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A New Justification of Wang Transform Operator in Financial Risk Analysis

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Abstract

One of the most widely used (and most successful) methods for pricing financial and insurance instruments under risk is the Wang transform method. In this paper, we provide a new explanation for the empirical success of Wang's method – by providing a new simpler justification for the Wang transform.

Pricing under risk: a problem. The price of most financial instruments unpredictably (randomly) fluctuates. As a result of these unpredictable fluctuations, investing in a financial instrument is risky: there is a always a possibility that the price of this particular instrument will go down, and as a result, the investors will suffer losses.

In many practical situations, based on the prior performance of a financial instrument (and on the additional information that we may have), we can get a pretty good understanding of the probabilities of different future prices. In other words, we usually know the probability distribution of the future price X. This distribution can be described, e.g., by describing the probabilities that the future price does not go below a certain threshold, i.e., by the *decumulative distribution function*

$$S(x) \stackrel{\text{def}}{=} \text{Prob}(X > x). \tag{1}$$

Alternatively, we can use the cumulative distribution function

$$F(x) \stackrel{\text{def}}{=} \text{Prob}(X \leq x) = 1 - S(x). \tag{2}$$

Once we know this probability distribution, what is a reasonable price of this financial instrument?

First approximation to pricing under risk: a straightforward application of the traditional probability approach. A seemingly natural approach to pricing can be obtained from a straightforward application of the traditional probability approach; see, e.g., [11]. If we invest in similar financial instruments, with a similar probability distribution, again and again, then on average, due to the Law of Large Numbers, our average gain per investment

$$\frac{X_1 + \ldots + X_n}{n} \tag{3}$$

will be equal to the mean (expected) value $E[X]$ of the corresponding random variable X. Thus, from the traditional probability approach, it seems reasonable to use this mean value as the fair price of the financial instrument.

In terms of the decumulative distribution function $S(x)$, this mean $E[X]$ has the form

$$E[X] = \int_L^\infty \text{Prob}(X > x)\,dx + L = \int_L^\infty S(x)\,dx + L, \tag{4}$$

where L is the smallest possible price. Usually, we take $L = 0$, so the formula (4) takes the simplified form

$$E[X] = \int_0^\infty \text{Prob}(X > x)\,dx = \int_0^\infty S(x)\,dx. \tag{5}$$

Comment. Strictly speaking, according to decision theory [3, 4, 7, 9, 10], we must take the mean not of the future price itself, but of the *utility* corresponding to the future price. This difference is important because people's utilities are non-linearly related to their gains and losses. So, strictly speaking, instead of the distribution of the prices, we should start with the corresponding distribution of utilities. For simplicity, in the future, we will simply talk about future prices (gains, etc.), but all our considerations are applicable to utilities as well.

Need to go beyond the first approximation. According to the above straightforward application of the traditional probability approach, the fair price of the financial instrument should be equal to the expected value of the future price. In real life, however, most people prefer to get, e.g., a \$1 to a situation in which, with probability 1/2, we get \$2 and with probability 1/2, we get 0.

One may claim that this preference is caused by the non-linearity of the corresponding utility function, but the same preference can be observed if we use utils (units of utility) instead of dollars:

- On the one hand, theoretically, getting 1 util with certainty should be equivalent to getting 2 units with probability 1/2 and 0 utils with probability 1/2.

- However, in practice, in this choice, most people prefer 1 util with certainty.

In other words, risk decreases the fair price of a financial instrument: the more risk, the smaller the price.

For insurance instruments, a similar effect of risk moves the price in the opposite direction: we need to pay extra insurance premium for risk: the larger the risk, the larger this premium and thus, the larger the resulting price.

How can we describe the resulting fair pricing under risk?

Wang transform: an efficient empirical method for pricing under risk. In his papers [13, 14], S. S. Wang proposed to replace the expression (5) with a modified formula

$$\int_0^\infty g(S(x))\,dx, \tag{6}$$

where the *Wang transform* $g(y)$ is defined by the formula

$$g(y) = \Phi(\Phi^{-1}(y) + \alpha), \tag{7}$$

in which α is a constant, and $\Phi(y)$ is the cumulative distribution function of the standard normal distribution (with 0 mean and standard deviation 1).

