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Cantor Sets, Cantorvals, and Their Topological Structure

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to my

FAMILY

with love

CANTOR SETS, CANTORVALS, AND THEIR TOPOLOGICAL STRUCTURE

by

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THESIS

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I love you Anabel!

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Abstract

Abstract. With interesting topological properties, the Cantor set is worth studying for itself. In other areas, topological structures arise that are in fact homeomorphic to the Cantor set. In particular, we see sets that are homeomorphic to the Cantor set which result from the subsums of particular series, as well as linear combinations of algebraic sums of Cantor sets. These also result in what has been termed a Cantorval, which we also investigate.

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Chapter 1

Introduction

1.1 Historical Background

Books have been written in the topic of Cantor sets alone. Discovered by Henry John Stephen Smith, the Cantor set is an example of a perfect, nowhere dense set. It was thoroughly studied by Georg Cantor, and played a vital role in the foundations of general topology. It has been used to construct sets with interesting unintuitive properties, counterexamples, and because of its self similarity properties, it has found applications in dynamical systems, biology , chemistry, physics geology, and other areas.

1.2 Conventions

We will denote a standard Cantor set with constant ratio of dissection r as $\mathfrak{C}_r$ from now on, furthermore we will denote the standard middle thirds cantor set $\mathfrak{C}_{\frac{1}{3}}$, as just $\mathfrak{C}$, i.e.,

$$\mathfrak{C} = \mathfrak{C}_{\frac{1}{3}}$$

from now on.

As is common in topological literature we will denote the closed interval from 0 to 1 as $\mathbb{I}$, i.e.,

$$\mathbb{I} = [0, 1]$$

1.3 Preliminaries

Definition 1 *A set is said to be* ***perfect*** *if it is closed and has no isolated points.*

Definition 2 *Given two topological spaces X and Y, Then a function $f : X \to Y$ is a* ***homeomorphism*** *if and only if it is one-to-one, onto, continuous and has a continuous inverse.*

Definition 3 *A* ***topological property*** *is a property of a topological space which is invariant under homeomorphisms.*

In other words, it can be said that a topological property is a property of a space that can be expressed using open sets.

Example. Examples of common topological properties include:

1. Metrizability,
2. Compactness,
3. Connectedness,
4. Countability conditions, and
5. Separation.

To prove that two spaces are not homeomorphic, it is sufficient to show they do not share a topological property.

Example. The intervals $(0, 1)$ and $[0, 1]$ are not homeomorphic since we can remove 0 or 1 from $[0, 1]$ and it remains connected, but removing any point form $(0, 1)$ makes the interval disconnected, and since connectedness is a topological property it follows that they are not homeomorphic.

Theorem 1 *(Tychonoff). A non-empty product $\prod_{i \in I} X_i$ is compact if and only if each factor X_i is compact.*

1.4 The Cantor Set

Recall that The Cantor Set $\mathfrak{C}$ is obtained by successively removing the open middle thirds from $\mathbb{I}$, and the closed intervals that remain as we iterate this process, i.e.,

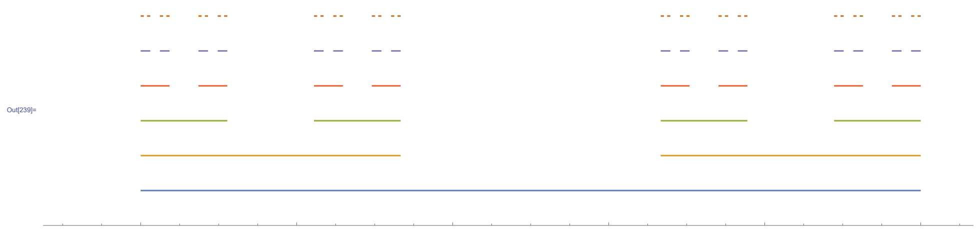

Figure 1.1: Classic ternary Cantor set.

and then taking the intersection of all the sets generated in the process. Alternatively, we can represent $\mathfrak{C}$ as the set of points $x \in \mathbb{I}$ having a ternary expansion without

1's, i.e.,

$$x = \sum \frac{x_i}{3^i}, x_i \in \{0, 2\}$$

Furthermore, the Cantor set is self similar, compact, totally disconnected, perfect, and uncountable, yet with measure zero.

Chapter 2

Cantor Set Theory

Theorem 2 *$\mathfrak{C}_{\frac{1}{3}}$ is homeomorphic to* $\{0,1\}^{\mathbb{N}}$

Proof. Let $x \in \mathfrak{C}_{\frac{1}{3}}$, then x can be written as,

$$x = \sum \frac{x_i}{3^i},\ x_i \in \{0,2\}.$$

Now, let

$$f(x) = \left(\frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}, \ldots\right),$$

then it follows that f defines a continuous function from all of $\mathfrak{C}_{\frac{1}{3}}$ to all of $\{0,1\}^{\mathbb{N}}$.

Now, let $y \in \{0,1\}^{\mathbb{N}}$, then y is the the form

$$y = (y_1, y_2, y_3, \ldots).$$

Additionally let,

$$f^{-1}(y) = \sum_{n=1}^{\infty} \frac{2y_n}{3^n},$$

then it follows that f defines a continuous function from all of $\{0,1\}^{\mathbb{N}}$ to all of $\mathfrak{C}_{\frac{1}{3}}$. □

In his book Willard[15] introduces the following two theorems, of which the latter is a powerful result.

Theorem 3 *Let X be a totally disconnected compact metric space. Then*

1. For each $n = 0, 1, 2, \ldots$ there is a finite open cover $\mathscr{U}_n$ of X by disjoint open sets of diameter $< \frac{1}{2^n}$ such that $\mathscr{U}_{n+1} < \mathscr{U}_n$ for each $n \geq 0$

2. If $Y_0 \overset{f_1}{\longleftarrow} Y_1 \overset{f_2}{\longleftrightarrow} \cdots$ is the derived sequence of any such sequence $\mathscr{U}_0, \mathscr{U}_1, \ldots$ of covers, then X is homeomorphic to the resulting inverse limit space Y_∞ .

Theorem 4 *Every compact metric space X is a continuous image of the Cantor set.*

Proof. Let $\mathscr{U}_1, \mathscr{U}_2, ...$ be a sequence of finite covers of X by the closures of open sets, the sets $\mathscr{U}_n$ of being of diameter $< \frac{1}{2^n}$, such that $\mathscr{U}_n$¡ $\mathscr{U}_{n-1}$ for $n = 2, 3, ...$. Say $\mathscr{U}_n = \left\{U_{n_1}, \ldots, U_{n_{k_n}}\right\}$ For each $U_{1_i} \in \mathscr{U}_1$, define $V_{1_i} = \{(u, i) \mid u \in U_{1_i}\}$ so that $V_1 = V_{1_1} \cup \cdots \cup V_{1_{k_1}}$ is the disjoint union of the $U_{1_i} \in \mathscr{U}_1$. Now each $U_{2_i} \in \mathscr{U}_2$ is contained in some $U_{1_k} \in \mathscr{U}_1$. Define $V_{1_{i_j}} = \{(u, i, j) \mid u \in U_{2_i}\}$ whenever $U_{2_j} \subset U_{1_i}$, and let $V_2 = \cup_{j=1}^{k_2} \cup_{U_{2_j} \subset U_{1_i}} V_{2_{i_j}}$. Each U_{2_j} occurs in the disjoint union once for each U_{1_i} such that $U_{2_j} \subset U_{1_i}$. Now define $\mathrm{f}_2 : V_2 \to V_1$ by $\mathrm{f}_2((u, i, j)) = (u, i)$. Then f_2 is continuous on each piece $V_{2_{i_j}}$ and thus continuous on V_2. Also, there is a map $\varphi_1 : V_1 \longrightarrow X$ defined by $\varphi_1(u, i) = u$ and a map $\varphi_2 : V_2 \longrightarrow X$ defined by $\varphi_2(u, i, j) = u$. Continuing the process this result in a pair of inverse sequences, and a mapping between them, where i is the identity map on X.

$$
\begin{array}{ccccccc}
 & \longrightarrow & V_3 & \overset{f_3}{\to} & V_2 & \overset{f_2}{\to} & V_1 \\
\cdots & & \downarrow \varphi_3 & & \downarrow \varphi_2 & & \downarrow \varphi_1 \\
 & \longrightarrow & X & \overset{i}{\to} & X & \overset{i}{\to} & X
\end{array}
$$

The result is a map $\varphi : V_\infty \longrightarrow X$ of the inverse limit spaces, which is continuous and onto because X and each V_n is a compact Hausdorff space and each φ_n is continuous and onto.

