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USE OF DETERMINISTIC TRAFFIC ASSIGNMENT ALGORITHMS IN STOCHASTIC NETWORKS: ANALYSIS OF E QUIVALENT LINK DISUTILITY FUNCTIONS

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ABSTRACT

At present, in practice, most traffic assignment tasks are performed by using deterministic network (DN) models, which assume that the link travel time is uniquely determined by the link volume and link capacity. In reality, for the same link volume and link capacity, a link may have different travel times. However, the corresponding stochastic network (SN) models are not widely used because they are much more computationally complex than the DN models. In the past research, it was shown that in the important particular case, when the link travel time follows Gamma distribution, the traffic assignment problem for SN can be reformulated in terms of deterministic equivalent link disutility function. Thus in this case the traffic assignment can be solved by the standard Frank-Wolfe algorithm. In this paper, we show that a similar equivalent link disutility function exists in the general case of an arbitrary distribution of link travel time. Therefore, we can use the Frank-Wolfe algorithm in the general SN case, both for

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the risk averse and risk prone driver route choice behavior. We also provide an explicit expression for this equivalent link disutility function in terms of the link volume and link capacity.

Keywords: traffic assignment, route choice, utility function, stochastic network, user equilibrium

1. Introduction

In traffic assignment, a road network is usually modeled as a set of nodes connected by unidirectional links. A set of vehicles, also known as users, travelers, or drivers, are to be loaded into the network and travel from their origin nodes to the destination nodes. A modeler is always interested in seeking the traffic flow distribution in a network, i.e., the volume of traffic in the links, such that no user can improve his/her travel time by unilaterally changing his/her route. This state of flow distribution is called the User Equilibrium (UE) condition. In assigning traffic flow to a network, among the assumptions made are the characteristics of link travel times and of drivers' knowledge of the link travel times.

At present, a large number of transportation network models are based on the assumption that travel time in a link is a deterministic function of the link's characteristics (such as free-flow travel time and link capacity) and link volume. A network with such a deterministic link travel time function is called a Deterministic Network (DN). In reality, empirical data shows that for the same traffic flow in a link, we have variations in travel time. These variations are due to the difference in vehicle mix, difference in driver reactions, weather, incident conditions, etc. These variations are small when the traffic flow is light but they become much larger as the link becomes more congested. A natural way to model such variation is to consider travel time in a link as a probability distribution, with mean and variance expressed as functions of the link characteristics and link volume. A network with such probabilistic link travel times is called a Stochastic Network (SN).

Most transportation network models assume that the drivers have perfect knowledge of the link travel times (in the deterministic case) or of the probabilities of different values of link travel times (in the stochastic case). The resulting state of the transportation network is called Deterministic User Equilibrium (DUE). In reality, a driver's knowledge is usually somewhat imperfect. The driver's perception of a (deterministic or stochastic) link travel time may be slightly different from the actual travel time. Some transportation network models take this perception error into account by modeling it as a normal distribution with zero mean. Due to these perception errors the selected routes of the drivers vary stochastically. The resulting state of the transportation network is called Stochastic User Equilibrium (SUE).

Based on the assumptions in link travel times and drivers' perception on the link travel times, traffic assignment models may therefore be classified into four types: Deterministic Network-Deterministic User Equilibrium (DN-DUE), Deterministic Network-Stochastic User Equilibrium (DN-SUE), Stochastic Network-Deterministic User Equilibrium (SN-DUE), Stochastic Network-Stochastic User Equilibrium (SN-SUE) (Chen and Recker 2000).

The DN-DUE is the simplest, the easiest to understand, and the most widely accepted traffic assignment model. It assumes that drivers have perfect knowledge of the deterministic link travel times (with a given flow distribution) in the network, and they always select the paths that have the shortest travel times between their origins and destinations. This model was originally formulated by Beckman et al. (1956) and can be solved by the Frank-Wolfe algorithm (Sheffi, 1985). In DN-SUE, the network's link travel times are deterministic (with a given flow distribution), but they may be perceived differently by different drivers. Due to the error in

travel time perception, drivers will always select what they perceive as the shortest paths but these may not be the actual shortest paths. The DN-SUE model was originally formulated by Daganzo and Sheffi (1977). A popular solution algorithm for the DN-SUE model is the Method of Successive Averages proposed by Sheffi and Powell (1982).

In DN, the travel time is uniquely determined by the path; a driver selects the path connecting an origin and a destination with the shortest travel time. As we have mentioned, in reality, the travel time is not uniquely determined by the path; each driver selects the path with the lowest expected value of the disutility. Such SN models were first studied by Mirchandani and Soroush (1987). In particular, the SN-DUE assumes that drivers have perfect knowledge of the degree of variation in link travel times, and they factor this variation in their route choice decisions. While DN-DUE and DN-SUE models are used by many transportation modelers, only few papers (Mirchandani and Soroush, 1987; Tatineni, et al., 1997; Chen and Recker, 2000; Chen et al. 2000) used SN models because these models are much more computationally complex than the DN models. It is known that, under certain conditions, the SN-DUE model can be solved by the Frank-Wolf algorithm simply by replacing the link travel time function with a suitable equivalent link disutility function (see for examples Mirchandani and Soroush, 1987; and Tatineni, 1996). The main objective of this paper is to extend this possibility to a more realistic set of conditions.

