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A New Simplified Derivation of Nash Bargaining Solution

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Abstract

In the 1950s, the Nobel Prize winner John F. Nash has shown that under certain conditions, the best solution to the bargaining problem is when the product of the (increase in) utilities is the largest. Nash's derivation assumed that we are looking for strategies that assign a single situation to each bargaining situation. In this paper, we propose a simplified derivation of Nash bargaining solution that does not requires this assumption.

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1 Need for Cooperative Decision Making: Reminder

One of the main objectives of decision making theory and game theory is to provide reasonable solutions to situations in which different participants have different preferences. In particular, in cooperative games, we consider situations in which, in principle, if all the participants cooperate, they can reach situations which are preferable to the status quo for each of them.

To describe solutions to such problems, we first need to describe the preferences of individual players. In the standard decision making theory (see, e.g., [5, 6, 8]), these preferences are described by the corresponding utility values. These numerical values come from the observation that we can have a numerical scale for describing the person's preferences by using probabilities in the following way. We select two alternatives:

- a very negative alternative A_0; e.g., an alternative in which the decision maker loses all his money (and/or loses his health as well), and
- a very positive alternative A_1; e.g., an alternative in which the decision maker wins several million dollars.

Based on these two alternatives, we can, for every value $p \in [0,1]$, consider a randomized alternative $L(p)$ in which we get A_1 with probability p and A_0 with probability $1-p$.

When $p=1$, we get the favorable alternative A_1 (with probability 1), and when $p=0$, we get the unfavorable alternative A_0. In general, the larger the probability p of the favorable alternative A_1, the more preferable is the corresponding randomized alternative $L(p)$. Thus, the corresponding randomized alternatives ("lotteries") $L(p)$ form a continuous 1-D scale ranging from the very negative alternative A_0 to the very positive alternative A_1.

So, it is reasonable to gauge the preference of an arbitrary alternative A by comparing it to different alternatives $L(p)$ from this scale until we find A's place on this scale, i.e., the value $p \in [0,1]$ for which, to this decision maker, the alternative A is equivalent to $L(p)$: $L(p) \sim A$. This value is called the *utility* $u(A)$ of the alternative A in the standard decision making theory.

In our definition, the numerical value of the utility depends on the selection of the alternatives A_0 and A_1: e.g., A_0 is the alternative whose utility is 0 and A_1 is the alternative whose utility is 1. It is possible to show that if we use a different set of alternatives A_0' and A_1', then the new utility value $u'(A)$ is obtained from the original one by a linear transformation $u'(A) = \lambda \cdot u(A) + b$.

In particular, if only consider scales in which, as A_0, we take the status quo point S, then we get $u(S) = u'(S) = 0$ and thus, $b = 0$. In this case, utility is defined modulo re-scaling $u'(A) = \lambda \cdot u(A)$, with $\lambda > 0$.

When we have several (n) participants, each alternative A is characterized by n utility values $u_1(A), \ldots, u_n(A)$ that describe the preference of the corresponding participants. Based on these values, we need to select one (or several) alternatives.

2 Nash Bargaining Solution and Its Original Derivation

In his 1950 paper [7], the Nobel Prize winner John F. Nash proposed to select an alternative for which the product $\prod\limits_{i=1}^{n} u_i$ of the utility values is the largest possible. This solution is called the *Nash bargaining solution*.

This solution does not change if we re-scale all the utilities, i.e., if we replace

each utility combination $(u_1, \ldots, u_n)$ with a new combination

$$(\lambda_1 \cdot u_1, \ldots, \lambda_n \cdot u_n).$$

Indeed, after this re-scaling, the objective function is simply multiplied by a positive constant $\lambda_1 \cdot \ldots \cdot \lambda_n$ and thus, it attains the largest value on exactly the same alternative(s).

This solution is also *symmetric* in the sense that it does not change if we rename the original participants, i.e., for example, if we replace each combination $(u_1, u_2, u_3, \ldots, u_n)$ with a combination $(u_2, u_1, u_3, \ldots, u_n)$. Nash has proven that if we have a strategy that assigns, to every set of possible alternatives, a single alternative, and if this strategy is invariant with respect to re-scaling and re-naming, then this strategy must coincide with the Nash bargaining solution.