For financial instruments, when the presence of risk decreases the price, we take $\alpha \le 0$. For estimating insurance premiums, when the presence of risk increases the price, we should take $\alpha \ge 0$.

This method turned out to be highly efficient in practice. As a result, this method (and its modifications and extensions) is now one of the main methods for pricing under risk; see, e.g., [5].

How can we explain this empirical success? There exist several justifications of the Wang transform method, justifications that show that this method is uniquely determined by some reasonable properties. The first justifications was proposed by Wang himself, in [15]; several other justifications are described in [8].

Limitations of this explanation. The main limitations of the existing justifications of the Wang transform is that these justifications are too complicated, too mathematical, and not very intuitive.

What we do in this paper. In this paper, we provide a new, more intuitive justification of the Wang transform.

Difference between subjective and objective probabilities. How can we explain the fact that, in spite of the seeming naturalness of the mean, people's pricing under risk is usually different from the mean $E[X]$?

A natural explanation comes from the observation that the mean $E[X]$ is based on the "objective" probabilities (frequencies) of different events, while people use "subjective" estimates of these probabilities when making decisions. For a long time, researchers thought that subjective probabilities are approximately equal to the objective ones, and often they are equal. However, in 1979, a classical paper by D. Kahneman and A. Tversky ([6], reproduced in [12]) showed that our "subjective" probability of different events is, in general, different from the actual ("objective") probabilities (frequencies) of these events. This difference is especially drastic for events with low probabilities and probabilities close to 1 – i.e., for the events that are most important when estimating risk.

Specifically, Kahnemann and Tversky has shown that there is a one-to-one correspondence $g(y)$ between objective and subjective probabilities, so that once we know the (objective) probability $\mathrm{Prob}(E)$ of an event E, the subjective probability $\mathrm{Prob}_{\mathrm{subj}}(E)$ of this event E is (approximately) equal to

$$\mathrm{Prob}_{\mathrm{subj}}(E) = g(\mathrm{Prob}(E)). \tag{8}$$

This idea was further explored by other researchers; see, e.g., [1] and references therein.

First step of our justification: justification of the Wang-type formula. Let us apply the Kahneman-Tversky idea to the events $X > x$ corresponding to different values x. For such an event E, its objective probability is equal to the decumulative distribution function $S(x)$. Thus, the corresponding subjective probability of this event is equal to

$$\mathrm{Prob}_{\mathrm{subj}}(X > x) = g(S(x)). \tag{9}$$

If we compute the "subjective" mean $E_{\mathrm{subj}}[X]$ of X based on these subjective probabilities, we get

$$E_{\mathrm{subj}}[X] = \int_0^\infty \mathrm{Prob}_{\mathrm{subj}}(X > x)\,dx = \int_0^\infty g(S(x))\,dx. \tag{10}$$

This is exactly the expression (6) for pricing under risk, with the only difference that we do yet have an explicit expression for the transform $g(y)$.

The derivation of Wang's expression (7) for the transform $g(y)$ constitutes the second part of our justification.

Properties of the Wang transform: reminder. Before we start the second step of our justification, let us recall the known properties of the Wang transform [13]:

- The result $g(S(x))$ of applying the Wang transform to the decumulative distribution function $S(x)$ of a normal distribution is also a decumulative distribution function of a normal distribution. This property is very important since in practice, many distributions are normal; see, e.g., [11].

- The result $g(S(x))$ of applying the Wang transform to the decumulative distribution function $S(x)$ of a lognormal distribution is also a decumulative distribution function of a lognormal distribution. This property is also very important because many stock returns and other financial quantities have a lognormal distribution.