In principle, it is possible to consider an even more realistic SN-SUE model which adds drivers' perception errors into the link travel time variations. However, according to Chen and

Recker (2000), the SN-DUE model is quite suitable for modeling of peak hour traffic because regular commuters have a good knowledge of the mean and variance of peak hour travel times.

The outline of this paper is as follows. In section 2, we describe the previous related work. In section 3, we emphasize the limitations of the previous work: that the possibility to use equivalent link disutility function is limited to a specific case of a Gamma distribution and that there is no ready to use expression for this equivalent link disutility function in terms of link volume and link capacity. The need to overcome these limitations is the motivation for our current research. In section 4, we prove the possibility to use the equivalent link disutility function in the general stochastic case, and we provide the expression for these functions in terms of mean and variance of link travel time. In section 5, we describe the desired properties of the equivalent link disutility function; in section 6, we illustrate these desired properties on the example of the DN, and in section 7 we use these properties to derive the expression for the equivalent link disutility function in terms of link volume and link capacity. In section 8, we analyze the resulting route choice behavior for risk averse and risk prone drivers. Finally, in an auxiliary section 9, we provide an alternative derivation of the expression for the variance of link travel time in terms of link volume and link capacity. Section 10 summarizes this paper.

2. Previous Related Work

In the deterministic case, drivers select a route r with the smallest value of the route travel time $t_r = \sum_{i \in r} t_i$. To describe the deterministic link travel time t_i, the most popular function used by transportation modelers is the Bureau of Public Road (BPR) function:

$$t_i = t_i^f \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^{\beta} \right] \quad (1)$$

where t_i is the travel time in link *i*, t_i^f is the free-flow travel time in link *i*, v_i is the volume in link *i*, c_i is the capacity of link *i*, and α and β are constants. The t_i^f is computed by dividing l_i, the length of link *i*, by u_i^f, the free-flow speed of link *i*. Typical values of α and β are 0.15 and 4 respectively.

In the stochastic case, it is reasonable to assume that formula (1) describes the average link travel time $\bar{t}_i$. To describe the difference between the actual link travel time t_i and its mean value $\bar{t}_i$, the authors of Mirchandani and Soroush, 1987; Tatineni, 1996; Tatineni, et al., 1997; Chen and Recker, 2000; Chen et al. 2000 used empirical evidence according to which the actual distribution of t_i can be described (with reasonable accuracy) by the Gamma distribution with a lower bound equal to t_i^f. To describe SN-SUE models, they also made an empirical based assumption that the perception errors have normal distribution with mean equal to 0.

The drivers route selection depends on how the drivers react to the travel time uncertainty. This is particularly important if drivers have constraints in the time of arrival (e.g., scheduled events, work starting times) with heavy penalties for late arrivals. Mirchandani and Soroush (1987), Tatineni et al. (1997) and Chen and Recker (2000) describe three types of such behavior: risk averse, risk prone and risk neutral. The term risk here refers to the risk of a late arrival at the

destination. A risk averse driver prefers a route with longer expected travel time but smaller variation to a route with faster expected travel time but higher variation. That is, he/she would rather use the route with longer travel time (and depart early) to lower the risk of arriving late. On the contrary, a risk prone driver would select the route with a faster travel time but higher variation. A driver with risk neutral behavior does not consider travel time variation in his/her route choice decision.

According to decision theory (see for example Watson and Buede, 1994), in the stochastic case, a rational decision maker maximizes the expected value of his/her utility function, or equivalently minimizes the expected value of the disutility function. In particular, for the stochastic traffic assignment problem, given a choice of routes $r \in R$ connecting an origin-destination pair, a driver will select the route r' which has the smallest expected disutility $E[DU_r]$

$$E[DU_{r'}] = \min_{r \in R}\{E[DU_r]\} \tag{2}$$

For the drivers with risk neutral behavior, the route disutility function DU_r is equal to the route travel time t_r. Therefore the expected route disutility $E[DU_r]$ is equal to the average route travel time $\bar{t}_r$. The route travel time t_r, for a route r which is made up of L links, is equal to the sum of the link travel times: $t_r = t_1 + ... + t_L$. So, the average route travel time is equal to the sum of the average link travel times: $\bar{t}_r = \bar{t}_1 + ... + \bar{t}_L$. Thus selecting a route with the smallest

$E[DU_r]$ is equivalent to selecting a route with the smallest value of the sum of $\bar{t}_i$. Hence, a risk neutral driver can be described by an equivalent link disutility function $DU_i = \bar{t}_i$.

For describing risk averse and risk prone behavior, the most commonly used disutility functions are the exponential functions (Watson and Buede, 1994). Such functions have been used by Tatineni et al. (1997) and Chen and Recker (2000) to represent the risk averse and risk prone behaviors in a SN:

$$DU_r = \begin{cases} b_1[exp(\omega t_r) - 1] & \text{for risk averse drivers} \\ b_2[1 - exp(-\varphi t_r)] & \text{for risk prone drivers} \end{cases} \tag{3}$$

where t_r is the route travel time, and $b_1, b_2, \omega, \varphi$ are positive constants. Given a choice of routes $r \in R$ connecting an origin-destination pair, a driver will select the route r' which has the smallest expected disutility $E[DU_r]$.

Under these assumptions, it was shown that selecting a route with the smallest value of $E[DU_r]$ is equivalent to selecting a route with the smallest value of the sum $du_r = \sum_{i \in r} DU_i$ for some values DU_i. This minimized expression is similar to the minimized expression $t_r = \sum_{i \in r} t_i$ in the deterministic case. Thus, it is reasonable to call DU_i an equivalent link disutility function.