Each V_n is a compact metric space, being a disjoint union of a finite number of compact metric spaces. Let d_n be the metric on V_n induced by the metrics on the U_{n_j}. Also, if $(x_1, x_2, ...) \in V_\infty$, then we must have $\varphi(x_1) = \varphi_2(x_2) = ...$, and, if z_x denotes this common value, then for any$(y_1, y_2, ...) \in V_\infty$, $d_n(x_n, y_n) \geq d(z_x, z_y)$.

Now, we would like to show V_∞ is the Cantor set. It is compact because each V_n is compact, and metric because it is a subset of the metric space V_n. If $x = (x_0, x_1, ...)$ and $y = (y_0, y_1, ...)$ are distinct points of V_∞, then for some n, $x_n \neq y_n$. Now x_n and y_n must correspond to distinct points of X (under $\varphi_n : V_n \longrightarrow X$), say to z_x and z_y. Now if d_m is the metric on V_m, then clearly $d_m(x_m, y_m) \geq (z_x, z_y)$ for all $m \geq n$. Since the diameters of the sets $V_{m_1}, \ldots, V_{m_{k_n}}$, which compose V_m approach 0 as $m \longrightarrow \infty$, it follows that beyond some point N, x_m and y_m belong to different sets of V_m; say, $x_m \in V_{m_1}$, $y_m \notin V_{m_1}$. But V_{m_1} is open–closed in V_m, and hence $\{(z_0, z_1, ...) \in V_\infty \mid Z_m \in V_{m_1}\}$ is an open-closed nhood of x in V_∞ not containing y. Thus V_∞ is totally disconnected. But V_∞ need not, in general, be perfect. However, if $\mathfrak{C}_\infty$ is the Cantor set, $V_\infty \times \mathfrak{C}_\infty$ is a perfect, totally disconnected compact metric space which has V_∞, and hence X, for a continuous image. □

2.1 Interesting Example: The Knaster and Kuratowski Fan

The cantor set also gives rise to other interesting constructions, as in an example due to Knaster and Kuratowski, which is of a connected space **K** and a point p in **K** such that $\mathbf{K} - \{p\}$ is totally disconnected!

Example. Recall that $\mathfrak{C}$ is obtained by deleting a countable collection of open intervals from $\mathbb{I}$. Let Q be the set of endpoints of these intervals, and set $P = \mathfrak{C} - Q$. Let $p \in R^2$ be the point $\left(\frac{1}{2}, \frac{1}{2}\right)$, and for each $x \in \mathfrak{C}$, denote by L_x the straight-line segment joining p and x. Define

$$\begin{aligned} L_x^* &= \{(x_1, x_2) \in L_x \mid x_2 \in \mathbb{Q}\}, \; if \; x \in Q \\ L_x^* &= \{(x_1, x_2) \in L_x \mid x_2 \in \mathbb{R} - \mathbb{Q}\}, \; if \; x \in p \end{aligned}$$

Then the subspace$K = \cup_{x \in \mathfrak{C}_\infty} L_x^*$ is connected, while $K - \{p\}$ is totally disconnected.

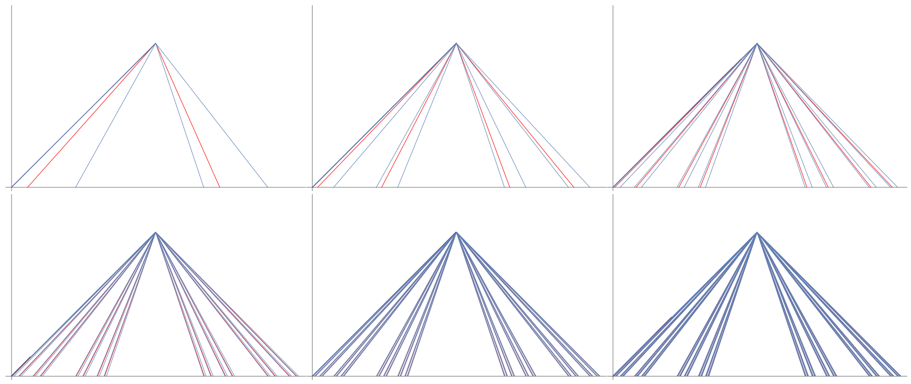

Figure 2.1: The Knaster and Kuratowski Fan

For more on this example the book 'Counter Examples in Topology'[12] has an exceptionally nice treatment on this example.

2.2 Another Interesting Example: The Devils Staircase

Another interesting construction is seen in the Cantor function, also known as, The Devil's Staircase.

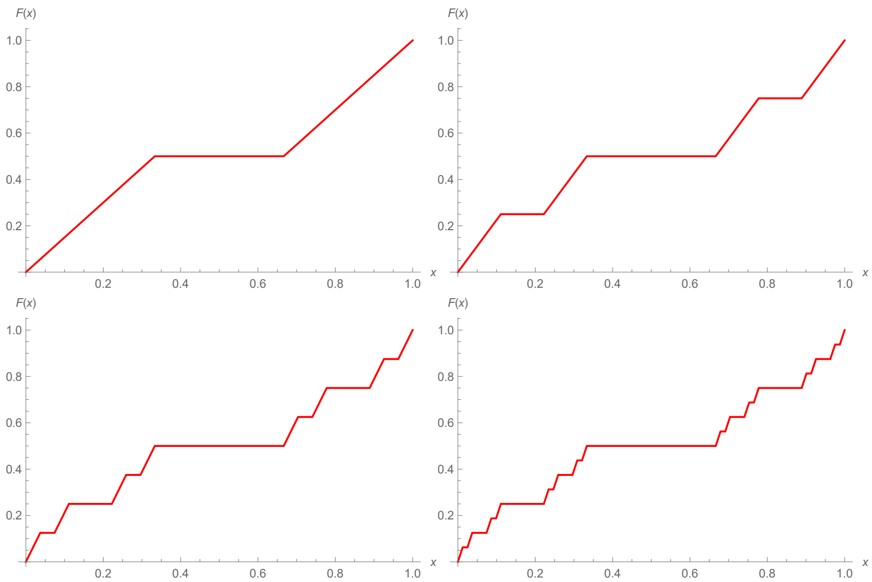

Figure 2.2: Devil's Staircase

Let $f : \mathbb{I} \to \mathbb{I}$ such that $f(0) = 0$ and $f(1) = 1$, and

$$f_1(x) = \begin{cases} (3/2)x & \text{for } 0 \leq x \leq 1/3 \\ 1/2 & \text{for } 1/3 < x < 2/3 \\ (3/2)x - 1/2 & \text{for } 2/3 \leq x \leq 1 \end{cases}$$

$$f_2(x) = \begin{cases} (1/2)f_1(3x) & \text{for } 0 \leq x \leq 1/3 \\ f_1(x) & \text{for } 1/3 < x < 2/3 \\ (1/2)f_1(3x-2) + 1/2 & \text{for } 2/3 \leq x \leq 1 \end{cases}$$

$$\vdots$$

$$f_n(x) = \begin{cases} (1/2)f_{n-1}(3x) & \text{for } 0 \leq x \leq 1/3 \\ f_1(x) & \text{for } 1/3 < x < 2/3 \\ (1/2)f_n(3x-2) + 1/2 & \text{for } 2/3 \leq x \leq 1 \end{cases}$$

What is particularly interesting about this function is that it has a slope of zero almost everywhere, i.e., on a set of measure one, yet rises from the point $(0, 0)$ to the point $(1, 1)$. For more on this example Royden[11] has a nice brief treatment.

