green's function corresponding to (1.1) where G is defined as follows:

$$G(t,x) = \begin{cases} U(t)PU^{-1}(s) & \text{for} \quad s \leq t \\ -U(t)PU^{-1}(s) & \text{for} \quad s \geq t \end{cases}$$

take $f \in C(\mathbb{R}, X)$ and set

$$u(t) = \int_{-\infty}^{\infty} G(t,s)f(s)ds.$$

A straightforward calculation shows that u is strongly differentiable and satisfies $Lu = f$; using the bounds implied by the exponential dichotomy (3.4), it readily follows that

$$\|u\|_{L^\infty(\mathbb{R},X)} \leq \left(\frac{N_1}{\nu_1} + \frac{N_2}{\nu_2} \right) \|f\|_{L^\infty(\mathbb{R},X)}, \tag{4.16}$$

hence, u is bounded on $\mathbb{R}$.

To prove that u is indeed continuous, we write

$$\begin{aligned} & u(t+h) - u(t) \\ & = \int_{-\infty}^{t+h} U(t+h)PU^{-1}(s)f(s)ds - \int_{t+h}^{\infty} U(t+h)(I-P)U^{-1}(s)f(s)ds \\ & \quad + \int_{t}^{\infty} U(t)(I-P)U^{-1}(s)f(s)ds - \int_{-\infty}^{t} U(t)PU^{-1}(s)f(s)ds \end{aligned} \tag{4.17}$$

$$
\begin{aligned}
&= [U(t+h) - U(t)] \int_{-\infty}^{t+h} PU^{-1}(s)f(s)ds \\
&\quad - \int_{t+h}^{\infty} [U(t+h) - U(t)](I-P)U^{-1}(s)f(s)ds + \int_{-\infty}^{t+h} U(t)PU^{-1}(s)f(s)ds \\
&\quad - \int_{t+h}^{\infty} U(t)(I-P)U^{-1}(s)f(s)ds - \int_{-\infty}^{t} U(t)PU^{-1}(s)f(s)ds \\
&\quad + \int_{t}^{\infty} U(t)(I-P)U^{-1}(s)f(s)ds \\
&= [U(t+h) - U(t)] \int_{-\infty}^{t+h} PU^{-1}(s)f(s)ds - \int_{t+h}^{\infty} [U(t+h) - U(t)](I-P)U^{-1}(s)f(s)ds \\
&\quad + \int_{t}^{t+h} PU(t)U^{-1}(s)f(s)ds \quad + \int_{t}^{t+h} U(t)(I-P)U^{-1}(s)f(s)ds.
\end{aligned}
$$

It is easy to see that the norm of (4.17) is bounded above by

$$
\|U(t+h) - U(t)\| \left\|U^{-1}(t)\right\| \frac{c}{\alpha} \|f\| + \left\|U(t+h)U^{-1}\right\| \|f\| \frac{c}{\alpha} \left|(e^{\alpha(t+h)} - e^{\alpha t})\right| e^{-\alpha t} +
$$
$$
\|U(t+h) - U(t)\| \left\|U^{-1}(t)\right\| \frac{c}{\alpha} \|f\| + \left\|U(t+h)U^{-1}\right\| \|f\| \frac{c}{\alpha} \left|(-e^{-\alpha(t+h)} + e^{-\alpha t})\right| e^{\alpha t}
$$

which shows that as $h \to 0$, $U(t+h) \to U(t)$.

Finally, we verify that u is a solution to the differential equation (3.3) by direct computation: Fix $t \in \mathbb{R}, h \in \mathbb{R}$, and set