Second part of our justification: main idea. Normal distributions are ubiquitous. It is therefore reasonable to require that when we have an (objectively) normal distribution, its subjective perception should also be normal. In other words, it is reasonable to require that the result $g(S(x))$ of applying the transform $g(y)$ to the (decumulative distribution function of a) normal distribution $S(x)$ is also a (decumulative distribution function of a) normal distribution. We will show that the Wang transform is the only function satisfying this property.

In financial applications, lognormal distributions are also frequent. Thus, it is also reasonable to require that when we have an (objectively) lognormal distribution, its subjective perception should also be lognormal. In other words, it is reasonable to require that the result $g(S(x))$ of applying the transform $g(y)$ to a (decumulative distribution function of a) lognormal distribution $S(x)$ is also a (decumulative distribution function of a) lognormal distribution. We will show that the Wang transform is the only function satisfying this property.

These two results will complete our justification of the Wang transform.

Theorem 1. *Let a function $g : [0,1] \to [0,1]$ satisfy the following two properties:*

- *for a decumulative distribution function $S(x)$ of a normal distribution, the function $g(S(x))$ is also a decumulative distribution function of a normal distribution; and*

- *for every decumulative distribution function $S(x)$, we have*

$$\int_0^\infty g(S(x))\,dx \le E[X] = \int_0^\infty S(x)\,dx. \tag{11}$$

Then, $g(y)$ is the Wang transform (7) with $\alpha \le 0$.

Comment. The second property means that because of the possible risk, the fair price is either equal to the mean or smaller than the mean. In other words, it means that when the expected returns are equal, the presence of risk decreases the attractiveness of the financial instrument.

For an *insurance* instrument, we have the opposite inequality: when we estimate the insurance premium, we need to pay extra for the risk. In this case, we have a similar result:

Theorem 2. *Let a function $g : [0, 1] \to [0, 1]$ satisfy the following two properties:*

- *for a decumulative distribution function $S(x)$ of a normal distribution, the function $g(S(x))$ is also a decumulative distribution function of a normal distribution; and*
- *for every decumulative distribution function $S(x)$, we have*

$$\int_0^\infty g(S(x))\,dx \geq E[X] = \int_0^\infty S(x)\,dx. \tag{12}$$

Then, $g(y)$ is the Wang transform (7) with $\alpha \geq 0$.

Proof of Theorem 1. Our formulas deal with decumulative distribution functions, and the first condition of Theorem 1 deals with normal distributions. Because of this, let us first recall how to describe the decumulative distribution functions corresponding to the normal distributions. Let us start with the standard normal distribution, with 0 mean and standard deviation 1. This distribution is symmetric, i.e., its probability density function $\rho(x)$ is even: $\rho(x) = \rho(-x)$. Because of this symmetry, the decumulative distribution function $S_0(x) = 1 - \Phi(x)$ of the standard normal distribution satisfies the property

$$S_0(x) = \text{Prob}(X > x) = \int_x^\infty \rho(t)\,dt =$$

$$\int_{-\infty}^{-x} \rho(t)\,dt = \text{Prob}(X \leq -x) = \Phi(-x). \tag{13}$$

An arbitrary normally distributed random variable X, with the mean a and the standard deviation σ, can be reduced to the standard normal distribution. Specifically, for each normally distributed random variable X, the expression

$$Y \stackrel{\text{def}}{=} \frac{X - a}{\sigma} \tag{14}$$

is also normally distributed, with 0 mean and standard deviation 1. Thus, for this new random variable Y, we have $\text{Prob}(Y > y) = S_0(y)$.

We are interested in the decumulative distribution function $S(x)$ corresponding to the normally distributed random variable X. By definition of the decumulative distribution function, this means that we are interested in the probabilities that $X > x$ for different values x. For an arbitrary real number x, the inequality $X > x$ is equivalent to $X - a > x - a$ and thus, to

$$Y = \frac{X - a}{\sigma} > y \stackrel{\text{def}}{=} \frac{x - a}{\sigma}. \tag{15}$$