3 What We Plan to Do

In this paper, we show that when we derive the Nash bargaining solution, we do not need to assume that only one alternative is selected. The new derivation is even simpler than the original one.

Comments.

- It is interesting to mention that while the derivation of Nash solution was only found in 1950, the solution itself was already proposed in economics in 1930 [11]. The mathematical equivalence of Zeuthen's and Nash's solutions was shown in [3].
- It is also worth mentioning that since the original 1950 paper, many other alternative derivations of Nash bargaining solution and its generalizations have been proposed; see, e.g., [1, 2, 4, 9, 10] and references therein. Some of these derivations show that it is not necessary to assume that we simply select between pairs of alternatives; some other derivations show what happens if we do not require re-naming invariance (i.e., if we do not assume that all the participants are equal), etc.

4 New Derivation: Description

We want to make a selection between different alternatives characterized by different values of the utility vector $\vec{u} \stackrel{\text{def}}{=} (u_1, \ldots, u_n) \in \left(\mathbb{R}_0^+\right)^n$, where $\mathbb{R}_0^+$ denotes the set of all non-negative real numbers. To describe this selection, let us first describe the equivalence relation $\vec{u} \sim \vec{v}$ between such vectors meaning that the alternatives corresponding to $\vec{u}$ and $\vec{v}$ are equally reasonable to select.

In terms of this equivalence relation, the invariance requirements take the following form.

Definition 4.1

- *We say that an equivalence relation* $\sim$ *is* scale-invariant *if for every* $\lambda_1 > 0, \ldots, \lambda_n > 0$, $(u_1, \ldots, u_n) \sim (v_1, \ldots, v_n)$ *implies*

$$(\lambda_1 \cdot u_1, \ldots, \lambda_n \cdot u_n) \sim (\lambda_1 \cdot v_1, \ldots, \lambda_n \cdot v_n).$$

- *We say that an equivalent relation* $\sim$ *is* symmetric *if* $(u_1, \ldots, u_n) \sim (u_{\pi(1)}, \ldots, u_{\pi(n)})$ *for every permutation* π.

Comment. In particular, for $\pi : 1 \leftrightarrow 2$, we have

$$(u_1, u_2, u_3, \ldots, u_n) \sim (u_2, u_1, u_3, \ldots, u_n).$$

Theorem 4.2 *For every scale-invariant symmetric equivalence relation* $\sim$, *we have* $(u_1, \ldots, u_n) \sim (u_1 \cdot \ldots \cdot u_n, 1, \ldots, 1)$.

Proof. Due to symmetry, we have $(1, u_2, u_3, \ldots, u_n) \sim (u_2, 1, u_3, \ldots, u_n)$. By applying scale-invariance with $\lambda_1 = u_1$, we conclude that $(u_1, u_2, u_3, \ldots, u_n) \sim (u_1 \cdot u_2, 1, u_3, \ldots, u_n)$. By applying a similar idea to the 1st and the 3rd participants, we get $(u_1 \cdot u_2, 1, u_3, u_4, \ldots, u_n) \sim (u_1 \cdot u_2 \cdot u_3, 1, 1, u_4, \ldots, u_n)$ and thus, since $\sim$ is the equivalence relation, that

$$(u_1, u_2, u_3, u_4 \ldots, u_n) \sim (u_1 \cdot u_2 \cdot u_3, 1, 1, u_4, \ldots, u_n).$$

By applying this same idea to the 1st and 4th, 1st and 5th, etc., we get the desired equivalence. The theorem is proven.

Discussion. Because of the theorem, the quality of each alternative is uniquely determined by Nash's product of the utilities – in the sense that every two alternatives with the same product value are equivalent.

If we have two alternative with two different values of the product $U < V$, then U is equivalent to the vector $(U^{1/n}, \ldots, U^{1/n})$ with the same product, and V is equivalent to the vector $(V^{1/n}, \ldots, V^{1/n})$. Here $U^{1/n} < V^{1/n}$, hence for the second vector, every participant gets a larger utility. Thus, the second vector is clearly better.

Thus, we should select an alternative for which the product attains the largest possible value. Nash's idea has been indeed justified – without the original Nash's assumption that there is only one selected vector of utilities.

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