In particular, in the SN-DUE case, when there is no perception error, for risk averse drivers the equivalent link disutility function takes the following form (Tatineni, 1996; Tatineni, et al., 1997)

$$DU_i = \bar{t}_i + c\sigma_{t_i}^2\left(\frac{1}{2}+\frac{1}{3}c\frac{\sigma_{t_i}^2}{\bar{t}_i - t_i^f}\right) \tag{4}$$

where $\sigma_{t_i}^2$ is the variance of the link travel time and c is a constant determined by the parameters of the exponential disutility function. In the derivation of this formula, the authors assume that the difference between the actual link travel time t_i and that average travel time $\bar{t}_i$ is reasonably small, so we can ignore higher order terms in $(t_i - \bar{t}_i)$.

The fact that the users preferences can be expressed in the form of minimizing the expression $du_r = \sum_{i \in r} DU_i$ allows us to use Frank-Wolf algorithm to solve the traffic assignment problem in the stochastic case as well (Tatineni et al., 1997; Chen and Recker, 2000).

3. Motivation for the Current Research

The above solution to the stochastic traffic assignment problem requires first that we assume that the distribution of link travel time is Gamma, and that for every link, we know the variance of the link travel time. In practice, the actual link travel time distribution can be somewhat different from Gamma, and very frequently we do not know the variances of the travel time of all the links.

Our first motivation for this paper is to show that the equivalent link disutility function DU_i (incorporating average link travel time and travel time variation) can be used to describe drivers' route choice preference in the general SN-DUE case, without the need to assume Gamma distribution. Our second motivation is to provide formulas which estimate the variances of the link travel times and the expression for the equivalent link disutility DU_i, so that we will be able to apply the Frank-Wolfe algorithm even if we do not have the empirical information about the variance.

4. Possibility to Use Equivalent Link Disutility Functions in a General Stochastic Network

In this section, we show that the equivalent link disutility function DU_i can be used to describe drivers' route choice preference in the general SN-DUE case, without the need to assume Gamma distribution.

4.1 Risk Averse Behavior: Possibility to Use Equivalent Link Disutility Functions

According to the SN-DUE model, a driver selects a route with the minimum value of the expected disutility $E[DU_r]$. If we "rescale" the disutility function, i.e., consider an auxiliary function $A_r = g(E[DU_r])$ for some monotonically increasing function $g(x)$, then minimizing $E[DU_r]$ is equivalent to minimizing A_r. We will use this property to simplify the decision making in the SN-DUE model.

In particular, for risk averse drivers, following Equation (3), we have $E[DU_r] = b_1(A_r - 1)$, where

$$A_r = E[exp(\omega t_r)] \tag{5}$$

Therefore, $A_r = g(E[DU_r])$ for $g(x) = (x/b_1) + 1$. Since b_1 >0, the function $g(x)$ is monotonically increasing and therefore, minimizing $E[DU_r]$ is equivalent to minimizing A_r.

The route travel time t_r is composed of link travel times t_i: $t_r = \sum_{i=1}^{L} t_i$. In a SN, link travel times (t_i) are considered to be independent random variables. Thus, the auxiliary expression $A_r = E[exp(\omega t_r)]$ can be expressed as

$$A_r = E[exp(\omega t_r)] = E[exp(\omega(t_1 + t_2 + ... + t_L))] = E[exp(\omega t_1) exp(\omega t_2) ... exp(\omega t_L)]$$

$$= E[exp(\omega t_1)] \cdot E[exp(\omega t_2)] \cdot \ldots \cdot E[exp(\omega t_L)] \tag{6}$$

Drivers will select the route that minimizes $E[DU_r]$; this is equivalent to minimizing A_r. Since $ln(x)$ is a monotonically increasing function, this choice is, in its turn, equivalent to selecting the route that minimizes $ln(A_r)$. Here

$$ln(A_r) = ln\{E[exp(\omega t_1)]\} + ln\{E[exp(\omega t_2)]\} + \ldots + ln\{E[exp(\omega t_L)]\} \tag{7}$$

Let us perform one more rescaling, to make this expression similar to that of the DN. A DN can be viewed as a particular case of a SN, in which all travel times t_i and t_r are deterministic. In a DN, the above expression reduces to

$$ln(A_r) = ln[exp(\omega t_1)] + ln[exp(\omega t_2)] + \ldots + ln[exp(\omega t_L)] = \omega(t_1 + t_2 + \ldots + t_L) = \omega t_r \tag{8}$$

In a DN, we select a route with the smallest route travel time t_r. For convenience, let us rescale the objective function $ln(A_r)$ one more time so that for DN, the rescaled objective function will coincide with t_r. Specifically, we consider $du_r = \frac{1}{\omega} ln(A_r)$ instead of $ln(A_r)$, both for the DN and for the SN. In this case, for the DN we have $du_r = t_r$.