Thus, the probability $S(x) = \text{Prob}(X > x)$ is equal to the probability that $Y > y$, i.e., to $S_0(y)$. Substituting $y = \dfrac{x - a}{\sigma}$ into this formula, we conclude that

$$S(x) = S_0\left(\frac{x - a}{\sigma}\right). \tag{16}$$

According to the formulation of Theorem 1, the result $g(S(x))$ of applying the function $g(y)$ to a decumulative distribution function $S(x)$ corresponding to the normal distribution also leads to a function corresponding to the normal distribution. In particular, for the function $S_0(x)$ corresponding to the standard normal distribution, the expression $S(x) = g(S_0(x))$ also corresponds to normal distribution. Expressions $S(x)$ corresponding to a normal distributions have the form (16). Thus, we conclude that

$$g(S_0(x)) = S_0\left(\frac{x - a_0}{\sigma_0}\right) \tag{17}$$

for some values a_0 and σ_0 Since $S_0(x) = \Phi(-x)$, we get

$$g(\Phi(-x)) = \Phi\left(-\frac{x - a_0}{\sigma_0}\right). \tag{18}$$

To find $g(y)$ for a given number y, we first find x for which $\Phi(-x) = y$. In terms of the inverse function Φ^{-1}, this value x takes the form $-x = \Phi^{-1}(y)$, i.e.,

$$x = -\Phi^{-1}(y). \tag{19}$$

Substituting this expression for x into the right-hand side of the formula (18), we conclude that

$$g(y) = \Phi\left(-\frac{-\Phi^{-1}(y) - a_0}{\sigma_0}\right) = \Phi\left(\frac{\Phi^{-1}(y) + a_0}{\sigma_0}\right). \tag{20}$$

To complete the proof, we must show that $\sigma_0 = 1$ and that $a_0 \le 0$. For that, let us use the second property of the function $g(y)$. For every number $y \in [0, 1]$, let us consider a random variable X which is equal to 1 with probability y and to 0 with the remaining probability $1 - y$. The mean value $E[X]$ of this random variable is equal to $y \cdot 1 + (1 - y) \cdot 0 = y$. The subjective probability of $X = 1$ equal to $g(y)$, so the corresponding integral $\int g(S(x))\, dx$ is equal to $g(y)$.

For such random variables, the requirement that the integral is always smaller than or equal to $E[X]$ means that $g(y) \le y$ for all y, i.e., that

$$\Phi\left(\frac{\Phi^{-1}(y) + a_0}{\sigma_0}\right) \le y. \tag{21}$$

For every real value x, this inequality must hold for the value $y = \Phi(x)$, for which $\Phi^{-1}(y) = x$. For this value, (21) turns into

$$\Phi\left(\frac{x + a_0}{\sigma_0}\right) \le \Phi(x). \tag{22}$$

Since the function $\Phi(x)$ is monotonically increasing, the inequality (22) is equivalent to

$$\frac{x + a_0}{\sigma_0} \le x \tag{23}$$

for all x. Multiplying both sides by $\sigma_0 > 0$, we get

$$x + a_0 \le \sigma_0 \cdot x. \tag{24}$$

Moving terms containing x to the right-hand side, we get

$$a_0 \le (\sigma_0 - 1) \cdot x. \tag{25}$$

If $\sigma_0 < 1$, then this inequality is violated for $x \to +\infty$, when its right-hand side tends to $-\infty$. If $\sigma_0 > 1$, then this inequality is violated for $x \to -\infty$, when its right-hand side also tends to $-\infty$. Thus, the only possibility for this inequality to be satisfied for all possible values x is to have $\sigma_0 = 1$.

For $\sigma_0 = 1$, the inequality (23) turns into

$$x + a_0 \le x. \tag{26}$$

After we subtract x from both sides of this inequality, we get the desired inequality $a_0 \le 0$.

The theorem is proven. Thus, the use of the Wang transform has been justified.

Proof of Theorem 2 is similar to the proof of Theorem 1.

