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# Equidecomposability (scissors congruence) of polyhedra in R^3 and R^4 is algorithmically decidable: Hilbert's 3rd Problem revisited

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# EQUIDECOMPOSABILITY (SCISSORS CONGRUENCE) OF POLYHEDRA IN $\mathbb{R}^3$ AND $\mathbb{R}^4$ IS ALGORITHMICALLY DECIDABLE: HILBERT'S 3rd PROBLEM REVISITED

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**Hilbert's third problem: brief reminder.** It is known that in a plane, every two polygons $P$ and $P'$ of equal area $A(P) = A(P')$ are *scissors congruent* (*equidecomposable*) – i.e., they can be both decomposed into the same finite number of pair-wise congruent polygonal pieces: $P = P_1 \cup \ldots \cup P_p$, $P' = P'_1 \cup \ldots \cup P'_p$, and $P_i \sim P'_i$.

In one of the 23 problems that D. Hilbert formulated in 1900 as a challenge to the 20 century mathematics, namely, in Problem No. 3, Hilbert asked whether every two polyhedra $P$ and $P'$ with the same volume $V(P) = V(P')$ are equidecomposable (Hilbert 1900).

This problem was the first to be solved: already in 1900, Dehn proved (Dehn 1900, Dehn 1901) that there exist a tetrahedron of volume 1 which is not scissors congruent with a unit cube. He proved it by showing that for every additive function $f : \mathbb{R} \to \mathbb{R}$ from real numbers to real numbers for which $f(2\pi) = 0$, the expression $D_f(P) \stackrel{\text{def}}{=} \sum\limits_{i=1}^{n} \ell_i \cdot f(\alpha_i)$ is invariant under scissors congruence, where $\ell_i$ is the length of the $i$-th edge, and $\alpha_i$ is the (dihedral) angle between the corresponding faces.

Sydler (Sydler 1965) proved that, vice versa, if two polyhedra $P$ and $P'$ have the same volume and the same Dehn invariants $D_f(P) = D_f(P')$ for all $f(x)$, then $P$ and $P'$ scissors congruent. This proof has been later simplified and generalized; see, e.g., (Boltianskii 1978), (Neumann 1998), (Kellerhals 2000), and (Benko 2007). A similar result is known for polyhedra in $\mathbb{R}^4$; see, e.g., (Boltianskii 1978).

**A natural algorithmic question.** Let us consider polyhedra which can be constructed by geometric constructions. It is well known that for such polyhedra, all vertices have *algebraic* coordinates (i.e., values which are roots of polynomials with integer coefficients); see, e.g., (Courant and Robbins 1996).

A natural question arises: is there an *algorithm* for checking whether two given polyhedra with algebraic coordinates are scissors congruent? This question was raised, e.g., in (Mohanty 2004).

**This question is non-trivial.** At first glance (see, e.g., (Boltianskii 1978)), the answer to the above question is straightforward: let us compute and compare Dehn invariants. However, this idea does not automatically lead to an algorithm:

- first, there are infinitely many additive functions $f(x)$ – hence, infinitely many Dehn invariants, and
- second, for each of these functions $f(x)$, we must check equality of real numbers – and in general, check equality of real numbers is not algorithmically possible; see, e.g., (Turing 1937, Kreinovich et al. 1998).

To be more precise, checking equality of *algebraic* numbers is decidable; see, e.g., (Tarski 1951, Mishra 1997, Basu et al. 2006). However, the angles $\alpha_i$ are not, in general, algebraic numbers: however, their sines and cosines are algebraic numbers. It is known that adding trigonometric functions like $\sin(x)$ to elementary geometry enables us to describe integers (as values for which $\sin(\pi \cdot x) = 0$) and thus, we get first order arithmetic and (due to Gödel's theorem) undecidability.

**Towards an algorithm for checking equidecomposability.** Let us denote the sides and angles describing $P$ by $\ell_1, \ldots, \ell_n$, $\alpha_1, \ldots, \alpha_n$, and the sides and angles describing $P'$ by $\ell'_1, \ldots, \ell'_{n'}$, $\alpha'_1, \ldots, \alpha'_{n'}$. To describe values of all possible additive functions $f(x)$ on the angles $\alpha_i$ and $\alpha'_i$, it is sufficient to find a $2\pi$-*basis* $B = \{\beta_1, \ldots, \beta_m\} \subseteq$