Chapter 3

Series

3.1 Motivation For The Use Of Series

3.1.1 A simple divergent series

Modern mathematics is filled with the study of sequences and series. Because of the significance of some of these sequences and series, it is of particular interest to know what the sums of the terms of all subsequence are, and if there are any particular patterns or properties of the set of all these sums. We find that frequently we discover some very interesting properties, or at the very least some insightful information or pattern.

Take for example the following series,

$$\sum_{i=1}^{n}\left[(k(2k+1)+k+i)^2-(k(2k+1)+i)^2\right]$$

a divergent series whose sums I was investigating because they lead to interesting identities.

First observe that if we let n=k, then for all k,

$$\sum_{i=1}^{k}\left[(k(2k+1)+k+i)^2-(k(2k+1)+i)^2\right] \quad = \quad (k(2k+1))^2$$

which we can rewrite as follows,

$$\begin{aligned}
\sum_{i=1}^{k}\left[(k(2k+1)+k+i)^2-(k(2k+1)+i)^2\right]-(k(2k+1))^2 &= 0\\
\Rightarrow \sum_{i=1}^{k}(k(2k+1)+k+i)^2-\sum_{i=1}^{k}(k(2k+1)+i)^2-(k(2k+1))^2 &= 0\\
\Rightarrow \sum_{i=k+1}^{2k}(k(2k+1)+i)^2-\sum_{i=0}^{k}(k(2k+1)+i)^2 &= 0\\
\Rightarrow \sum_{i=0}^{k}(k(2k+1)+i)^2 &= \sum_{i=k+1}^{2k}(k(2k+1)+i)^2
\end{aligned}$$

which for$k=1$ yields,

$$3^2+4^2 = 5^2$$

$k=2$ yields,

$$10^2+11^2+12^2 = 13^2+14^2$$

$k=3$ yields,

$$21^2+22^2+23^2+24^2 = 25^2+26^2+27^2$$

$k=4$ yields,

$$36^2+37^2+38^2+39^2+40^2 = 41^2+42^2+43^2+44^2$$

$k=10$ yields,

$$\begin{aligned}
&210^2+211^2+212^2+213^2+214^2+215^2+216^2+217^2+218^2+219^2+220^2\\
&=221^2+222^2+223^2+224^2+225^2+226^2+227^2+228^2+229^2+230^2
\end{aligned}$$

$k=997$ yields,

$$1989015^2+1989016^2+\cdots+1990012^2=1990013^2+1990014^2+\cdots+1991009^2$$

containing 998 terms on the left hand side, and 997 on the right hand side, and being composed of consecutive integers from beginning to end, resulting in an interesting way to generate these interesting patterns, but we want more than just this.

3.1.2 An example with a convergent sequence:

The following example due to Nitecki[7] provides an algoritheorem to finding the sums of all subsequences of a convergent sequence. We will formally define the set of these sums later.

Example. Consider,

$$\sum_{i=n+1}^{\infty} x_i,$$

and let,

$$x_k = \frac{1}{3^k}.$$

Then if we include all terms we get,

$$\sum_{i=1}^{\infty} x_i = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2},$$

but if we don't include any we get,

$$\sum_{i=1}^{\infty} 0 \quad = \quad 0$$

Hence, it follows that the set of all possible "subsums" of $\sum_{i=n+1}^{\infty} x_i$ when $x_k = \frac{1}{3^k}$ is contained in the interval from $\left[0, \frac{1}{2}\right]$, call it $\mathfrak{C}_0$.

However, what if we excluded the first term of this sum? Then

$$R_1 = \sum_{i=2}^{\infty} x_i = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \frac{1}{6},$$

thus it follows that the set of all "subsums" in this example that exclude the first term is contained in the interval $\left[0, \frac{1}{6}\right]$. Now, reintroducing the first term, $\frac{1}{3}$, translates $\left[0, \frac{1}{6}\right]$ to $\left[\frac{1}{3}, \frac{1}{2}\right]$, and hence the set of all "subsums" belongs not only to the interval $\left[0, \frac{1}{2}\right]$, but more precisely to, $\left[0, \frac{1}{6}\right] \cup \left[\frac{1}{3}, \frac{1}{2}\right]$. If we let $J_0 = \left[0, \frac{1}{6}\right]$, and $J_1 = \left[\frac{1}{3}, \frac{1}{2}\right]$ we can rewrite this set as,

$$\mathfrak{C}_1 \quad = \quad J_0 \cup J_1.$$

What if we now excluded the first and second term of this sum? Then

$$R_2 = \sum_{i=3}^{\infty} x_i = \frac{\frac{1}{27}}{1-\frac{1}{3}} = \frac{1}{18},$$

thus it follows that the set of all "subsums" in this example that exclude the first and second term is contained in the interval $\left[0, \frac{1}{18}\right]$. Now, reintroducing the second term, $\frac{1}{9}$, translates $\left[0, \frac{1}{18}\right]$ to $\left[\frac{1}{9}, \frac{1}{6}\right]$, and reintroducing the first term, $\frac{1}{3}$, translates $\left[0, \frac{1}{18}\right]$ and $\left[\frac{1}{9}, \frac{1}{6}\right]$ to $\left[\frac{1}{3}, \frac{7}{18}\right]$ and $\left[\frac{4}{9}, \frac{1}{2}\right]$, hence the set of all "subsums" belongs not only to the interval $\mathfrak{C}_1 = J_0 \cup J_1$, but more precisely to, $\left[0, \frac{1}{18}\right] \cup \left[\frac{1}{9}, \frac{1}{6}\right] \cup \left[\frac{1}{3}, \frac{7}{18}\right] \cup \left[\frac{4}{9}, \frac{1}{2}\right]$. If we let $J_{00} = \left[0, \frac{1}{18}\right]$, $J_{01} = \left[\frac{1}{9}, \frac{1}{6}\right]$, $J_{10} = \left[\frac{1}{3}, \frac{7}{18}\right]$, and $J_{11} = \left[\frac{4}{9}, \frac{1}{2}\right]$ we can rewrite this set as,

$$\mathfrak{C}_2 = J_{00} \cup J_{01} \cup J_{10} \cup J_{11},$$

and so far here is what it looks like,

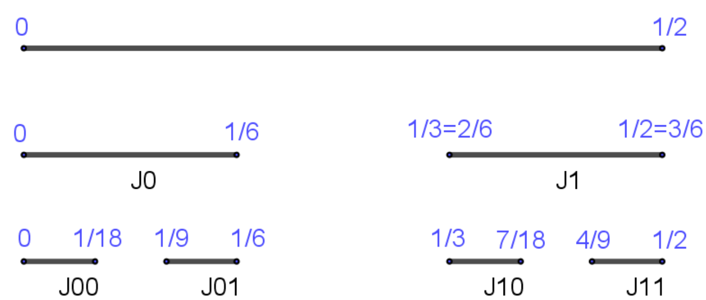

Figure 3.1: Subsum set for $x_n = \frac{1}{3}$

a Cantor set except on the interval from 0 to $\frac{1}{2}$ rather than 0 to 1. Thus, these types of sums will be of great interest to us throughout.

Now, notice that J_0 is the interval produced when we exclude the first term, and J_1 when we re-introduce the first term. Furthermore J_{00} is the interval that results from excluding the first and second term, J_{01} when we re-introduce the second term, J_{10} when we re-introduce the first term, but not the second, and J_{11} when we introduce both the first and second term.

3.2 Subsum of the General Geometric Series

In the previous section we saw a particular example of the subsum of a geometric sequence. We can now generalize.