$$
\begin{aligned}
&\frac{u(t+h) - u(t)}{h} - A(t)u(t) \\
&\quad = \frac{1}{h}\Bigg(\int_{-\infty}^{t+h} U(t+h)PU^{-1}(s)f(s)ds - \int_{t+h}^{\infty} U(t+h)(I-P)U^{-1}(s)f(s)ds \\
&\qquad\qquad - \int_{-\infty}^{t} U(t)PU^{-1}(s)f(s)ds + \int_{t}^{\infty} U(t)(I-P)U^{-1}(s)f(s)ds\Bigg) \\
&\qquad - A(t)u(t) \\
&\quad = \frac{1}{h}\Bigg(\int_{-\infty}^{t+h} [U(t+h) - U(t)]PU^{-1}(s)f(s)ds \\
&\qquad\qquad - \int_{t+h}^{\infty} [U(t+h) - U(t)](I-P)U^{-1}(s)f(s)ds \\
&\qquad\qquad + \int_{-\infty}^{t+h} U(t)PU^{-1}(s)f(s)ds - \int_{t+h}^{\infty} U(t)(I-P)U^{-1}(s)f(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \int_{-\infty}^{t} U(t)PU^{-1}(s)f(s)ds + \int_{t}^{\infty} U(t)(I-P)U^{-1}(s)f(s)ds\Big) \\
&- A(t)u(t) \\
&= \frac{1}{h}\Big([U(t+h) - U(t)] \int_{-\infty}^{t+h} PU^{-1}(s)f(s)ds \\
&\qquad - [U(t+h) - U(t)] \int_{t+h}^{\infty} (I-P)U^{-1}(s)f(s)ds \\
&\qquad + \int_{t}^{t+h} U(t)PU^{-1}(s)f(s)ds + \int_{t}^{t+h} U(t)(I-P)U^{-1}(s)f(s)ds\Big) \\
&- A(t)u(t) \\
&= \frac{U(t+h)-U(t)}{h}\left[\int_{0}^{t} PU^{-1}(s)f(s)ds - \int_{t}^{\infty} (I-P)U^{-1}(s)f(s)ds\right] \\
&+ \frac{1}{h}\int_{t}^{t+h} U(t+h)U^{-1}(s)f(s)ds - A(t)u(t).
\end{aligned}
$$

Since the Cauchy operator $U(t)$ satisfies the identity

$$\frac{d}{dt}U(t) = A(t)U(t)$$

and

$$\lim_{h\to 0}\frac{1}{h}\int_{t}^{t+h} U(t+h)U^{-1}(s)f(s)ds = f(t),$$

we conclude that

$$\frac{d}{dt}u(t) = A(t)u(t),$$

as claimed. □

It has been established that $u \in C(\mathbb{R}, X)$ and hence that L is onto.(Note that it is implicit in the previous calculations that $u \in D(L)$. Now for $\mu \in D(L)$, $L\mu = 0$ implies that μ is a bounded solution of the differential equation (3.3), which is exponentially dichotomic and hence admits only the one bounded solution, i.e. $u \equiv 0$. Hence L is invertible; inequality (4.16) shows that the inverse is bounded and that its norm is subject to the bound

$$\left\|L^{-1}\right\| \leq \left(\frac{N_1}{\nu_1} + \frac{N_2}{\nu_2}\right). \tag{4.18}$$

Sufficiency follows from Lemmas (4.2), (4.3), and (4.4).

Chapter 5

Roughness of Exponential Dichotomy

In this chapter, we address the issue of roughness of exponential dichotomy. This problem has attracted many mathematicians. See for example [2, 4, 5]. Using theorem (4.1), we show that, under the hypothesis (exponential growth), exponential dichotomy is proved under suitable L^∞ perturbations. More precisely, we prove the following theorem:

Theorem 5.0.1. *Under condition (*4.1*), if (*1.1*) is exponentially dichotomic, so is the perturbed equation*

$$\frac{d}{dt}x = (A(t) + B(t))x \tag{5.1}$$

for any strongly continuous operator function $B : \mathbb{R} \to \mathcal{B}(X)$ with

$$\|B(t)\|_{L^\infty(\mathbb{R},\mathcal{B}(X))} < \left(\frac{N_1}{\nu_1} + \frac{N_2}{\nu_2}\right)^{-1}. \tag{5.2}$$

Proof. Since (1.1) is exponentially dichotomic, then L is invertible and (4.2) holds; The (unbounded) operator

$$S = L - B(t) : \mathcal{D}(L) \to C(\mathbb{R}, X)$$

is invertible (see [[3],Sect. IV,2, Remark 2.22]). The sufficiency in Theorem 4.1 shows that (5.1) is Exponentially Dichotomic, as claimed. □

We underline the fact that the dichotomic constants for the perturbed equation (5.1) can be tracked down in the previous proof and that they depend only on the dichotomic constants of the original differential equation and the L^∞ norm of $B(t)$.

Remark 5.0.1. Condition (4.1) cannot be omitted in Theorem (4.1), see [1] for a counter-example.

Remark 5.0.2. The L^∞-bound (5.2) is optimal, see [5]. The roughness property of Exponential Dichotomy has been proved by methods different from the ones employed in this work, in the absence of condition (4.1). (See [5], [6]).

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Curriculum Vitae

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