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Symmetry Operator for the Maxwell Equations

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REPORT

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ABSTRACT

In this work is shown that when the background space-time is type D solution of the Einstein vacuum equations with or without cosmological constant, a symmetry operator can be found for the Maxwell equations, as a consequence of that each component of the electromagnetic *spinor* satisfies a decoupled equation and that all the vacuum type D metrics admit a two index Killing *spinor*. Besides, the Maxwell equations are solved when the background space-time is the Carter A metric.

Contents

Chapter 1

Introduction

When a set of differential equations that describe a physical system can be solved by the method of separation of variables, the symmetries appeared like operators [1]; this operators are such that produce a mapping into the solution space itself. For instances the symmetry operators are operators that commute with the differential operator that is present in the system of differential equations. These operator had been obtained for the equations that describe fields massless spin $\frac{1}{2}, 1$ and $\frac{3}{2}$[2-5], and the Dirac equation too [6-7]. In this report show that the symmetry operator introduced in the Ref.[3], without proof for the Maxwell equations is a solution type$D$ of the Einstein equations in vacuum with or without cosmological constant, can be obtain trough the followings properties: a)each component of the electromagnetic *spinor* satisfy a differential equation decoupled of second order and b)that all the solutions type$D$with or without cosmological constant have a Killing *spinor* with two index.

In the chapter 2 is given the abstract of the *spinorial* formalism and the Newman-Penrose notation, and the other conventions that will be use in all this report. In the chapter 3 is studied the condition under the differential operator satisfy an equation decoupled for each component of the electromagnetic *spinor*, is founded that operator in a background space-time of type$D$.Besides is written the decoupled equations in a covariant way and the symmetry operator obtained for the Maxwell equations. Following the Ref.[8] and in the background space-time Carter A metric, in the chapter 4 is shown that for the symmetry operator is possible obtain a complete solution of the Maxwell equations from the solution of one decoupled equation with components with weight spin maxima for the electromagnetic *spinor*.

Chapter 2

Spinorial Formalism

2.1 Spinors

The spinors are elements of a vectorial space complex of dimension two with an inner antisymmetry given by

$$\{\psi, \zeta\} \equiv \epsilon_{AB} \psi^A \zeta^B, \tag{2.1}$$

where ψ^A and ζ^B represent the components of the 1-spinors ψ and ζ respectively, the index A and B take the values 0 and 1, while that the matrix ϵ_{AB} has the property defined by $\epsilon_{AB} = \epsilon^{AB}$. The rule of rise and low the index is

$$\psi_B = \psi^A \epsilon_{AB} and \psi^A = \psi_B \epsilon^{AB}. \tag{2.2}$$

Under a spin transformation by the matrix $({L_A}^B)$ is element of the group $SL(2, C)$, with the respective transformation rules.The spinors with more indexes have a similar spin transformation given by

$$\Psi'_{AB...D} = {L_A}^E {L_B}^F ... {L_D}^G \Psi_{EF...G}. \tag{2.3}$$

The relation among tensorial and spinorial components is determinate by

$$T^{A\dot{A}B\dot{B}\cdots}{}_{C\dot{C}\cdots} = {\sigma_\alpha}^{A\dot{A}} {\sigma_\beta}^{B\dot{B}} ... {\sigma^\gamma}_{C\dot{C}} T^{\alpha\beta\cdots}_{\gamma\cdots}, \tag{2.4}$$

By notation the Greek letters of the tensorial index have two spinorial indexes one with dot and other without dot, and the mixed objet ${\sigma_\alpha}^{A\dot{A}}$ the Greek letter take four values and the ${\sigma_\alpha}^{A\dot{A}}$ matrix is a Hermitic matrix $(2 * 2)$ connected with the metric tensor with signature -2.