Theorem 5 *For* $x_k \in \left[0, \frac{1}{2}\right)$ *the subsum set* $\sum \{x_i\}_{i=1}^{\infty}$ *is a cantor set, and for* $x_k \in \left[\frac{1}{2}, 1\right)$ *the subsum set* $\sum \{x_i\}_{i=1}^{\infty}$ *is an interval.*

First we observe that for $x_k < \frac{1}{2}$ segments in our construction are disjoint, but if $x_k \in \left[\frac{1}{2}, 1\right)$ the segments in our construction intersect so we consider two cases in our construction.
If $x_k < \frac{1}{2}$

Consider $\sum_{i=1}^{\infty} \varepsilon_i x_i,\ \varepsilon_i \in \{0, 1\}\,,\ \forall\, i \in \mathbb{N}$, and let

$$x_k = r \quad , \quad r \ \in (0, 1)\,.$$

Then for $\varepsilon_k = 1$, for all k,

$$\sum_{i=1}^{\infty} r^i \quad = \quad \frac{r}{1-r},$$

and for $\varepsilon_k = 0$, for all k,

$$\sum_{i=1}^{\infty} 0 \quad = \quad 0$$

Hence, it follows that the set of all subsums of (4.2) when $x_k = r$ is contained in the interval from $\left[0, \frac{r}{1-r}\right]$, call it $\mathfrak{C}_0$.

However, what if we excluded the first term of this sum? Then

$$R_1 = \sum_{i=2}^{\infty} r^i = \frac{r^2}{1-r}$$

thus it follows that the set of all subsums in this example that exclude the first term is contained in the interval $[0, R_1]$. Now, reintroducing the first term, r, translates $[0, R_1]$ to $\left[r, \frac{r}{1-r}\right]$, and hence the set of all subsums belongs not only to the interval $\left[0, \frac{r}{1-r}\right]$, but more precisely to,$[0, R_1] \cup \left[r, \frac{r}{1-r}\right]$. If we let $J_0 = [0, R_1]$, and $J_1 = \left[r, \frac{r}{1-r}\right]$ we can rewrite this set as,

$$\mathfrak{C}_1 \quad = \quad J_0 \cup J_1.$$

What if we now excluded the first and second term of this sum? Then

$$R_2 = \sum_{i=3}^{\infty} r^i = \frac{r^3}{1-r}$$

thus it follows that the set of all subsums in this example that exclude the first and second term is contained in the interval $[0, R_2]$. Now, reintroducing the second term, r^2, translates $[0, R_2]$ to $[r^2, R_2^2 + r^2]$, and reintroducing the first term, r, translates $[0, R_2]$ and $[r^2, R_2^2 + r^2]$ to $[r, R_2]$ to $[r^2, R_2^2 + r^2]$, hence the set of all subsums belongs not only to the interval $\mathfrak{C}_1 = J_0 \cup J_1$, but more precisely to ,$[0, R_2] \cup [r^2, R_2^2 + r^2] \cup [r, R_2] \cup [r^2, R_2^2 + r^2]$. If we let $J_{00} = [0, R_2]$, $J_{01} = [r^2, R_2^2 + r^2]$, $J_{10} = [r, R_2]$, and $J_{11} = [r^2, R_2^2 + r^2]$. Letting $J_{0^n} = [0, R_n]$, and $\xi_1 \cdots \xi_n \in \{0, 1\}^n$ we can rewrite this set as,

$$\begin{aligned}
\mathfrak{C}_2 &= J_{00} \cup J_{01} \cup J_{10} \cup J_{11}, \\
\mathfrak{C}_3 &= J_{000} \cup J_{001} \cup J_{010} \cup J_{011} \cup J_{100} \cup J_{101} J_{110} \cup J_{111}, \\
&\vdots \\
\mathfrak{C}_n &= J_{0^{(n)}} \cup J_{0^{(n)}+1_{(2)}} \cup \cdots \cup J_{\xi_1 \cdots \xi_n} \cup J_{\xi_1 \cdots \xi_n + 1_{(2)}} \cup \cdots \cup J_{0^{(n)}+(2^n-1)_{(2)}} \\
&= \bigcup_{\xi_1 \cdots \xi_n \in \{0,1\}^n} J_{\xi_1 \cdots \xi_n}
\end{aligned}$$

We now observe that,

$$\begin{aligned}
R_{n-1} &= \sum_{i=n}^{\infty} r^i \\
&= \frac{r^n}{1-r},
\end{aligned}$$

and

$$\begin{aligned} R_n &= \sum_{i=n+1}^{\infty} r^i \\ &= \frac{r^{n+1}}{1-r}, \\ &= rR_{n-1} \end{aligned}$$

so that if $r < \frac{1}{2}$, $J_{\xi_1 \cdots \xi_{n-1} 0} \cap J_{\xi_1 \cdots \xi_{n-1} 1} = \emptyset$, thus

$$\mathfrak{C}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{C}_n$$

resulting in a cantor set as follows.

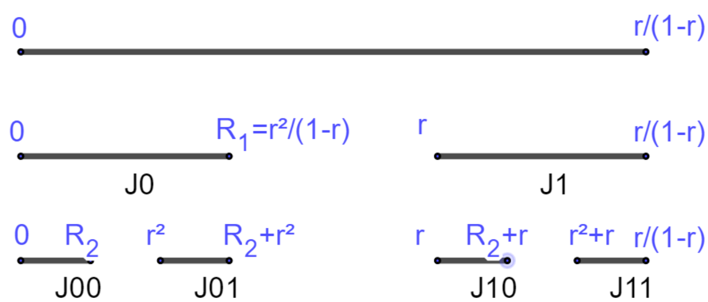

Figure 3.2: Subsum set for $x_n = r$, $r \in \left(0, \frac{1}{2}\right)$

However, if $r \geq \frac{1}{2}$, $J_{\xi_1 \cdots \xi_{n-1} 0} \cap J_{\xi_1 \cdots \xi_{n-1} 1} \neq \emptyset$, which means the intervals overlap, and

$$\mathfrak{C}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{C}_n$$

resulting in an interval as follows.

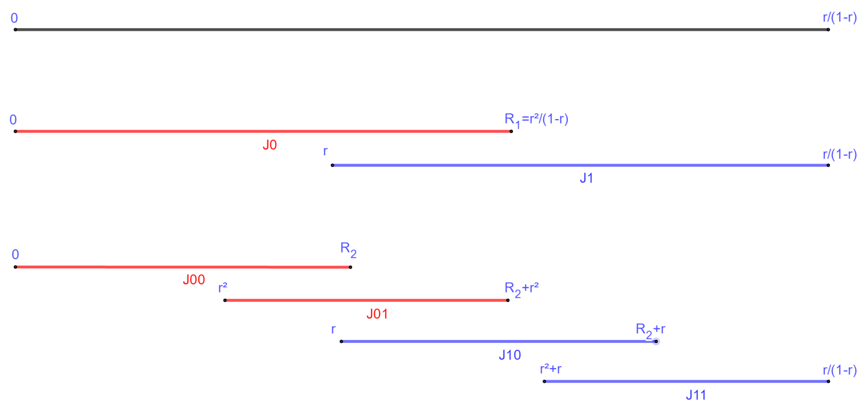

Figure 3.3: Subsum set for $x_n = r$, $r \in \left[\frac{1}{2}, 1\right)$

3.3 Basic Properties

The following well known result is very useful when manipulating series, namely,

Theorem 6 *Given a sequence $\{x_n\}$ of positive numbers, the sum of its corresponding series $\sum_{i=1}^{\infty} x_n$ is not changed by rearranging the order of the terms.*

As a result of this last Theorem given any sequence $\{x_n\}$ of positive numbers, we can rearrange the terms if necessary, such that for all $n \in \mathbb{N}$, $x_{i+1} \leq x_i$ so that $\{x_n\}$ is a decreasing sequence.

3.4 Divergent Sequences

Theorem 7 *Every real number x can be written as the sum of a subsequence of $\left\{\frac{1}{n}\right\}$,the harmonic sequence.*

Theorem 8 *If* $\{x_n\}$ *is a positive null sequence for which* $\sum_{i=1}^{\infty} x_i$ *diverges, then every* $x > 0$ *is the sum of some subsequence of* $\{x_n\}$.