In the general SN case, since $g(x) = \frac{x}{\omega}$ is a monotonically increasing function, selecting a route based on du_r is equivalent to selecting a route based on $ln(A_r)$, and thus equivalent to selecting a route based on $E[DU_r]$. From Equation (7), we conclude that the new objective

function du_r can be expressed as $du_r = DU_1 + ... + DU_L$, where $DU_i = \frac{1}{\omega} ln\{E[exp(\omega t_i)]\}$. Thus, the drivers preference in SN-DUE is equivalent to selecting a route with the smallest value of the sum $du_r = \sum_{i \in r} DU_i$. So we get the desired equivalence with the equivalent link disutility function $DU_i = \frac{1}{\omega} ln\{E[exp(\omega t_i)]\}$. Therefore, selecting a route in a SN is very similar to selecting a route in a DN, but with link disutility $DU_i = \frac{1}{\omega} ln\{E[exp(\omega t_i)]\}$ instead of link travel time.

4.2 Risk Averse Behavior: Expression for the Equivalent Link Disutility Functions in Terms of Mean and Variance of Link Travel Time

Let us reformulate this expression for DU_i in terms of mean and variance of t_i. In a SN the actual travel time t_i in link *i* can be expressed as the sum of the mean travel time $\bar{t}_i$ and the deviation from its mean:

$$t_i = \bar{t}_i + (t_i - \bar{t}_i) \tag{9}$$

It follows that

$$exp(\omega t_i) = exp(\omega \bar{t}_i) exp(\omega(t_i - \bar{t}_i)) \tag{10}$$

Hence

$$E[exp(\omega t_i)] = exp(\omega \bar{t}_i) E[exp(\omega(t_i - \bar{t}_i))] \tag{11}$$

Usually $\omega(t_i - \bar{t}_i)$ is small, so we can expand the exponential function into the Taylor series and only keep the first three terms in this expansion

$$exp(\omega(t_i - \bar{t}_i)) = 1 + \omega(t_i - \bar{t}_i) + \frac{\omega^2 (t_i - \bar{t}_i)^2}{2} + ... \tag{12}$$

Therefore

$$E[exp(\omega(t_i - \bar{t}_i))] \approx 1 + \omega E[t_i - \bar{t}_i] + \frac{\omega^2}{2} E[(t_i - \bar{t}_i)^2] \tag{13}$$

By definition, $E[t_i - \bar{t}_i]$ =0 and $E[(t_i - \bar{t}_i)^2] = \sigma_{t_i}^2$ which is the variance of t_i . Substituting Equation (13) into Equation (11), we obtain

$$E[exp(\omega t_i)] = exp(\omega \bar{t}_i)\left[1 + \frac{\omega^2}{2}\sigma_{t_i}^2\right] \tag{14}$$

The link disutility function thus becomes

$$DU_i = \frac{1}{\omega} ln\{E[exp(\omega t_i)]\} = \frac{1}{\omega} ln\left\{exp(\omega \bar{t}_i)\left[1 + \frac{\omega^2}{2}\sigma_{t_i}^2\right]\right\}$$

$$= \bar{t}_i + \frac{1}{\omega} ln\left[1 + \frac{\omega^2}{2}\sigma_{t_i}^2\right] \tag{15}$$

Using the Taylor series expansion of $ln(1+z)=z+...$ we obtain

$$DU_i \approx \bar{t}_i + \frac{\omega}{2}\sigma_{t_i}^2 \tag{16}$$

We have shown that, if the all drivers in a network follow the same risk averse behavior, solving for DUE in a SN is similar to solving for DUE in a DN, except that we replace t_i in a DN with DU_i in a SN. Note that the first term $\bar{t}_i$ in DU_i is the same as Equation (1), the BPR function. Thus, it can be said that, in a SN with risk averse behavior, the additional term in the route choice decision for drivers is the link travel time variance, scaled by a factor $\omega/2$. The magnitude of ω reflects the sensitivity of the drivers in avoiding the risk. Risk averse drivers will avoid links that have high $\sigma_{t_i}^2$. Note that, if σ_{t_i} =0, the SN-DUE model is reduced to a DN-DUE model.

4.3 Risk Prone Behavior: Possibility to Use Equivalent Link Disutility Functions

According to the SN-DUE model, a driver selects a route with the smallest possible value of the expected disutility $E[DU_r]$

$$E[DU_{r'}] = \min_{r \in R}\{E[DU_r]\} \tag{17}$$

For risk prone drivers, according to Equation (3), $E[DU_r] = b_2(1-B_r)$ where

$$B_r = E\left[exp\left(-\varphi t_r\right)\right] \tag{18}$$

Thus, minimizing $E\left[DU_r\right]$ is equivalent to maximizing B_r. Since the link travel times t_i are independent random variables, we conclude that for a route consisting of L links, we have

$$B_r = E\left[exp\left(-\varphi t_r\right)\right] = E\left[exp\left(-\varphi t_1\right)\right]\cdot E\left[exp\left(-\varphi t_2\right)\right]\cdot \ldots \cdot E\left[exp\left(-\varphi t_L\right)\right] \tag{19}$$

Selecting a route according to Equation (17) is equivalent to selecting a route that maximizes B_r. This choice, in its turn, is equivalent to selecting the route that minimizes $du_r = -\frac{1}{\varphi} ln\left(B_r\right)$. Here

$$du_r = -\frac{1}{\varphi} ln\left\{E\left[exp\left(-\varphi t_1\right)\right]\right\} - \frac{1}{\varphi} ln\left\{E\left[exp\left(-\varphi t_2\right)\right]\right\} - \ldots - \frac{1}{\phi} ln\left\{E\left[exp\left(-\varphi t_L\right)\right]\right\} \tag{20}$$

Thus, for risk prone behavior, the drivers preference in SN-DUE is equivalent to selecting a route with the smallest value of the sum $du_r = \sum_{i \in r} DU_i$. So we get the desired equivalence with the equivalent link disutility function $DU_i = -\frac{1}{\varphi} ln\left\{E\left[exp\left(-\varphi t_i\right)\right]\right\}$. Therefore, selecting a route in a SN is very similar to selecting a route in a DN, but with link disutility $DU_i = -\frac{1}{\varphi} ln\left\{E\left[exp\left(-\varphi t_i\right)\right]\right\}$ instead of link travel time.