For the case of a tensorial field antisymmetric with spinorial components, here the tensor is $F_{\mu\nu} = -F_{\nu\mu}$ then the spinorial components are represented by$F_{A\dot{A}B\dot{B}}$ such that $F_{A\dot{A}B\dot{B}} = -F_{B\dot{B}A\dot{A}}$ then can be written

$$F_{A\dot{A}B\dot{B}} = \epsilon_{AB}\psi_{\dot{A}\dot{B}} + \varphi_{AB}\epsilon_{\dot{A}\dot{B}}, \tag{2.5}$$

where $\psi_{\dot{A}\dot{B}} = \psi_{(\dot{A}\dot{B})}$ is symetrric, is the same for the spinor without dots, for instance if $F_{\mu\nu}$ is a tensor antisymmetry then is written by

$$F_{A\dot{A}B\dot{B}} = \varphi_{AB}\epsilon_{\dot{A}\dot{B}} + \epsilon_{AB}\varphi_{\dot{A}\dot{B}}. \tag{2.6}$$

The electromagnetic field with six components is reduced to three complex components in a base given and are denoted for $\varphi_0 \equiv \varphi_{AB}o^A o^B$, $\varphi_1 \equiv \varphi_{AB}o^A$-$\iota^B$ and $\varphi_2 \equiv \varphi_{AB}\iota^A\iota^B$.

In analogy is possible built the respective spinor for the Riemann curvature tensor, the spinorial components of this tensor is given by [9]

$$R_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = X_{ABCD}\epsilon_{\dot{A}\dot{B}} + \Phi_{AB\dot{C}\dot{D}}\epsilon_{\dot{A}\dot{B}}\epsilon_{CD} + \Phi_{\dot{A}\dot{B}CD}\epsilon_{AB}\epsilon_{\dot{C}\dot{D}} + X_{\dot{A}\dot{B}\dot{C}\dot{D}}\epsilon_{AB}\epsilon_{CD}, \tag{2.7}$$

with

$$X_{ABCD} = \frac{1}{4}R_{A\dot{E}BC\dot{F}D}{}^{\dot{E}\dot{F}} = \Psi_{ABCD} + \Lambda(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}), \Phi_{AB\dot{C}\dot{D}} = \frac{1}{4}R^{\dot{E}F}_{A\dot{E}BF\dot{C}\dot{D}} \tag{2.8}$$

Where $\Psi_{ABCD} = \Psi_{(ABCD)}$ represent the spinorial components of the Weyl tensor, $\Phi_{AB\dot{C}\dot{D}} = \Phi_{(AB)(\dot{C}\dot{D})}$ that represent the spinorial components of the Ricci tensor without trace and $\Lambda = \frac{R}{24}$ where R is the scalar curvature.

The classification of the Weyl spinor is given in six cases to know:

Type I or $\{1111\}$.This the case algebraic general, four principal directions null.

Type II or $\{211\}$:This case algebraic special where two principal directions are merged.

Type D or $\{22\}$.This case two principal directions are merged but both different.

Type III or $\{31\}$. Three principal directions are merged.

Type N or $\{4\}$. The four principal directions null are merged.

Type O or conformal flat space, has not principal directions null.

This clasification also is known like Petrov Classification.

2.2 Covariant differentiation

In general relativity with the properties of the tensors and spinors is necessary to give the definition of covariant differentiation. The symbol ∇_μ,or their respect operator differential $\nabla_{A\dot{A}}$ that will be used in this report, and is defined by

$$\nabla_{A\dot{A}}K_B = \partial_{A\dot{A}}K_B - \Gamma^C_{BA\dot{A}}K_C, \tag{2.9}$$

where $\Gamma^C_{BA\dot{A}}$ are called spin coefficients . This differential has the following properties

a)Linear

$\nabla_\mu(S^{\cdots} + T^{\cdots}) = \nabla_\mu S^{\cdots}T^{\cdots} + S^{\cdots}\nabla_\mu T^{\cdots}$.

b)Leibnitz Rule

$\nabla_(S^{\cdots}T^{\cdots}) = \nabla_\mu(S^{\cdots})T^{\cdots} + S^{\cdots}\nabla_\mu(T^{\cdots})$.

c) Partial over scalars

$\nabla_\mu\phi = \partial_\mu\phi = \phi_{,\mu}$.

d) Real

$(\nabla_\mu S^{\cdots}) = \nabla_\mu(S...)$.

e)$\nabla_\mu\epsilon_{AB} = 0$, and $\nabla_\mu\sigma^{B\dot{B}}_\mu = 0$.