Chapter 4

Cantorvals

4.1 Early Work

Following the notation of Nitecki[9] we define the subsum set of a null sequence.

Definition 4 *Let where $\xi = \xi_1 \xi_2 \cdots$, then the* ***subsum set*** *of a null sequence $\{x_n\}$, where $x_n \to 0$ as $n \to \infty$ is the collection*

$$\sum \{x_i\}_{i=1}^{\infty} \tag{4.1}$$

of all numbers of the form

$$x(\varepsilon) = \sum_{i=1}^{\infty} \varepsilon_i x_i,\, \varepsilon_i \in \{0,1\},\, \forall\, i \in \mathbb{N} \tag{4.2}$$

for which the series converges.

Definition 5 *Let*

$$R_n = \sum_{i=n+1}^{\infty} x_i$$

denote the n-th "tail" of 4.2

In 1914 the following results were discovered by Kakeya[5].

Theorem 9 *(Kakeya) The subsum set $\sum \{x_i\}_{i=1}^{\infty}$ is,*

1. a perfect set,
2. the finite union of closed intervals if and only $x_n \leq R_n$ for all n, or for n sufficiently large, or

3. homeomorphic to the Cantor set if $x_n > R_n$ for n sufficiently large.

He also conjectured $\sum \{x_i\}_{i=1}^{\infty}$ is nowhere dense, and as a result homeomorphic to the Cantor set, if $x_n > R_n$ for infinitely many n.

The first counterexample to this conjecture was given by Weinstein and Shapiro[14] without proof in 1980, namely

$$\sum_{i=1}^{\infty}\left[\varepsilon_{5i-4}\left(\frac{8}{10^i}\right)+\varepsilon_{5i-3}\left(\frac{7}{10^i}\right)+\varepsilon_{5i-2}\left(\frac{6}{10^i}\right)+\varepsilon_{5i-1}\left(\frac{5}{10^i}\right)+\varepsilon_{5i}\left(\frac{4}{10^i}\right)+\right],$$

where $\varepsilon_i \in \{0,1\}$, $\forall\, i \in \mathbb{N}$.

Shortly after, in 1984, Ferens[3] gave another example which included proof, namely,

$$\sum_{i=1}^{\infty}\left[\varepsilon_{5i-4}\left(\frac{7}{27^i}\right)+\varepsilon_{5i-3}\left(\frac{6}{27^i}\right)+\varepsilon_{5i-2}\left(\frac{5}{27^i}\right)+\varepsilon_{5i-1}\left(\frac{4}{27^i}\right)+\varepsilon_{5i}\left(\frac{3}{27^i}\right)+\right],$$

where $\varepsilon_i \in \{0,1\}$, $\forall\, i \in \mathbb{N}$.

However, not long after in 1988, Guthrie and Nymann[4] provided a simple counter example of a subsum set that is not the finite union of closed intervals, yet has non-empty interior, namely,

$$T = \sum_{i=1}^{\infty}\left[\varepsilon_{2i-1}\left(\frac{3}{4^i}\right)+\varepsilon_{2i}\left(\frac{2}{4^i}\right)\right],\ \varepsilon_i \in \{0,1\},\ \forall\, i \in \mathbb{N} \tag{4.3}$$

and went on to described the topological structure of the set of subsums $\sum \{x_i\}_{i=1}^{\infty}$ as follows:

Theorem 10 *(Guthrie-Nymann) If $\sum \{x_i\}_{i=1}^{\infty}$ is the set of subsums of a positive term convergent series $\sum_{i=1}^{\infty} x_i$, then $\sum \{x_i\}_{i=1}^{\infty}$ is one of the following:*

1. A finite union of closed intervals,

2. homeomorphic to the Cantor set, or

3. homeomorphic to the set of all subsums of T (4.3)

4.2 Current Work

In 1994 Mendez and Oliveira also studied sets of the type described in condition (3) of theorem 9 *(Kakeya)*, and called them Cantorvals, thus we can define a Cantorval as follows:

Definition 6 *A **symmetric Cantorval** is a nonempty compact set $S \subset \mathbb{R}$ such that,*

1. $S = \overline{S^{\circ}}$
2. The endpoints of any nontrivial component of S are accumulation points of trivial components of S.

and as a result the theorem of Guthrie and Nyman [4] can be rewritten as follows

Theorem 11 *(Guthrie-Nymann) If $0 < x_{i+1} \leq x_i$, and*

$$x(\varepsilon) = \sum_{i=1}^{\infty} \varepsilon_i x_i,\, \varepsilon_i \in \{0,1\},\, \forall\, i \in \mathbb{N} \tag{4.4}$$

then its subsum set is either,

1. a Cantor set,
2. the finite union of disjoint closed intervals, or
3. a symmetric Cantorval

Proof. (Due to Guthrie[4]) Suppose that E is neither a finite union of intervals nor homeomorphic to the Cantor set. Then it is clear the complement of E must contain infinitely many intervals. E must contain infinitely many intervals as well, for there were only finitely many, then either $E \cap [0, \varepsilon)$ is an interval for some $\varepsilon > 0$ or $E \cap [0, \varepsilon)$ contains no interval for some $\varepsilon > 0$. If the former is true, the there is some tail of $\sum a_a$ which has an interval as its set of subsums, and therefore E would be a finite union of intervals. If the latter holds, then $E \cap [0, \varepsilon]$ is homeomorphic to the Cantor set, and there is a tail of $\sum a_n$ which has the Cantor set as its set of subsums. Thus E would be homeomorphic to the Cantor set.

This is again a contradiction to our initial supposition. Thus E contains infinitely many intervals.

In fact, $E \cap [a, b]$ cannot be homeomorphic to the Cantor set for any $a, b \in E$, since every tail of $\sum a_n$ must have intervals in in its set of subsums. Suppose then that, for some $x \in E$,

$$E \cap (x, x + \varepsilon) = \emptyset \quad \text{for some } \varepsilon > 0$$

Then, since E is perfect, $E \cap (x - \varepsilon, x) \neq \emptyset$ for every $\varepsilon > 0$, and therefore there are intervals in E arbitrarily close to x.

We now define a strictly increasing mapping f from the union of all intervals of T onto the union of all intervals in E . We can define the mapping inductively. Begin by mapping the longest interval in T in a strictly increasing way onto the longest interval in E . There can be at most finitely many intervals of the same length in either set, so we may choose the left-most interval in case no one interval is longest.

After the nth step, $2^n - 1$ intervals $[\alpha_j, \beta_j] \subset T\,(1j2^n - 1, \beta_j < \alpha_{j+1})$ will have been identified, in a strictly increasing way, with intervals $\left[\alpha'_j, \beta'_j\right] \subset E$. Now repeat the above process on each subset of T lying in $[\beta_j, \alpha_{j+1}]\,(j < 2^n - 1)$ or in $[0, \alpha_1]$ or in $[\beta_{2n-1}, 5/3]$. That is, map the longest interval in every such portion of T to the longest interval in E lying in $\left[\beta'_j, \alpha'_{j+1}\right]\,(j < 2^n - 1)$ or in $[0, \alpha'_1]$ or in $\left[\beta'_{2n-1}, \sum a_k\right]$, respectively.