4.4 Risk Prone Behavior: Expression for the Equivalent Link Disutility Functions in Terms of Mean and Variance of Link Travel Time

Let us reformulate this expression for DU_i in terms of mean and variance of t_i. By following the same procedure as in the risk averse case, we can show that

$$-\frac{1}{\varphi} ln\{E[exp(-\varphi t_i)]\} = -\frac{1}{\varphi} ln\left\{exp(-\varphi \bar{t}_i)\left[1+\frac{\varphi^2}{2}\sigma_{t_i}^2\right]\right\} = \bar{t}_i - \frac{1}{\varphi} ln\left[1+\frac{\varphi^2}{2}\sigma_{t_i}^2\right] \tag{21}$$

Therefore, we can write

$$DU_i = \bar{t}_i - \frac{1}{\varphi} ln\left[1+\frac{\varphi^2}{2}\sigma_{t_i}^2\right] \tag{22}$$

Using the Taylor series expansion for $ln(1+x)$, we obtain

$$DU_i \approx \bar{t}_i - \frac{\varphi}{2}\sigma_{t_i}^2 \tag{23}$$

Equation (23) may be interpreted as follows. A risk prone driver will consider the average link travel times $(\bar{t}_i)$ as well as the variance of link travel times $(\sigma_{t_i}^2)$ in his/her route choice decision. If there are choices of two links with the same average travel time, a risk prone driver prefers the link with the higher variance. The higher the variance, the more favorable the link is to the risk prone driver. Therefore, the link disutility function has the link variance term, weighted by $(-\varphi/2)$.

5. Desired Properties of Equivalent Link Disutility Functions

Note that a route r is made up of a series of L connected links $i=1,\ldots,L$. We have already shown that we can assign, to every link i, a value DU_i in such a way that the drivers preference is equivalent to selecting a route with the smallest value of the sum $du_r = \sum_{i=1}^{L} DU_i$. In other words, the equivalent link disutility function satisfies the property

(P1) It must be mathematically consistent with the route disutility function, in the sense that it leads to the same routing decision (it may however have a different form than the route disutility function).

Property P1 ensures that the equivalent link disutility function describes the same route choice behavior as the original route disutility function.

It is also desirable that the equivalent link disutility function satisfies the following properties:

(P2) If we sub-divide a link into a series of shorter links, the equivalent disutility of the original link must be equal to the sum of the equivalent disutilities of the shorter links.

(P3) The equivalent link disutility function must be a monotonically increasing and continuously differentiable function of link volume.

Property P2 ensures that drivers' route choice and network flow remain the same irrespective of the resolution of network representation. Property P3 ensures that the equivalent link disutility function is consistent with common sense: the higher the link volume, the less preferable it is to the drivers, and small changes in the link volume lead to small changes in the driver's preference.

The consistency in UE flow pattern irrespective of the resolution in network representation is important in many practical applications. Many transportation planning models divide the geographical area to be analyzed into zones, depending on the land-use patterns. The zones in the geographical border (or buffer zones) are usually larger than the zones in the center business district. Naturally, the modeling details are often sized according to the zone dimension. Zones covering larger areas are likely to have longer links. On the other hand, smaller zones are likely to have shorter links and higher node density. Many traffic assignment algorithms use the geographical and topological information of the nodes and links converted from a Geographical Information Systems (GIS) database. To be geographically correct in representing a curve road segment which has a uniform geometry, intermediate nodes are inserted between the two ends of the segment so that it can be represented by a series of piecewise linear links. If the additive property of the link disutility is not preserved, such division of a link into a series of smaller links may produce different UE flow patterns after traffic assignment.

A consistent equivalent link disutility function can be placed instead of the deterministic link travel time function in the existing traffic assignment models (such as TransCAD (Caliper, 2005)) and thus enable us to use these models for SN-DUE applications.

6. Illustrating the Desired Properties on the Example of Deterministic Link Travel Time Function

We first use a commonly used deterministic link travel time function to illustrate the concepts of P2 and P3. As we have mentioned, the most popular deterministic link travel time function used by transportation modelers is the BPR function in Equation (1). The t_i^f is computed by dividing l_i, the length of link *i*, by u_i^f, the free-flow speed of link *i*. For a route *r* which is made up a series of *L* links $(i = 1,...,L)$, the route travel time is $t_r = \sum_{i=1}^{L} t_i$. In short, the route travel time is the arithmetic sum of the link travel times, with the latter represented by the BPR function.

Since α >0 and β >0, t_i is a monotonically increasing and continuously differentiable function of v_i, i.e., the BPR function satisfies P3.