Theorem 3. *Let a function $g : [0, 1] \to [0, 1]$ satisfy the following two properties:*

- *for a decumulative distribution function $S(x)$ of a lognormal distribution $S(x)$, the function $g(S(x))$ is also a decumulative distribution function of a lognormal distribution; and*
- *for every decumulative distribution function $S(x)$, we have*

$$\int_0^\infty g(S(x))\,dx \le E[X] = \int_0^\infty S(x)\,dx. \tag{27}$$

Then, $g(y)$ is the Wang transform (7) with $\alpha \le 0$.

Theorem 4. *Let a function $g : [0, 1] \to [0, 1]$ satisfy the following two properties:*

- *for a decumulative distribution function $S(x)$ of a lognormal distribution $S(x)$, the function $g(S(x))$ is also a decumulative distribution function of a lognormal distribution; and*
- *for every decumulative distribution function $S(x)$, we have*

$$\int_0^\infty g(S(x))\,dx \geq E[X] = \int_0^\infty S(x)\,dx. \tag{28}$$

Then, $g(y)$ is the Wang transform (7) with $\alpha \geq 0$.

Proof. The proof of these results is similar to the proof of Theorem 1.

A possible alternative justification. A possible alternative justification comes from the fact that the Wang transform is not a single function: we have a family of transforms $g_\alpha(y)$ corresponding to different values α. Different values of α correspond to different degree of dependence on risk.

These transforms form a *transformation group* in the sense that the following three conditions are satisfied:

- First, the identity transform $g(y) = y$ is a particular case of the Wang transform: specifically, it corresponds to $\alpha = 0$.
- Second, the inverse operation to a Wang transform $g_\alpha(y)$ is also a Wang transform: $g_\alpha^{-1}(y) = g_\beta(y)$ for some β in the sense that

  $$g_\beta(g_\alpha(y)) = g_\beta(g_\alpha(y)) = y \tag{29}$$

  for all y; specifically, this is true for $\beta = -\alpha$.
- Third, the composition of Wang transforms is also a Wang transform: if we first apply a transform $g_\alpha(y)$, and then a transform $g_\beta(y)$, then this is equivalent to applying the Wang transform with a single value γ:

  $$g_\beta(g_\alpha(y)) = g_\gamma(y) \tag{30}$$

  for all y; specifically, this is true for $\gamma = \alpha + \beta$.

It is reasonable to require that we have a family of transformation that forms a 1-parametric group. It is also reasonable to require that the transformations are continuous, and that the dependence on the parameter is also continuous, i.e., that this is a *Lie group*; see, e.g., [2]. It is known that every 1-dimensional Lie group (i.e., a Lie group described by a single parameter) is (at least locally) isomorphic to the additive group of real numbers. In precise terms, this means that instead of using the original values of the parameter v describing different elements of this group, we can use a re-scaled parameter $\alpha = h(v)$ for some non-linear function h, and in this new parameter scale:

- the identity element corresponds to $\alpha = 0$;
- the inverse element to an element with parameter α is an element with parameter $-\alpha$: $g_{-\alpha} = g_\alpha^{-1}$; and
- the composition of elements with parameters α and β is an element with the parameter $\alpha + \beta$: $g_\alpha \circ g_\beta = g_{\alpha+\beta}$.

Let us denote $\Phi(x) \stackrel{\text{def}}{=} g_x(0.5)$. Then, for every $y = \Phi(x)$ and for every real number α, we get

$$g_\alpha(y) = g_\alpha(\Phi(x)) = g_\alpha(g_x(0.5)) = (g_\alpha \circ g_x)(0.5). \tag{31}$$

Because of our choice of parameters, we have

$$g_\alpha \circ g_x = g_{x+\alpha} \tag{32}$$

and thus, the formula (31) takes the form

$$g_\alpha(y) = g_{x+\alpha}(0.5). \tag{33}$$

By definition of the function $\Phi(x)$, this means that

$$g_\alpha(y) = \Phi(x + \alpha). \tag{34}$$

Here, $y = \Phi(x)$, hence $x = \Phi^{-1}(y)$ and thus,

$$g_\alpha(y) = \Phi(\Phi^{-1}(y) + \alpha). \tag{35}$$

This is exactly the Wang transform formula (7) – except that the function $\Phi(x)$ does not necessarily coincide with the cumulative distributive function of the standard normal distribution. (To justify Wang's specific choice of the function $\Phi(x)$, we may want to use our Theorems 1–4.)