Using this property obtained that the spin coefficients are such that

$$\Gamma_{ABC\dot{D}} = \Gamma_{BAC\dot{D}} \tag{2.10}$$

2.3 Bianchi and Ricci identities

The Riemann curvature tensor $R_{\mu\nu\alpha\beta}$ is given by the commutative derivatives over vector or tensors, then is possible obtain a connection with the spinors that represent to $R_{\mu\nu\alpha\beta}$ will be appeared so that commutator applied to the spinors.The Ricci identity without torsion in spinorial form is

$$\Delta_{\mu\nu}K^{\beta\alpha} = R_{\mu\nu\rho^\beta}K^{\rho\alpha} + R_{\mu\nu\rho^\alpha}K^{\beta\rho}. \tag{2.11}$$

In the spinorial formalism this equation is the same that is given at [9]

$$\Box_{AB}K^C = X_{ABE^C}K^E, and\Box_{\dot{A}\dot{B}}K^C = \Phi_{\dot{A}\dot{B}E^C}K^E, \tag{2.12}$$

where

$$\Box_{AB} \equiv \nabla_{\dot{X}(A} \nabla_{B)}{}^{\dot{X}}, and \Box_{\dot{A}\dot{B}} \equiv \nabla_{X(\dot{A}} \nabla_{\dot{B})}{}^{X}. \tag{2.13}$$

Is known the relation between the operator of covariant derivative and the curvature, called Bianchi identity and their tensorial representation is given by

$$\nabla_{[\alpha} R_{\beta\gamma]\delta\rho} = 0. \tag{2.14}$$

The spinorial representation given also in [9]

$$\nabla_{\dot{B}}{}^{A} X_{ABCD} = \nabla_{B}{}^{\dot{A}} \Phi_{CD\dot{A}\dot{B}}. \tag{2.15}$$

and their complex conjugate.

2.4 Newmann-Penrose Formalism

The Newmann-Penrose formalism for any tensor component are determinate with respect to the tetrada null

$$D \equiv l^{\mu}\partial_{\mu}, \Delta \equiv n^{\mu}\partial_{\mu}, \delta \equiv m^{\mu}\partial_{\mu}, \overline{\delta} \equiv \overline{m}^{\mu}\partial_{\mu}, \tag{2.16}$$

where D and Δ are reals, while that δ and $\overline{\delta}$ are conjugate complex and satisfy the following properties

$$l \cdot n = -m \cdot \overline{m} = 1, l \cdot m = l \cdot \overline{m} = n \cdot m = n \cdot \overline{m} = 0. \tag{2.17}$$

Given the tretada null the spin coefficients can be determinate of the following relations

$$\kappa = m^{\mu} l_{\mu;\nu} l^{\nu} = -l^{\mu} m_{\mu;\nu} l^{\nu},$$

$$\sigma = m^{\mu} l_{\mu;\nu} m^{\nu} = -l^{\mu} m_{\mu;\nu} m^{\nu},$$

$$\rho = m^{\mu} l_{\mu;\nu} \overline{m}^{\nu} = -l^{\mu} m_{\mu;\nu} \overline{m}^{\nu},$$

$$\tau = m^\mu l_{\mu;\nu} n^\nu = -l^\mu m_{\mu;\nu} n^\nu,$$

$$\varepsilon = \frac{1}{2}(n^\mu l_{\mu;\nu} l^\nu + m^\mu \overline{m}_{\mu;\nu} l^\nu),$$

$$\beta = \frac{1}{2}(n^\mu l_{\mu;\nu} m^\nu + m^\mu \overline{m}_{\mu;\nu} m^\nu),$$