When f is defined in this way, it is a strictly increasing mapping of the union of all intervals in T onto the union of all intervals in E . The property verified above that each point of E (and of T) is the limit of a sequence chosen from the intervals of the set allows us to extend m continuouly to all of T, and guarantees that the extension will be onto E. The extension will be strictly increasing and, therefore, one-to-one. Since T is compact, f is the desired homeomorphism. □

Theorem 12 *Any two symmetric Cantorvals are homeomorphic.*

Proof. Consider two Cantorvals $\mathfrak{C}$ and $\mathfrak{C}'$. Taking the longest and rightmost, or leftmost, component of each, there exists is a unique affine, order preserving homeomorphism between these components. By definition, there are other components of $\mathfrak{C}$ or $\mathfrak{C}'$, respectively, to the right and the left of the one chosen. Now, its complement is contained in two disjoint intervals, and the part of each Cantorval in each of these intervals is also a Cantorval. By repeated application of this method we can pair the longest nontrivial component of the chosen one in $\mathfrak{C}$ with the corresponding one in $\mathfrak{C}'$, and thus resulting in an order preserving correspondence, and homeomorphism between the nontrivial components of $\mathfrak{C}$ and $\mathfrak{C}'$, and therefore an order preserving continuous map from $\mathfrak{C}$ onto $\mathfrak{C}'$ resulting in a homeomorphism from all of $\mathfrak{C}$ onto all of $\mathfrak{C}'$. □

In 2011, Jones made a big leap by providing the following extension to the Guthrie-Nymann set which also yielded a continuum of subsum sets yielding Cantorvals.

Theorem 13 *The set of subsums*

$$\sum_{i=1}^{\infty}\left[\varepsilon_{4i-3}\left(\frac{3}{q^i}\right)+\varepsilon_{4i-2}\left(\frac{2}{q^i}\right)+\varepsilon_{4i-2}\left(\frac{4}{q^i}\right)+\varepsilon_{2i}\left(\frac{2}{q^i}\right)\right],\ \varepsilon_i\in\{0,1\}\,,\ \forall\, i\in\mathbb{N}$$

is a cantorval for all q such that

$$\frac{1}{5}\leq\sum_{i=1}^{\infty}<\frac{2}{9}.$$

But then in 2015, inspired by Kenyon[6], Nitecki[9] paved the way for additional contributions.

Theorem 14 *(Kenyon/Nitecki). Suppose we are given* $n\in\mathbb{N}$ *and* n *integers* $d_0, d_1, \ldots, d_{n-1}$ *such that*

$$d_i \equiv j \mod n$$

Then the set of "generalized base n expansions" using these "digits"

$$\mathcal{S}=\left\{\sum_{i=1}^{\infty}\frac{a_i}{n^i}|a_i\in\{d_0,\ldots,d_{n-1}\}\right\}$$

has nonempty interior.

More recently Bartoszewics et al.[1] contributed the following result.

Theorem 15 *(Bartoszewics) Let* $k_1 \geq k_2 \geq \cdots \geq k_m$ *be positive integers and* $K = \sum_{i=1}^{m} k_i$ *Assume that there exist positive integers* n_0 *and* n *such that each of numbers* $n_0, n_0 + 1, \ldots, n_0 + n$ *can be obtained by summing up the numbers* $k_1, k_2, \ldots, k_m$ *(i.e.* $n_0 + j = \sum_{i=1}^{m} \varepsilon_i k_i$ *with* $\varepsilon_i \in \{0, 1\}, j = 1, \ldots, n$*). If* $q \geq \frac{1}{n+1}$ *then* $E(k_1, \ldots, k_m; q)$ *has a nonempty interior. If* $q < \frac{k_m}{K+k_m}$ *then* $E(k_1, \ldots, k_m; q)$ *is not a finite union of intervals. Consequently, if*

$$\frac{1}{n+1} \leq q < \frac{k_m}{K + k_m}$$

then

$$E(k_1, \ldots, k_m; q)$$

is a Cantorval.

Also introducing a new notation in the following way: For any $q \in \left(0, \frac{1}{2}\right)$ the symbol $(k_1, k_2, \ldots, k_m; q)$ will be used to denote the sequence

$$\left(k_1, k_2, \ldots, k_m, k_1 q, k_2 q, \ldots, k_m q, k_1 q^2, k_2 q^2, \ldots, k_m q^2, \ldots\right).$$

Chapter 5

More Cantor Set Theory

Definition 7 *Let*

$$r = \frac{|J_{0^{(n)}}|}{|J_{0^{(n-1)}}|},$$

then

$$\mathfrak{C}_r$$

will be the Cantor set with ***rate of dissection*** *r at the nth step.*

Note that in this paper we only deal with central cantor sets having constant rate of dissection.

5.1 Properties

In the literature the authors all seem to justify that $\mathfrak{C}_{\frac{1}{3}}$ is perfect by saying that the property of being perfect is preserved under nested intersection. Nitecki makes this claim in three of his papers[7, 8, 9]. This however is not true, take for example

$$\bigcap_{i=1}^{\infty} [-n, n] = \{0\}$$

which is clearly not perfect as it is a singleton.

Theorem 16 $\mathfrak{C}_{\frac{1}{3}}$ is perfect.

Proof. Recall that

$$\mathfrak{C}_{\frac{1}{3}} = \bigcap_{n=1}^{\infty} \mathfrak{C}_n,$$

where

$$\mathfrak{C}_n = \bigcup_{\xi_1\cdots\xi_n\in\{0,1\}^n} J_{\xi_1\cdots\xi_n},$$

and $\mathfrak{C}_{n+1} \subset \mathfrak{C}_n$.

Since $\{0,1\}$ is compact, it follows by Tychonoff's theorem that the set $\{0,1\}^{\mathbb{N}}$ is compact, and Since $\mathfrak{C}_{\frac{1}{3}}$ is homeomorphic to $\{0,1\}^{\mathbb{N}}$, it follows that $\mathfrak{C}_{\frac{1}{3}}$ is compact. Finally by the Heine–Borel theorem $\mathfrak{C}_{\frac{1}{3}}$ is also closed.

Now, since $J_{0^{(n)}} = \left[0, R_n = \left(\frac{1}{3}\right)^n\right]$ it follows that $|J_{\xi_1\cdots\xi_n}| = \left(\frac{1}{3}\right)^n$. Now, choose n such that $\left(\frac{1}{3}\right)^n < \epsilon$, and let $x \in \mathfrak{C}_{\frac{1}{3}} \bigcap J_{\xi_1\cdots\xi_n}$. If we remove the middle third from $J_{\xi_1\cdots\xi_n}$, then $x \in \mathfrak{C}_{\frac{1}{3}} \bigcap J_{\xi_1\cdots\xi_n 0}$ or $x \in \mathfrak{C}_{\frac{1}{3}} \bigcap J_{\xi_1\cdots\xi_n 1}$. Assume without loss of generality that $x \in \mathfrak{C}_{\frac{1}{3}} \bigcap J_{\xi_1\cdots\xi_n 0}$ then there exists $y \in \mathfrak{C}_{\frac{1}{3}} \bigcap J_{\xi_1\cdots\xi_n 1}$such that $y \neq x$, thus it follows that

$$\begin{aligned} (y,x) &\subseteq J_{\xi_1\cdots\xi_n} \\ \Rightarrow |y-x| &\leq |J_{\xi_1\cdots\xi_n}| \\ &< \epsilon \end{aligned}$$

□

5.2 Cantor Set Arithmetic

The following ideas are ready to be introduced, and will help tie together the cantor sets, Cantorvals, and the series that led to them. For a brief review of algebraic, and algebraic sum properties, see Bhuiyan[2]

Definition 8 *For any set $S \subset \mathbb{R}$, and $n \subset \mathbb{N}$, we let*

$$\oplus_n S = \{s_1 + \ldots + s_n : s_i \in S (i = 1, \ldots, n)\}$$

denote the algebraic sum of n copies of S.

