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Why Tensors?

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Abstract. We show that in many application areas including soft constraints reasonable requirements of scale-invariance lead to polynomial (tensor-based) formulas for combining degrees (of certainty, of preference, etc.)

Partial orders naturally appear in many application areas. One of the main objectives of science and engineering is to help people select decisions which are the most beneficial to them. To make these decisions,

- we must know people's preferences,
- we must have the information about different events – possible consequences of different decisions, and
- since information is never absolutely accurate and precise, we must also have information about the degree of certainty.

All these types of information naturally lead to partial orders:

- For preferences, $a < b$ means that b is preferable to a. This relation is used in *decision theory*; see, e.g., [1].
- For events, $a < b$ means that a can influence b. This causality relation is used in *space-time physics*.
- For uncertain statements, $a < b$ means that a is less certain than b. This relation is used in logics describing uncertainty such as fuzzy logic (see, e.g., [3]) and in soft constraints.

Numerical characteristics related to partial orders. While an order may be a natural way of describing a relation, orders are difficult to process, since most data processing algorithms process numbers. Because of this, in all three application areas, numerical characteristics have appeared that describe the corresponding orders:

- in decision making, *utility* describes preferences:
$$a < b \text{ if and only if } u(a) < u(b);$$
- in space-time physics, *metric* (and time coordinates) describes causality relation;
- in logic and soft constraints, numbers from the interval $[0, 1]$ are used to describe degrees of certainty; see, e.g., [3].

Need to combine numerical characteristics, and the emergence of polynomial aggregation formulas.

- In decision making, we need to combine utilities $u_1, \ldots, u_n$ of different participants. Nobelist Josh Nash showed that reasonable conditions lead to $u = u_1 \cdot \ldots \cdot u_n$; see, e.g., [1, 2].
- In space-time geometry, we need to combine coordinates x_i into a metric; reasonable conditions lead to polynomial metrics such as Minkowski metric in which

$$s^2 = c^2 \cdot (x_0 - x'_0)^2 - (x_0 - x'_0)^2 - (x_1 - x'_1)^2 - (x_2 - x'_2)^2 - (x_3 - x'_3)^2$$

  and of a more general Riemann metric where $ds^2 = \sum\limits_{i,j} g_{ij} \cdot dx^i \cdot dx^j$.
- In fuzzy logic and soft constraints, we must combine degrees of certainty d_i in A_i into a degree d for $A_1 \,\&\, A_2$; reasonable conditions lead to polynomial functions like $d = d_1 \cdot d_2$.

In mathematical terms, polynomial formulas are tensor-related. In mathematical terms, a general polynomial dependence

$$f(x_1, \ldots, x_n) = f_0 + \sum_{i=1}^{n} f_i \cdot x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} \cdot x_i \cdot x_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f_{ijk} \cdot x_i \cdot x_j \cdot x_k + \ldots$$

means that to describe this dependence, we need a finite collection of tensors f_0, f_i, f_{ij}, f_{ijk}, ..., of different arity.

Towards a general justification of polynomial (tensor) formulas. The fact that similar polynomials appear in different application areas indicates that there is a common reason behind them. In this paper, we provide such a general justification.

We want to find a finite-parametric class F of analytical functions $f(x_1, \ldots, x_n)$ approximating the actual complex aggregation. It is reasonable to require that this class F be invariant with respect to addition and multiplication by a constant, i.e., that it is a (finite-dimensional) linear space of functions.

The invariance with respect to multiplication by a constant corresponds to the fact that the aggregated quantity is usually defined only modulo the choice of a measuring unit. If we replace the original measuring unit by a one which is λ times smaller, then all the numerical values get multiplied by this factor λ: $f(x_1, \ldots, x_n)$ is replaced with $\lambda \cdot f(x_1, \ldots, x_n)$.

Similarly, in all three areas, the numerical values x_i are defined modulo the choice of a measuring unit. If we replace the original measuring unit by a one which is λ times smaller, then all the numerical values get multiplied by this factor λ: x_i is replaced with $\lambda \cdot x_i$. It is therefore reasonable to also require that the finite-dimensional linear space F be invariant with respect to such re-scalings, i.e., if $f(x_1, \ldots, x_n) \in F$, then for every $\lambda > 0$, the function

$$f_\lambda(x_1, \ldots, x_n) \stackrel{\text{def}}{=} f(\lambda \cdot x_1, \ldots, \lambda \cdot x_n)$$

also belongs to the family F.

Under this requirement, we prove that all elements of F are polynomials.