$$\alpha = \frac{1}{2}(n^\mu l_{\mu;\nu} \overline{m}^\nu + m^\mu \overline{m}_{\mu;\nu} \overline{m}^\nu),$$

$$\gamma = \frac{1}{2}(n^\mu l_{\mu;\nu} n^\nu + m^\mu \overline{m}_{\mu;\nu} n^\nu),$$

$$\pi = n^\mu \overline{m}_{\mu;\nu} l^\nu = -\overline{m}^\mu n_{\mu;\nu} l^\nu,$$

$$\mu = n^\mu \overline{m}_{\mu;\nu} m^\nu = -\overline{m}^\mu n_{\mu;\nu} m^\nu,$$

$$\lambda = n^\mu \overline{m}_{\mu;\nu} \overline{m}^\nu = -\overline{m}^\mu n_{\mu;\nu} \overline{m}^\nu,$$

$$\nu = n^\mu \overline{m}_{\mu;\nu} n^\nu = -\overline{m}^\mu n_{\mu;\nu} n^\nu \tag{2.18}$$

The commutators among the directional derivative is given with respect to the null tetrada equation (2.17)

$$\Delta D - D\Delta = (\gamma + \overline{\gamma})D + (\varepsilon + \overline{\varepsilon})\Delta - (\overline{\tau} + \pi)\delta - (\tau + \overline{\pi})\overline{\delta},$$

$$\delta D - D\delta = (\overline{\alpha} + \beta - \overline{\pi})D + \kappa\Delta - (\overline{\rho}) + \varepsilon - \overline{\varepsilon})\delta - \sigma\overline{\delta},$$

$$\delta\Delta - \Delta\delta = -\overline{\nu}D + (\tau - \overline{\alpha} - \beta)\Delta + (\mu - \gamma + \overline{\gamma})\delta + \overline{\lambda\delta},$$

$$\overline{\delta}\delta - \delta\overline{\delta} = (\overline{\mu} - \mu)D + (\overline{\rho} - \rho)\Delta + (\alpha - \overline{\beta})\delta + (\beta - \overline{\alpha})\overline{\delta}, \tag{2.19}$$

In the Newmann-Penrose formalism, the ten independient components of Weyl tensor are given by five complex scalars defined by

$$\Psi_0 = C_{\mu\nu\rho\sigma} l^\mu m^\nu l^\rho m^\sigma,$$

$$\Psi_1 = C_{\mu\nu\rho\sigma} l^\mu n^\nu l^\rho m^\sigma,$$

$$\Psi_2 = C_{\mu\nu\rho\sigma} l^\mu m^\nu \overline{m}^\rho n^\sigma,$$

$$\Psi_3 = C_{\mu\nu\rho\sigma} l^\mu n^\nu \overline{m}^\rho n^\sigma,$$

$$\Psi_4 = C_{\mu\nu\rho\sigma} n^\mu \overline{m}^\nu n^\rho \overline{m}^\sigma, \tag{2.20}$$

Also can be written the traceless Ricci tensor given by the scalar

$$\Phi_{00} = -\frac{1}{2}R_{11}; \Phi 01 = \overline{\Phi}_{10} = -\frac{1}{2}R_{13} = -\frac{1}{2}\overline{R}_{14};$$

$$\Phi_{11} = -\frac{1}{4}(R_{12} + R_{34}); \Phi 12 = \overline{\Phi}_{21} = -\frac{1}{2}R_{23} = -\frac{1}{2}\overline{R}_{24};$$

$$\Phi_{22} = -\frac{1}{2}R_{22}; \Phi 02 = \overline{\Phi}_{20} = -\frac{1}{2}R_{33} = -\frac{1}{2}\overline{R}_{44};$$

$$\Lambda = \frac{1}{24}R = \frac{1}{12}(R_{12} - R_{34}); \tag{2.21}$$

Ricci Identities

$$D\delta - \overline{\delta}\kappa = (\rho^2 + \sigma\overline{\sigma}) + \rho(\varepsilon + \overline{\varepsilon}) - \overline{\kappa}\tau - \kappa(3\alpha + \overline{\beta} - \pi) + \Phi_{00},$$