Theorem 17 $\oplus_2 \mathfrak{C}_{\frac{1}{3}} = \mathfrak{C}_{\frac{1}{3}} + \mathfrak{C}_{\frac{1}{3}} = [0,2]$

Proof. Let $c \in \mathbb{I}$. Then for all c its ternary expansion is,

$$c = \sum_{i=1}^{\infty} \frac{c_i}{3^i},\ c_i \in \{0, 1, 2\}$$

Let $c_i = a_i + b_i$ where $a_i, b_i \in \mathbb{N}$. It follows that

$$\begin{aligned}
\mathbb{I} &= \sum \left\{ \frac{c_i}{3^i} \right\}_{i=1}^{\infty},\ c \in \{0, 1, 2\} \\
&= \sum \left\{ \frac{a_i}{3^i} + \frac{b_i}{3^i} \right\}_{i=1}^{\infty},\ a_i, b_i \in \{0, 1\} \\
&= \sum \left\{ \frac{a_i}{3^i} \right\}_{i=1}^{\infty} + \sum \left\{ \frac{b_i}{3^i} \right\}_{i=1}^{\infty} \\
&= \frac{1}{2}\mathfrak{C} + \frac{1}{2}\mathfrak{C}
\end{aligned}$$

therefore

$$\begin{aligned}
\mathfrak{C} + \mathfrak{C} &= 2\mathbb{I} \\
&= [0, 2]
\end{aligned}$$

□

Theorem 18 $\ominus_2 \mathfrak{C}_{\frac{1}{3}} = \mathfrak{C}_{\frac{1}{3}} - \mathfrak{C}_{\frac{1}{3}} = [-1, 1]$

Proof. .

$$\begin{aligned}
\mathfrak{C}_{\frac{1}{3}} - \mathfrak{C}_{\frac{1}{3}} &= \left(\mathfrak{C}_{\frac{1}{3}} - \frac{1}{2} \right) + \left(\frac{1}{2} - \mathfrak{C}_{\frac{1}{3}} \right) \\
&= \left(\mathfrak{C}_{\frac{1}{3}} - \frac{1}{2} \right) + \left(\mathfrak{C}_{\frac{1}{3}} - \frac{1}{2} \right) \\
&= \mathfrak{C}_{\frac{1}{3}} + \mathfrak{C}_{\frac{1}{3}} - 1 \\
&= [0, 2] - 1 \\
&= [-1, 1]
\end{aligned}$$

□

5.3 Basic Operations

Theorem 19 *Let $\mathfrak{C}$ be a Cantor set, and $a, b \in \mathbb{R}$, then*

1. $\mathfrak{C} + a = \mathfrak{C}$
2. $a\mathfrak{C} + b\mathfrak{C} = a\mathfrak{C} + (-b)\mathfrak{C} + b\sup(\mathfrak{C})$
3. $a\mathfrak{C} + b\mathfrak{C} = a\left(\mathfrak{C} + \frac{b}{a}\mathfrak{C}\right)$

5.4 Algebraic Sum of Subsum Sets

In 1995 Nymann[10] proposed the following interesting theorem about the algebraic sums of subsum sets $\sum \{x_i\}_{i=1}^{\infty}$.

Theorem 20 *(Nymann) There is a positive integer m for which $\oplus_m \left(\sum \{x_i\}_{i=1}^{\infty}\right)$ is a finite union of intervals if and only if*

$$\limsup \frac{x_n}{R_n} < \infty.$$

Moreover, the smallest positive integer for which $\oplus_m \left(\sum \{x_i\}_{i=1}^{\infty}\right)$ is a finite union of intervals is the smallest integer m such that $x_n/R_n \leq m$ for all but a finite number of integers n.

Proof. Proof: Let $m \in \mathbb{N}$ be fixed. Construct $\{c_n\}$ as follows

$$c_{(q-1)m+1} = c_{(q-1)m+2} = \cdots = c_{qm} = a_q$$

for $q = 1, 2, 3, \ldots$ We observe that

$$\sum_{i=1}^{\infty} c_i$$

converges for all n, if $0 < c_{n+1} \leq c_n$, and $\oplus_m \left(\sum \{x_i\}_{i=1}^{\infty}\right)$ is also the set of subsums of

$$\sum_{i=1}^{\infty} c_i$$

Now, by Kakeya's theorem, $\oplus_m \left(\sum \{x_i\}_{i=1}^{\infty}\right)$ is a finite union of intervals if and only if $c_n \leq R_n$ for all but finitely many index values. If $m \neq nq$,for some q, them this inequality is true. If $n = mq$, then $c_n = a_q$ and $R_n = mr_q$. Thus $\oplus_m \left(\sum \{x_i\}_{i=1}^{\infty}\right)$ is an interval if and only if $a_q \leq mr_q$ for all but a finite number of integer values of q, and the conclusion follows. □

It follows from the theorem that the smallest integer m such that $x_n/R_n \leq m$ for all n is the smallest integer for which $\oplus_m \left(\sum \{x_i\}_{i=1}^{\infty}\right)$ is an interval, and in the same paper Nymann[10] provides the following example:

Example. The Cantor Ternary set is the set of subsums of $\sum x_n$, where $x_n = 2/3^n$. By the previous theorem,

$$\frac{x_n}{r_n} = \frac{2/3^n}{\sum_{k=n+1}^{\infty} 2/3^k} = 2$$

Thus, once again showing$\mathfrak{C} + \mathfrak{C} = \oplus_2 \mathfrak{C}$ is an interval.

Theorem 21 *Assume $x_{n+1}/x_n < x \leq 1$ for all n. Then $E + xE$ is an interval if and only if*

$$\frac{(x_n - r_n)}{r_{n-1}} \leq x \leq \frac{r_n}{(x_n - r_n)}$$

for all n for which $x_n > r_n$.

Now, we can talk about Cantor sets in terms of the the subsums of definition 1, and analogous to theorem 11, the following theorem is formulated.

Theorem 22 *Let $a, b \in \mathbb{R}$, then*

1. $\sum \{x_i\}_{i=1}^{\infty} + \sum_{i=1}^{\infty} x_n = \sum \{x_i\}_{i=1}^{\infty}$

2. $a\left(\sum \{x_i\}_{i=1}^{\infty}\right) + b\left(\sum \{x_i\}_{i=1}^{\infty}\right) = a\left(\sum \{x_i\}_{i=1}^{\infty}\right) + (-b)\left(\sum \{x_i\}_{i=1}^{\infty}\right) + b\sum_{i=1}^{\infty} x_n$

3. $a\left(\sum \{x_i\}_{i=1}^{\infty}\right) + b\left(\sum \{x_i\}_{i=1}^{\infty}\right) = a\left[\left(\sum \{x_i\}_{i=1}^{\infty}\right) + \frac{b}{a}\left(\sum \{x_i\}_{i=1}^{\infty}\right)\right]$

Chapter 6

Connecting The Dots

6.1 Bringing It Together

Theorem 23 *(Agüero) For all integers $n \geq 3$,*

$$\frac{1}{n-1}\mathfrak{C}_{\frac{1}{n}} \quad = \quad \sum_{i=1}^{\infty} \frac{\varepsilon_i}{n^i},\ \varepsilon_i \in \{0,1\}\,.$$

Theorem 24 *(Agüero) For all integers $n \geq 3$,*

$$\oplus_{n-1}\mathfrak{C}_{\frac{1}{n}} = [0, n-1]$$

Proof. Let $b \in \mathbb{I}$. Then for all b its n-nary expansion is,

$$b = \sum_{i=1}^{\infty} \frac{b_i}{n^i},\ b_i \ \in \ \{0,1,2,\ldots,n-1\}$$

Let $b_i = a_{i_1} + a_{i_2} + \cdots + a_{i_{n-1}}$ where $a_i, b_i \in \mathbb{N}$. It follows that

$$\begin{aligned}
\mathbb{I} &= \sum \left\{\frac{b_i}{n^i}\right\}_{i=1}^{\infty},\ c \ \in \ \{0,1,2,\ldots,n-1\} \\
&= \sum \left\{\frac{a_{i_1}}{n^i} + \cdots + \frac{a_{i_{n-1}}}{n^i}\right\}_{i=1}^{\infty},\ a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}} \ \in \ \{0,1\} \\
&= \sum \left\{\frac{a_{i_1}}{n^i}\right\}_{i=1}^{\infty} + \cdots + \sum \left\{\frac{a_{i_{n-1}}}{n^i}\right\}_{i=1}^{\infty} \\
&= \frac{1}{n-1}\mathfrak{C}_{\frac{1}{n}} + \cdots + \frac{1}{n-1}\mathfrak{C}_{\frac{1}{n}}
\end{aligned}$$

therefore

$$\begin{aligned}
\mathfrak{C}_{\frac{1}{n}} + \cdots + \mathfrak{C}_{\frac{1}{n}} &= (n-1)\mathbb{I} \\
&= [0, n-1]
\end{aligned}$$