$A \stackrel{\text{def}}{=} \{\alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_{n'}\}$ of the set $A$ over rational numbers $Q$, i.e., a set $B$ whose union with $\{2\pi\}$ is $Q$-linearly independent, and for which $\alpha_i = r_{i0} \cdot (2\pi) + \sum\limits_{j=1}^{m} r_{ij} \cdot \beta_j$ and $\alpha'_i = r'_{i0} \cdot (2\pi) + \sum\limits_{j=1}^{m} r'_{ij} \cdot \beta_j$ for some rational numbers $r_{ij}$ and $r'_{ij}$. The values $f(\alpha_i)$ and $f(\alpha'_i)$ of an arbitrary additive function $f(x)$ are uniquely determined by its values $f(\beta_j)$, so it is sufficient to consider $m$ functions $f_1(x), \ldots, f_m(x)$ for which $f_j(\beta_j) = 1$ and $f_j(\beta_{j'}) = 0$ for all $j' \neq j$. For each function $f_j(x)$, the equality of Dehn invariants means that $\sum\limits_{i=1}^{n} \ell_i \cdot r_{ij} = \sum\limits_{i=1}^{n'} \ell'_i \cdot r'_{ij}$. Values on both sides of this equality are rational combinations of algebraic numbers, hence these values are themselves algebraic numbers, and we have already mentioned that checking equality of algebraic numbers is algorithmically decidable.

Thus, to algorithmically check equidecomposability, it is sufficient to be able to algorithmically perform two things:

- find a $2\pi$-basis $B \subseteq A$, and
- find the corresponding coefficients $r_{ij}$ and $r'_{ij}$.

To find the basis, let us first check whether the set $A \cup \{2\pi\}$ is $Q$-linearly independent. This is equivalent to checking whether

$$r_0 \cdot (2\pi) + \sum_{i=1}^{n} r_i \cdot \alpha_i + \sum_{i=1}^{n'} r'_i \cdot \alpha'_i = 0$$

for some rational values $r_i$, i.e., equivalently, to checking whether $k_0 \cdot (2\pi) + \sum\limits_{i=1}^{n} k_i \cdot \alpha_i + \sum\limits_{i=1}^{n'} k'_i \cdot \alpha'_i = 0$ for some integers $k_i$. This, in turn, is equivalent to checking whether $\prod\limits_{i=1}^{n} z_i^{k_i} \cdot \prod\limits_{i=1}^{n'} (z'_i)^{k'_i} = 1$ for some integers $k_i$ and $k'_i$, where $z_i \stackrel{\text{def}}{=} \exp(\mathrm{i} \cdot \alpha_i) = \cos(\alpha_i) + \mathrm{i} \cdot \sin(\alpha_i)$. We have already mentioned that the values $z_i$ and similar values $z'_i$ are algebraic numbers, hence we can use the algorithm from (Ge 1993)

(see also (Ge 1994, Babai et al. 1995, Buchmann and Eisenbrand 1999, Derksen et al. 2005)) to check whether such integers $k_i$ and $k_i'$ exist.

If $A$ is not a $2\pi$-basis, then we delete elements of $A$ one by one and check – until we find a $2\pi$-basis. There are only finitely many subsets of $A$, so we can algorithmically try them all.

Once we find a $2\pi$-basis $B$, we must algorithmically find, for every $i$, the values $r_{ij}$ for which $\alpha_i = r_{i0}\cdot(2\pi)+\sum\limits_{j=1}^{m} r_{ij}\cdot\beta_j$, i.e., equivalently, the integers $k$, $k_0$, and $k_j$ for which $k\cdot\alpha_i = k_0\cdot(2\pi)+\sum\limits_{j=1}^{m} k_j\cdot\beta_j$. This condition is equivalent to $z_i^k = \prod\limits_{j=1}^{m} t_j^{k_j}$, where $z_i$ and $t_j \stackrel{\rm def}{=} \exp(\mathrm{i}\cdot\beta_j)$ are algebraic numbers. We know that $B$ is a $2\pi$-basis, so the desired combination of $k$, $k_0$, and $k_j$ exists. For each combination, checking the above condition simply means checking the equality of two algebraic numbers; thus, we can find $k$, $k_0$, and $k_j$ – and hence, $r_{ij} = k_j/k_0$ – by simply testing all possible combinations.

Thus, we indeed have an algorithm for checking whether two given polyhedra $P$ and $P'$ are equidecomposable.

**Auxiliary question: how can we find the actual scissor transformation?** Once we know that the polyhedra are scissors congruent, how can we find the actual scissor transformations?

Sydler's proof is not fully constructive; see, e.g., (Mohanty 2004). Let us show that there exists an algorithm which is applicable to any two polyhedra $P$ and $P'$ (with algebraic coordinates of its vertices) that are known to be scissors congruent, and which produces the corresponding scissor operations.

**Tarski's theory of elementary geometry: the basis for the desired algorithm.** The existence of our algorithm easily follows from the fact that many related formulas can be described in the *first order theory of real numbers* (also known as *elementary geometry*): a theory in which :

- *variables* run over real numbers,

- *terms* $t$, $t'$ re obtained from variables by addition and multiplications,
- *elementary formulas* are of the type $t = t'$, $t < t'$, $t \leq t'$, and
- arbitrary formulas are constructed from the elementary ones by adding logical connectives $\&$ ("and"), $\vee$ ("or"), $\neg$ ("not"), and quantifiers $\forall x$ and $\exists x$.