$$D\sigma - \delta\kappa = \sigma(\rho + \overline{\rho} + 3\varepsilon - \overline{\varepsilon}-) - \kappa(\tau - \overline{\pi} + \overline{\alpha} + 3\beta) + \Psi_0,$$

$$D\tau - \Delta\kappa = \rho(\tau + \overline{\pi}) + \sigma(\overline{\tau} + \pi) + \tau(\varepsilon - \overline{\varepsilon}) - \kappa(3\gamma + \overline{\gamma}) + \Psi_1 + \Phi_{01},$$

$$D\alpha - \overline{\delta}\varepsilon = \alpha(\rho + \overline{\varepsilon} - 2\varepsilon) + \beta\overline{\sigma} - \overline{\beta}\varepsilon - \kappa\lambda - \overline{\kappa}\gamma + \pi(\varepsilon + \rho) + \Phi_{10},$$

$$D\beta - \delta\varepsilon = \sigma(\alpha + \pi) + \beta(\overline{\rho} - \overline{\varepsilon}) - \kappa(\mu + \gamma) - \varepsilon(\overline{\alpha} - \overline{\pi}) + \Psi_1,$$

$$D\gamma - \Delta\varepsilon = \alpha(\tau + \overline{\pi}) + \beta(\overline{\tau} + \pi) - \gamma(\varepsilon + \overline{\varepsilon}) - \varepsilon(\gamma + \overline{\gamma}) + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Lambda,$$

$$D\lambda - \overline{\delta}\pi = (\rho\lambda + \overline{\sigma}\mu) + \pi(\pi + \alpha - \overline{\beta}) - \nu\overline{\kappa} - \lambda(3\varepsilon - \overline{\varepsilon}) + \Phi_{20},$$

$$D\mu - \delta\pi = (\overline{\rho}\mu + \sigma\lambda) + \pi(\overline{\pi} - \overline{\alpha} + \beta) - \mu(\varepsilon + \overline{\varepsilon}) - \nu\kappa + \Psi_2 + 2\Lambda,$$

$$\nu - \Delta\pi = \mu(\pi + \overline{\tau}) + \lambda(\overline{\pi} + \tau) + \pi(\gamma - \overline{\gamma}) - \nu(3\varepsilon + \overline{\varepsilon}) + \Psi_3 + \Phi_{21},$$

$$\Delta\lambda - \overline{\delta}\nu = -\lambda(\mu + \overline{\mu} + 3\gamma - \overline{\gamma}) + \nu(3\alpha + \overline{\beta} + \pi - \overline{\tau}) - \Psi_4,$$

$$\delta\rho - \overline{\delta}\sigma = \rho(\overline{\alpha} + \beta) - \sigma(3\alpha - \overline{\beta}) + \tau(\rho - \overline{\rho}) + \kappa(\mu - \overline{\mu}) - \Psi_1 + \Phi_{01},$$

$$\delta\alpha - \overline{\delta}\beta = (\mu\rho - \lambda\sigma) + \alpha\overline{\alpha} + \beta\overline{\beta} - 2\alpha\beta + \gamma(\rho - \overline{\rho}) + \varepsilon(\mu - \overline{\mu}) - \Psi_2 + \Phi_{11} + \Lambda,$$

$$\delta\lambda - \overline{\delta}\mu = \nu(\rho + \overline{\rho}) + \pi(\mu - \overline{\mu}) + \mu(\alpha + \overline{\beta}) + \lambda(\overline{\alpha} - 3\beta) - \Psi_3 + \Phi_{21},$$