We now illustrate the concept of P2. Suppose that we now divide link *i* into *n* consecutive sub-links $\{i_1, i_2,..., i_n\}$, with lengths $\{l_{i_i}, l_{i_2,...}, l_{i_n}\}$. Then, the volume, capacity, and free-flow speed of the sub-links are same as that of link *i*, i.e., $v_{i_1} = v_{i_2} = ... = v_{i_n} = v_i$, $c_{i_1} = c_{i_2} = ... = c_{i_n} = c_i$, and $u^f_{i_1} = u^f_{i_2} = ... = u^f_{i_n} = u_i^f$. The free-flow travel times of the sub-links are thus $\{t^f_{i_1}, t^f_{i_2},..., t^f_{i_3}\} = \left\{\frac{l_{i_1}}{u_i^f}, \frac{l_{i_2}}{u_i^f},..., \frac{l_{i_n}}{u_i^f}\right\}$. The travel time in link *i*, computed from the sum of the travel times in its sub-links, is

$$t_{i_1} + t_{i_2} + ... + t_{i_n} = t^f_{i_i}\left[1 + \alpha\left(\frac{v_i}{c_i}\right)^{\beta}\right] + ... + t^f_{i_n}\left[1 + \alpha\left(\frac{v_i}{c_i}\right)^{\beta}\right] = \left(t^f_{i_1} + ... + t^f_{i_n}\right)\left[1 + \alpha\left(\frac{v_i}{c_i}\right)^{\beta}\right]$$

$$= \left(\frac{l_{i_1} + \ldots + l_{i_n}}{u_i^f} \right) \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^{\beta} \right] = \left(\frac{l_i}{u_i^f} \right) \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^{\beta} \right] = t_i^f \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^{\beta} \right] = t_i \tag{24}$$

Therefore, if we divide an original link into shorter links and compute the travel times of the shorter links, then the sum of the travel times on the shorter links is the same as the original link travel time. Thus, by using the BPR function, the additive property of the link travel time is preserved, and the BPR function satisfies property P3.

7. Expression for the Equivalent Link Disutility Functions in Terms of Link Volume and Link Capacity

Let us use the property P2 to derive expression for the equivalent link disutility functions in terms of link volume and link capacity. In a SN, t_i, the travel time in link *i*, is a random variable. For this random variable t_i, the average travel time $\bar{t}_i$ can be estimated by the BPR function:

$$\bar{t}_i = t_i^f \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^{\beta} \right] \tag{25}$$

Note that, according to this formula, when v_i=0, we have $\bar{t}_i = t_i^f$. Moreover, in the absence of traffic flow, i.e., when v_i=0, the link travel time t_i should be equal to t_i^f (with probability=1.0). Other than these restriction on the average and on the free-flow travel time, we are not making

any other explicit assumption about the distribution of t_i; in this sense, the conclusions of this section are distribution-free.

It is natural to assume that, DU_i, the equivalent disutility of link *i* should depend on the free-flow travel time t_i^f and the relative average delay $d = \left(\bar{t}_i - t_i^f\right)/t_i^f$, i.e.,

$$DU_i = F\left(t_i^f, d\right) \tag{26}$$

where

$$d = \frac{\bar{t}_i - t_i^f}{t_i^f} = \alpha\left(\frac{v_i}{c_i}\right)^{\beta} \tag{27}$$

for some function $F\left(t_i^f, d\right)$. So, to describe an equivalent link disutility function, we must find the appropriate function $F\left(t_i^f, d\right)$.

One would expect a link which has a longer uncongested travel time to have a higher equivalent disutility; so, $F\left(t_i^f, d\right)$ must be an increasing function of t_i^f. One would also expect that as the link becomes more congested, the equivalent disutility would increase; so, $F\left(t_i^f, d\right)$ must also be an increasing function of *d*. In addition, the function $F\left(t_i^f, d\right)$ must satisfies the following conditions:

(i) In the deterministic case, we want our equivalent link disutility function to reduce to the standard link travel time function. We have already mentioned that when $v_i = 0$, then the travel time is deterministically determined $t_i = \bar{t}_i = t_i^f$, therefore

$$F\left(t_i^f, 0\right) = t_i^f \qquad (28)$$

(ii) We would like the equivalent link disutility function to satisfy the property P2: If we sub-divide a link into a series of shorter links, the equivalent disutility of the original link must be equal to the sum of the equivalent disutilities of the shorter links. If we sub-divide a link into two sub-links with free-flow travel times $t_{i_1}^f$ and $t_{i_2}^f$ respectively, then $v_{i_1} = v_{i_2} = v_i$, and $c_{i_1} = c_{i_2} = c_i$; so by Equation (27), the relative average delay *d* for both sub-links is the same as for the original link. Thus the desired property P2 takes the following form

$$F\left(t_{i_i}^f + t_{i_2}^f, d\right) = F\left(t_{i_1}^f, d\right) + F\left(t_{i_2}^f, d\right) \qquad (29)$$

Let us describe all the functions $F\left(t_i^f, d\right)$ which satisfy these conditions. First we analyze Equation (29). We fix a value *d* and introduce an auxiliary function $G(a) = F(a, d)$. In terms of this new function, the Equation (29) takes the form

$$G(a+b) = G(a) + G(b) \qquad (30)$$

We also know that $F\left(t_i^f, d\right)$ is an increasing function of t_i^f and therefore, $G(a)$ is an increasing function of *a*. It is known (Aczel, 2006) that every monotonically increasing function $G(a)$ which satisfies Equation (30) has the form $G(a) = k \cdot a$ for some k>0. For different *d*, the coefficient *k* may in general be different: $k = k(d)$. Thus we conclude that

$$DU_i = F\left(t_i^f, d\right) = t_i^f k(d) \tag{31}$$

From Equation (28), we know that for *d*=0 we have $F\left(t_i^f, d\right) = t_i^f$. Therefore $k(0)$=1.