Definition 1. *Let n be an arbitrary integer. We say that a finite-dimensional linear space F of analytical functions of n variables is* scale-invariant *if for every $f \in F$ and for every $\lambda > 0$, the function*

$$f_\lambda(x_1, \ldots, x_n) \stackrel{\text{def}}{=} f(\lambda \cdot x_1, \ldots, \lambda \cdot x_n)$$

also belongs to the family F.

Main result. *For every scale-invariant finite-dimensional linear space F of analytical functions, every element $f \in F$ is a polynomial.*

Proof. Let F be a scale-invariant finite-dimensional linear space F of analytical functions, and let $f(x_1, \ldots, x_n)$ be a function from this family F.

By definition, an analytical function $f(x_1, \ldots, x_n)$ is an infinite series consisting of monomials $m(x_1, \ldots, x_n)$ of the type

$$a_{i_1 \ldots i_n} \cdot x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}.$$

For each such term, by its *total order*, we will understand the sum $i_1 + \ldots + i_n$. The meaning of this total order is simple: if we multiply each input of this monomial by λ, then the value of the monomial is multiplied by λ^k:

$$m(\lambda \cdot x_1, \ldots \lambda \cdot x_n) = a_{i_1 \ldots i_n} \cdot (\lambda \cdot x_1)^{i_1} \cdot \ldots \cdot (\lambda \cdot x_n)^{i_n} =$$

$$\lambda^{i_1 + \ldots + i_n} \cdot a_{i_1 \ldots i_n} \cdot x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} = \lambda^k \cdot m(x_1, \ldots, x_n).$$

For each order k, there are finitely many possible combinations of integers $i_1, \ldots, i_n$ for which $i_1 + \ldots + i_n = k$, so there are finitely many possible monomials of this order. Let $P_k(x_1, \ldots, x_n)$ denote the sum of all the monomials of order k from the series describing the function $f(x_1, \ldots, x_n)$. Then, we have

$$f(x_1, \ldots, x_n) = P_0 + P_1(x_1, \ldots, x_n) + P_2(x_1, x_2, \ldots, x_n) + \ldots$$

Some of these terms may be zeros – if the original expansion has no monomials of the corresponding order. Let k_0 be the first index for which the term $P_{k_0}(x_1, \ldots, x_n)$ is not identically 0. Then,

$$f(x_1, \ldots, x_n) = P_{k_0}(x_1, \ldots, x_n) + P_{k_0+1}(x_1, x_2, \ldots, x_n) + \ldots$$

Since the family F is scale-invariant, it also contains the function

$$f_\lambda(x_1, \ldots, x_n) = f(\lambda \cdot x_1, \ldots, \lambda \cdot x_n).$$

At this re-scaling, each term P_k is multiplied by λ^k; thus, we get

$$f_\lambda(x_1, \ldots, x_n) = \lambda^{k_0} \cdot P_{k_0}(x_1, \ldots, x_n) + \lambda^{k_0+1} \cdot P_{k_0+1}(x_1, x_2, \ldots, x_n) + \ldots$$

Since F is a linear space, it also contains a function

$$\lambda^{-k_0} \cdot f_\lambda(x_1, \ldots, x_n) = P_{k_0}(x_1, \ldots, x_n) + \lambda \cdot P_{k_0+1}(x_1, x_2, \ldots, x_n) + \ldots$$

Since F is finite-dimensional, it is closed under turning to a limit. In the limit $\lambda \to 0$, we conclude that the term $P_{k_0}(x_1, \ldots, x_n)$ also belongs to the family F.

Since F is a linear space, this means that the difference

$$f(x_1, \ldots, x_n) - P_{k_0}(x_1, \ldots, x_n) =$$

$$P_{k_0+1}(x_1, x_2, \ldots, x_n) + P_{k_0+2}(x_1, x_2, \ldots, x_n) + \ldots$$

also belongs to F. If we denote, by k_1, the first index $k_1 > k_0$ for which the term $P_{k_1}(x_1, \ldots, x_n)$ is not identically 0, then we can similarly conclude that this term $P_{k_1}(x_1, \ldots, x_n)$ also belongs to the family F, etc.

We can therefore conclude that for every index k for which term $P_k(x_1, \ldots, x_n)$ is not identically 0, this term $P_k(x_1, \ldots, x_n)$ also belongs to the family F.

Monomials of different total order are linearly independent. Thus, if there were infinitely many non-zero terms P_k in the expansion of the function $f(x_1, \ldots, x_n)$, we would have infinitely many linearly independent function in the family F – which contradicts to our assumption that the family F is a finite-dimensional linear space.

So, in the expansion of the function $f(x_1, \ldots, x_n)$, there are only finitely many non-zero terms. Hence, the function $f(x_1, \ldots, x_n)$ is a sum of finitely many monomials – i.e., a polynomial.

The statement is proven.

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