Yet another alternative justification. Another alternative justification of the formula (6) comes from the fact that in many cases, an investor is most interested in his ability to have enough cash after a certain period of time. For example, a person who invests as part of his retirement plan, would like to have, at the moment of his or her retirement, enough money to spend on the basics during the retirement.

To describe this idea in precise mathematical terms, let us denote the corresponding minimal return by x. Then, what is most important to the investor is the probability that this return will be attained, i.e., the probability $\text{Prob}(X \geq x)$. This is, in effect, the decumulative distribution function $S(x)$. Thus, from this viewpoint, the utility of the investment to each investor can be characterized by the corresponding value $S(x)$.

Different investors have different thresholds x. So, for different investors, the same financial instrument is characterized by the values $S(x)$ corresponding to different values x. We need to somehow combine these utility values into a single numerical criterion – the price describing this particular financial instrument.

How can we combine different utility values? For example, if we have two investors with utilities a and b, what combination $a * b$ of these values should we use? It is reasonable to assume that this combination does not depend on the order in which we combine these utilities, i.e., in mathematical terms, that the corresponding combination function $a * b$ is commutative ($a * b = b * a$) and associative ($a * (b * c) = (a * b) * c$). It is also reasonable to assume that for each "positive" utility value u, there is a corresponding negative value v for which $u * v = 0$. In other words, it is reasonable to require that the set of real numbers (possible utility values) with the operation $*$ form a *group*.

We have already mentioned that every 1-dimensional Lie group is (at least locally) isomorphic to the group of real numbers. This isomorphism means that there exists a mapping $g(y)$ from the original utility scale to the new scale in which, for every two real numbers a and b, we have

$$g(a * b) = g(a) + g(b), \tag{36}$$

and thus,

$$a * b = g^{-1}(g(a) + g(b)). \tag{37}$$

Similarly, for n utilities $u_1, \ldots, u_n$, we have

$$u_1 * \ldots * u_n = g^{-1}(g(u_1) + \ldots + g(u_n)). \tag{38}$$

If we have n investors, with thresholds $x_1, \ldots, x_n$, then their utilities are equal to $u_1 = S(x_1)$, $\ldots$, $u_n = S(x_n)$ and therefore, the combined utility is equal to

$$u_1 * \ldots * u_n = g^{-1}(g(S(x_1) + \ldots + g(S(x_n))). \tag{39}$$

In real life, we have a large number of investors, with different values x_i, so it is reasonable to approximate the sum by the corresponding integral. As a result, the combined utility is equal to $u = g^{-1}(\int g(S(x))\,dx)$. To price the financial instrument, we must compare it with a simple investment X_v that provides a certain value v with guarantee. In other words, we must find the value v for which

$$g^{-1}\left(\int g(S(x))\,dx\right) = g^{-1}\left(\int g(S_v(x))\,dx\right), \tag{40}$$

where S_v is the decumulative distribution function corresponding to the guaranteed-return investment v. By applying the function $g(y)$ to both sides of the equality (40), we conclude that

$$\int g(S(x))\,dx = \int g(S_v(x))\,dx. \tag{41}$$

For the guaranteed investment, for every x, we have either $S_v(x) = 0$ or $S_v(x) = 1$ and thus, $g(S_v(x)) = S_v(x)$ for all x. So, for this investment, the integral $\int g(S_v(x))\,dx$ is thus equal to $\int S_v(x)\,dx$, i.e., to the mean return of this investment – which is exactly v. Thus,

$$\int g(S_v(x))\,dx = v. \tag{42}$$

So, according to the formula (41), the fair price v of an investment is determined by the formula

$$v = \int g(S(x))\,dx. \tag{43}$$

This is exactly the formula (6) – again, without specifying why the Wang transform is the best; to justify the use of the Wang transform, we can again use Theorems 1–4.

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