For typical values of α and β (see Equation (27)), we have $d << 1$. Thus we can use the Taylor series expansion

$$k(d) = 1 + a_1 d + a_2 d^2 + \ldots \tag{32}$$

and ignore the higher order terms, i.e., use an expression $k(r) = 1 + a_1 d + a_2 d^2$. Substituting the formula for *r* (Equation (27)) into this expression, we conclude that

$$DU_i \approx t_i^f \left[1 + a_1 \alpha \left(\frac{v_i}{c_i} \right)^{\beta} + a_2 \alpha^2 \left(\frac{v_i}{c_i} \right)^{2\beta} \right] \tag{33}$$

Equation (33) can also be expressed as

$$DU_i = t_i^f \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^{\beta} + (a_1 - 1)\alpha \left(\frac{v_i}{c_i} \right)^{\beta} + a_2 \alpha^2 \left(\frac{v_i}{c_i} \right)^{2\beta} \right]$$

$$= \bar{t}_i + t_i^f \left[(a_1 - 1)\alpha \left(\frac{v_i}{c_i} \right)^{\beta} + a_2 \alpha^2 \left(\frac{v_i}{c_i} \right)^{2\beta} \right] \tag{34}$$

Hence, we may view DU_i as consisting of two components: the deterministic component $\bar{t}_i$ and stochastic component $t_i^f [...]$. The stochastic component is due to the uncertainty in the link travel time in drivers' route choice process.

Our derivation (Equations (29) and (30)) has already ensured that the DU_i has property P2. Namely using the same logic as in Equation (24), it can also be shown that Equation (34) satisfies property P2.

8. Route Choice Behavior in Stochastic Networks

We have already shown that the property P1 is satisfied. In terms of our function DU_i, this property means that the driver preferences should be equivalent to selecting a route with the smallest value of the sum $du_r = \sum_{i=1}^{L} DU_i$.

Thus, the Equation (34) provides a convenient way of solving the SN-DUE model using the Frank-Wolf algorithm, provided that DU_i is a convex function of v_i. This property of DU_i

holds e.g. when $a_1 \geq 0$ and $a_2 \geq 0$; in the following text we will show that this is the case e.g. for risk averse drivers, drivers typical for the morning rush hours. The coefficients a_1 and a_2 can be estimated from user surveys, which will reflect the user behavior in response to uncertain link travel times.

Therefore, we can treat the SN-DUE model like a DN-DUE model simply by replacing the t_i and t_r in the DN-DUE model by DU_i and du_r respectively. In fact, we only need to replace t_i by DU_i in the solution algorithm!

8.1 Risk Averse Behavior

In section 4.2, we have shown that for risk averse behavior, the equivalent link disutility function has the form in Equation (16). By comparing this expression with Equation (34), we conclude that

$$\sigma_{t_i}^2 = t_i^f \frac{2}{\omega}\left[(a_1 - 1)\alpha\left(\frac{v_i}{c_i}\right)^{\beta} + a_2\alpha^2\left(\frac{v_i}{c_i}\right)^{2\beta}\right] \tag{35}$$

Therefore, if we assume that the equivalent link disutility function satisfies P2 then the variance $\sigma_{t_i}^2$ must have the form of Equation (35). The variance is always non-negative, $\sigma_{t_i}^2 \geq 0$, so the coefficients a_1 and a_2 must be such that

$$a_1 + a_2\alpha\left(\frac{v_i}{c_i}\right)^{\beta} \geq 1 \tag{36}$$

Note that the final expression for the equivalent link disutility function DU_i in Equation (33) contains only two new parameters a_1 and a_2. These parameters take into account both the dependence of the variance $\sigma_{t_i}^2$ and the parameter ω that describes the drivers' behavior.

8.2 Risk Prone Behavior

In this case, to ensure that Equation (23) from section 4.4 satisfies P2 and P3, we compare Equations (23) and (34) and deduce that

$$\sigma_{t_i}^2 = -t_i^f \frac{2}{\varphi}\left[(a_1 - 1)\alpha\left(\frac{v_i}{c_i}\right)^{\beta} + a_2\alpha^2\left(\frac{v_i}{c_i}\right)^{2\beta}\right] \tag{37}$$

To ensure that $\sigma_{t_i}^2 \geq 0$, we must have

$$a_1 + a_2\alpha\left(\frac{v_i}{c_i}\right)^{\beta} \leq 1 \tag{38}$$

Note that, even if the combination of a_1 and a_2 values satisfies Equation (38), it does not guarantee that Equation (37) is monotonically increasing function of v_i. To satisfy P3, the rate

of increase in $\left(\varphi/2\right)\sigma_{t_i}^2$ term Equation (37) with respect to v_i must be relatively small compared to the rate of increase of $\bar{t}_i$ with respect to v_i.

When the expression for DU_i is convex, we can use the Frank-Wolf algorithm to solve the corresponding traffic assignment problem. For strongly risk prone behavior with large φ, the function DU_i is no longer convex. However, in this paper, following Tatineni et al. (1997) and Chen and Recker (2000), we assume that all drivers in a network follow the same route choice behavior. Under this assumption, the coefficients in Equation (23) represent the average behavior in the network. Therefore the extremely risk prone behavior is highly unlikely and thus the actual expression for DU_i should be convex.