A well-known result by Tarski is that this theory of decidable, i.e., that there exists an algorithm which, given a formula from this language, decides whether this formula holds (Tarski 1951). The original Tarski's algorithm required an unrealistically large amount of computation time; however, later, faster algorithms have been invented; see, e.g., (Mishra 1997, Basu et al. 2006).

For existential formulas of the type $\exists x_1 \ldots \exists x_m\, F(x_1, \ldots, x_m)$, these algorithms (e.g., algorithms based on the Cylindrical Algebraic Decomposition ideas) not only return "true" or "false " – when the formula is true, they actually return the values $x_i$ which make the formula $F(x_1, \ldots, x_m)$ true.

*Comment.* The theoretical possibility of algorithmically returning such values $x_i$ comes from the known fact that every formula of elementary geometry holds in $\mathbb{R}$ if and only if it holds in the field of all algebraic numbers. Thus, if the above existential formula holds, there exist algebraic values $x_i$ for which $F(x_1, \ldots, x_m)$ is true. We can constructively enumerate all algebraic numbers, and for each combination $x_i$ of these numbers, we can check whether $F(x_1, \ldots, x_m)$ holds for this combination – until we find the values for which $F(x_1, \ldots, x_m)$ holds.

It is worth mentioning that this "exhaustive search" simply explains the theoretical possibility: the actual algorithms described in (Mishra 1997, Basu et al. 2006) are much faster.

**How we can apply the elementary geometry algorithms and find the scissor operations.** Let us show how the elementary geometry algorithms can be applied to find the scissor operations which transform given polyhedra $P$ and $P'$ into one another.

Each polyhedron can be decomposed into tetrahedra. So, without losing generality, we can assume that $P$ can be decomposed into tetrahedra which can be then moved one-by-one and reassembled into $P'$.

We say that $P$ and $P'$ are *$n$-scissors congruent* if they can be both decomposed into finitely many pair-wise congruent tetrahedra in such a way that a total number of all the vertices of all these tetrahedra does not exceed $n$.

Let's fix a coordinate system. Then a tetrahedron is described by the coordinates of its 4 vertices, i.e., by 12 (finitely many) real numbers. Let's denote these 12 numbers by a 12-dimensional vector $\vec{x}$. The coordinates of the vertices of the tetrahedra which form a decomposition are also real numbers (no more than $3n$ of them, because there are no more than $n$ vertices). Congruence can be expressed as equality of all the distances, which, in its turn, is equivalent to equality of their squares. So, it is expressible by an elementary formula of elementary geometry.

The fact that $P$ and $P'$ are $n$-scissors congruent (we will denote it by $s_n(P, P')$) means that there exist a decomposition of $P$ and a decomposition of $P'$ which are pair-wise congruent. Therefore, $s_n(P, P')$ is constructed from elementary formulas by applying existential quantifiers: $s_n(P, P') \leftrightarrow \exists x_1 \ldots \exists x_{m_n} F_n$. Hence, $s_n(P, P')$ is also a formula of elementary geometry. Thus, for every $n$, we can check whether $s_n(P, P')$ holds or not; see, e.g., (Kreinovich and Kosheleva 1994).

If we know that $P$ and $P'$ are scissors congruent, this means that they are $n$-scissors congruent for some $n$. We can thus (algorithmically) check the formula $s_1(P, P')$, $s_2(P, P')$, $\ldots$, until we find the first $n$ for which the existential formula $s_n(P, P') \equiv \exists x_1 \ldots \exists x_{m_n} F_n(x_1, \ldots, x_{m_n})$ holds.

For this $n$, as we have mentioned, the elementary geometry algorithms also return the values $x_i$ for which $F_n(x_1, \ldots, x_{m_n})$ holds, i.e., the coordinates of the tetrahedra which form the desired decompositions of $P$ and $P'$. The statement is proven.

**The above algorithms are not very fast, but it is OK because an algorithm for finding scissor operations cannot be very fast.** It is known that algorithms for deciding elementary geometry cannot be very fast: e.g., for purely existential formulas like the ones we use, there is an exponential lower bound $a^n$ for the number of computational steps.

However, this is OK, because a similar exponential lower bound exists for constructing scissor transformations – even for polygons; see, e.g., (Kosheleva and Kreinovich 1994).

**Remaining open problems.** Can we always check equidecomposability in exponential time? When can we check it faster? Can we translate Sydler's construction into faster algorithms? What happens in higher dimensions? in spherical and hyperbolic spaces?

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