$\square$

Theorem 25 *(Agüero) If for integers $k_1 \geq k_2 \geq \cdots \geq k_m$,*

$$\left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \bmod n \;=\; \{0, 1, \ldots, n-1\},$$

then

$$\left(\sum \left\{\left(\varepsilon_{m(t-1)+1}k_1 + \varepsilon_{m(t-1)+2}k_2 + \cdots + \varepsilon_{mt}k_m\right) q^i\right\}_{i=1}^\infty\right),\ \varepsilon_i \in \{0,1\}, \tag{6.1}$$

is a Cantorval if

$$\frac{1}{n+1} \leq q < \frac{k_m}{K + k_m}.$$

Proof. Let

$$p_j \;\equiv\; j \bmod n,$$

and

$$\begin{aligned}\left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \bmod n \;&=\; \{0, 1, \ldots, n-1\},\\ &=\; \{p_0 \bmod n, p_1 \bmod n, \ldots, p_{n-1} \bmod n\}.\\ &\quad\; \{n_0 + p_0 \bmod n, n_0 + p_1 \bmod n, \ldots, n_0 + p_{n-1} \bmod n\},\end{aligned}$$

where $n_0 = 0$. Thus, by theorem 15 *(Bartoszewics)*, for

$$k_j \bmod n \in \left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \bmod n \;=\; \{p_0 \bmod n, p_1 \bmod n, \ldots, p_{n-1} \bmod n\}$$

6.1 is a Cantorval. However, by theorem 14 *(Kenyon/Nitecki)*, 6.1 is once again a Cantorval for,

$$k_j \in \sum \{\alpha_i k_i\}_{i=1}^m \;=\; \{p_0, p_1, \ldots, p_{n-1}\}.$$

Hence, If for integers $k_1 \geq k_2 \geq \cdots \geq k_m$,

$$\left(\sum \{\alpha_i k_i\}_{i=1}^m\right) \bmod n \;=\; \{0, 1, \ldots, n-1\},$$

it follows that,

$$\left(\sum\left\{\left(\varepsilon_{m(t-1)+1}k_1+\varepsilon_{m(t-1)+2}k_2+\cdots+\varepsilon_{mt}k_m\right)q\right\}_{i=1}^{\infty}\right),\ \varepsilon_i\in\{0,1\},$$

is a Cantorval if

$$\frac{1}{n+1}\leq q<\frac{k_m}{K+k_m}.$$

□

Theorem 26 *(Agüero) If for integers $k_1 \geq k_2 \geq \cdots \geq k_m$,*

$$\left(\sum\{\alpha_i k_i\}_{i=1}^{m}\right)\bmod n \quad = \quad \{0,1,\ldots,n-1\},$$

then

$$k_1\mathfrak{C}_{\frac{1}{n}}+k_2\mathfrak{C}_{\frac{1}{n}}+\cdots+k_m\mathfrak{C}_{\frac{1}{n}},$$

is a Cantorval.

Proof. Recall that, $\frac{1}{n-1}\mathfrak{C}_{\frac{1}{n}}=\sum_{i=1}^{\infty}\frac{\varepsilon_i}{n^i},\ \varepsilon_i\in\{0,1\}$. Then

$$\begin{aligned}k_1\mathfrak{C}_{\frac{1}{n}}+k_2\mathfrak{C}_{\frac{1}{n}}+\cdots+k_m\mathfrak{C}_{\frac{1}{n}} &= \oplus_{i=1}^{m}k_i(n-1)\sum_{i=1}^{\infty}\frac{\varepsilon_i}{n^i},\ \varepsilon_i\in\{0,1\}\\ &= (n-1)\oplus_{i=1}^{m}k_i\sum_{i=1}^{\infty}\frac{\varepsilon_i}{n^i},\ \varepsilon_i\in\{0,1\}\end{aligned}$$

which, by theorem 25 *(Agüero)*, is a Cantorval. □

6.2 Guthrie-Nymann Set Extended and Generalized

Now that we are bringing the material together to obtain new results we observe that we are able to use the Cantor set equvalent when working with series, or vice versa, hence allowing us to produce new results. Take for example the Guthrie-Nymann set

$$T=\sum_{i=1}^{\infty}\left[\varepsilon_{2i-1}\left(\frac{3}{4^i}\right)+\varepsilon_{2i}\left(\frac{2}{4^i}\right)\right],\ \varepsilon_i\in\{0,1\},\ \forall i\in\mathbb{N}$$

Using Bartoszewics' notation,

$$T = \left(3, 2; \frac{1}{4}\right).$$

We will now extend this result.

Upon closer observation we see that

$$\left(\underbrace{3, \ldots, 3}_{k\ 3\text{'s}}, \underbrace{2, \ldots 2}_{k'\ 2\text{'s}}; \frac{1}{3k + 2k' - 1}\right)$$

for $k, k' \in \mathbb{N}$ is a Cantorval. First, we notice that the linear combinations of the terms are a linear of 2's and 3's. In 1884 Alexander[13] showed that if the $\gcd(a, b) = 1$, then every consecutive integer greater than $ab - a - b$ is a linear combination of a and b. For $a = 3$ and $b = 2$ we have $\gcd(3, 2) = 1$ therefore every consecutive integer $n > 1$, $n = 3k + 2k'$. Now, fix k and k', then for every $m = 3q + 2q'$, $n - m$ is a linear combination of 2's and 3's with the exception of when $m = 1$, again by Alexander's result. Hence, for $i = 1, \ldots, k$ and $j = 1, \ldots, k'$,

$$\left(\sum \{3i + 2j\}\right) = \{0, 2, 3, \ldots, 3k + 2k' - 1, 3k + 2k'\}.$$

It follows that,

$$\{0, 2, 3, \ldots, 3k + 2k' - 1, 3k + 2k'\} \bmod (3k + 2k' - 1) = \{1, 2, \ldots, 3k + 2k' - 2\},$$

and by theorem 25 *(Agüero)*,

$$\left(\underbrace{3, \ldots, 3}_{k\ 3\text{'s}}, \underbrace{2, \ldots 2}_{k'\ 2\text{'s}}; \frac{1}{3k + 2k' - 1}\right)$$

is a Cantorval.

In summary, the result can be restated as follows.

Theorem 27 *(Agüero)[The extended, and generalized, Guthrie-Nymann set] For $k, k' \in \mathbb{N}$,*

$$\left(\underbrace{3,\ldots,3}_{k\ 3\text{'s}},\underbrace{2,\ldots 2}_{k'\ 2\text{'s}};\frac{1}{3k+2k'-1}\right)$$

is a Cantorval.

Conjecture 1 *(Agüero) For* $r \in \left(\frac{1}{n}, \frac{1}{n-1}\right)$,

$$\oplus_{n-1}\mathfrak{C}_r = [0, n-1]$$

Chapter 7

Future Work

Here are some things we know, and some we don't:

1. All known examples of sequences generating Cantorvals as their achievement sets have been multigeometric sequences, the simplest which was found by (Guthrie-Nymann)[4], namely,

$$T = \sum_{i=1}^{\infty} \left[\varepsilon_{2i-1} \left(\frac{3}{4^i} \right) + \varepsilon_{2i} \left(\frac{2}{4^i} \right) \right], \ \varepsilon_i \in \{0,1\}, \ \forall\, i \in \mathbb{N}$$

2. It is not known whether all multigeometric sequences generating Cantorvals have been discovered.

3. We do not yet know if only multigeometric sequences produce Cantorvals, or if others exist.

4. Many conjectures remain without proof or counterexample.

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Curriculum Vitae

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