9. Expression for the Variance of Link Travel Time: Alternative Derivation

In the previous text, we have first derived the formulas for equivalent link disutility functions, and then used these formulas to derive the expressions of the variance of link travel time. Let us show that we can derive this expression for the variance directly.

It is natural to assume that the variance $\sigma_{t_i}^2$ of the travel time on link *i* should depend on the free-flow travel time t_i^f and the relative average delay $d = \left(\bar{t}_i - t_i^f\right)/t_i^f$, i.e.,

$$\sigma_{t_i}^2 = f\left(t_i^f, d\right) \tag{39}$$

(where d is as defined in Equation (27)) for some function $f\left(t_i^f, d\right)$. So, to describe an expression for the variance, we must find the appropriate function $f\left(t_i^f, d\right)$. As we have mentioned, in SN, it is usually assumed that link travel times are independent random variables. If we sub-divide a link into two sub-links with free-flow travel times $t_{i_1}^f$ and $t_{i_2}^f$ respectively, then $v_{i_1} = v_{i_2} = v_i$, $c_{i_1} = c_{i_2} = c_i$, and the relative average delay d for both sub-links is the same as for the original link. It is known that the variance of the sum $t_i^f = t_{i_1}^f + t_{i_2}^f$ of two independent random variables is equal to the sum of the corresponding variances $\sigma_{t_i}^2 = \sigma_{t_{i_1}}^2 + \sigma_{t_{i_2}}^2$. Thus

$$f\left(t_{i_i}^f + t_{i_2}^f, d\right) = f\left(t_{i_1}^f, d\right) + f\left(t_{i_2}^f, d\right) \tag{40}$$

Similarly to section 4, we thus conclude that

$$\sigma_{t_i}^2 = t_i^f g(d) \tag{41}$$

for some function $g(d)$. Since $d << 1$, by expanding $g(d)$ into a Taylor series, and ignoring the terms with d^3 and higher order, we get

$$\sigma_{t_i}^2 = t_i^f \left[a_0' + a_1'd + a_2'd^2 + ...\right] \approx t_i^f \left[a_0' + a_1'd + a_2'd^2\right] \tag{42}$$

In the absence of traffic flow, when $\bar{t}_i = t_i^f$ and d=0, we have $\sigma_{t_i}^2$=0; therefore we get a_0'=0. Thus

$$\sigma_{t_i}^2 = t_a^f \left[a_1' d + a_2' d^2 \right] \tag{43}$$

Substituting this expression into Equation (4), and ignoring the terms of d^3 and higher order, we get

$$DU_i \approx \bar{t}_i + t_i^f \left[\frac{1}{2} a_1' cd + \frac{1}{2} a_2' cd^2 + \frac{1}{3} (a_1')^2 c^2 d + \frac{2}{3} a_1' a_2' d^2 \right] \tag{44}$$

This expression can be rewritten as

$$DU_i = \bar{t}_i + t_i^f \left[a_1'' d + a_2'' d^2 \right]$$

$$= \bar{t}_i + t_i^f \left[a_1'' \alpha \left(\frac{v_i}{c_i} \right)^{\beta} + a_2'' \alpha^2 \left(\frac{v_i}{c_i} \right)^{2\beta} \right] \tag{45}$$

We can see that Equation (45) is indeed equivalent to Equation (34) (if one uses the appropriate matching coefficients). So, for risk averse drivers, our derivation of the expression for the equivalent link disutility function (Equation (34)) indeed applies to the empirical (Gamma) link travel time distribution. For risk prone drivers, similar derivation leads to the same conclusion.

10. Summary

At present, in practice, most traffic assignment tasks are performed by using deterministic network (DN) models, which assume that the link travel time is uniquely determined by the link volume and link capacity. In reality, for the same link volume and link capacity, a link may have different travel times. Stochastic network (SN) models, which take this difference into account, provide a more accurate description of driver behavior than the DN models. However, these models are not widely used because they are much more computationally complex than the DN models.

In the past research, it was shown that in the important particular case, when the link travel time follows Gamma distribution, the traffic assignment problem for SN can be reformulated in terms of deterministic equivalent link disutility function. Thus in this case the traffic assignment can be solved by the standard Frank-Wolfe algorithm.

In this paper, we show that a similar equivalent link disutility function exists in the general case of an arbitrary distribution of link travel time. Therefore, we can use the Frank-Wolfe algorithm in the general SN case. We also provide an explicit expression for this equivalent link disutility function in terms of the link volume and link capacity. This expression has two components: a deterministic component that accounts for the average link travel time (based on the well-known BPR function), and a stochastic component which is proportional to the variance of link travel time. This expression is independent of the resolution of the network representation.

The general expression for the equivalent link disutility function covers different risk taking behavior of the drivers. For risk averse drivers, the stochastic component in the link disutility function is linearly proportional to the link travel time variance. For risk prone drivers, the stochastic component in the link disutility function is linearly proportional to the negative value of the link travel time variance. For these two types of route choice behaviors, we have specified the constraints of the coefficients in the stochastic component of the link disutility function under which the traffic assignment problem can be solved by the Frank-Wolfe algorithm. We present arguments that these constraints are satisfied in real life.

Our work in this paper provides the justification for the use of a general equivalent link disutility function (which can be seen as an extension of the BPR function) so that we can solve the SN problem using the same approach as in the DN model.

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