$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\overline{\lambda}) + \mu(\gamma - \overline{\gamma}) - \overline{\nu}\pi + \nu(\tau - 3\beta - \overline{\alpha}) + \Phi_{22},$$

$$\delta\gamma - \Delta\beta = \gamma(\tau - \overline{\alpha} - \beta) + \mu\tau - \sigma\nu - \varepsilon\overline{\nu}) - \beta(\gamma - \overline{\gamma} - \mu) + \alpha\overline{\lambda} + \Phi_{12},$$

$$\delta\tau - \Delta\sigma = (\mu\sigma + \overline{\lambda}\rho) + \tau(\tau + \beta - \overline{\alpha}) - \sigma(3\gamma - \overline{\gamma}) - \kappa\overline{\nu} + \Phi_{02},$$

$$\Delta\rho - \overline{\delta}\tau = -(\rho\overline{\mu} + \sigma\lambda) + \tau(\overline{\beta} - \alpha - \overline{\tau}) + \rho(\gamma - \overline{\gamma}) + \nu\kappa - \Psi_2 - 2\Lambda,$$

$$\Delta\alpha - \overline{\delta}\gamma = \nu(\rho + \varepsilon) - \lambda(\tau + \beta) + \alpha(\overline{\gamma} - \overline{\mu}) + \gamma(\overline{\beta} - \overline{\tau}) - \Psi_3. \qquad (2.22)$$

Bianchi Identities in vacuum

$$D\Psi_1 - \overline{\delta}\Psi_0 = -3\kappa\Psi_2 + 2(\varepsilon + 2\rho)\Psi_1 + (\pi - 4\alpha)\Psi_0,$$

$$D\Psi_2 - \overline{\delta}\Psi_1 = -2\kappa\Psi_3 + 3\rho\Psi_2 + 2(\pi - \alpha)\Psi_1 - \lambda\Psi_0,$$

$$D\Psi_3 - \overline{\delta}\Psi_2 = -\kappa\Psi_4 - 2(\varepsilon - \rho)\Psi_3 + 3\pi\Psi_2 - 2\lambda\Psi_1,$$

$$D\Psi_4 - \overline{\delta}\Psi_3 = -(4\varepsilon - \rho)\Psi_4 + 2(2\pi + \alpha)\Psi_3 - 3\lambda\Psi_2,$$

$$\Delta\Psi_0 - \delta\Psi_1 = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2,$$

$$\Delta\Psi_1 - \delta\Psi_2 = \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - \tau\Psi_2 + 2\sigma\Psi_3,$$

$$\Delta\Psi_2 - \delta\Psi_3 = 2\nu\Psi_1 - 3\mu\Psi_2 - 2(\tau - \beta)\Psi_3 + \sigma\Psi_4,$$

$$\Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 - (\tau - 4\beta)\Psi_4. \tag{2.23}$$

2.5 Killing Spinors

The killing spinor with two indexes [10], is a spinorial field L_{AB} is such that satisfy the equation

$$\nabla_{\dot{A}(B} L_{C\dot{D})} = 0. \tag{2.24}$$

Killing spinor L_{AB} then satisfy for any vectorial complex field if this exist $K_{A\dot{B}}$ such that

$$\nabla_{A\dot{R}} L_{BC} = \frac{1}{3}(\varepsilon_{AB} K_{C\dot{R}} + \varepsilon_{AC} K_{B\dot{R}}). \tag{2.25}$$

When the Weyl spinor is not zero and the space-time type D has the solution

$$L_{AB} = -2\phi^{-1} o_{(A}\iota_{B)}, \tag{2.26}$$

Where o^A and ι^B are the principal spinors of Ψ_{ABCD} and ϕ is a complex function that satisfy

$$o^B \nabla_{A\dot{C}} o_B = o_A o^B \partial_{B\dot{C}} ln\phi,$$

$$\iota^B \nabla_{A\dot{C}} \iota_B = \iota_A \iota^B \partial_{B\dot{C}} ln\phi, \tag{2.27}$$

Under the conditions that the unique component different of zero of L_{AB} thus

$$\kappa = \sigma = \lambda = \nu = 0,$$

$$\rho = Dln\phi, \tau = \delta ln\phi, \pi = -\overline{\delta} ln\phi, \mu = -\Delta ln\phi. \tag{2.28}$$

Chapter 3

Symmetry Operator for the Maxwell equations

The Maxwell equation without sources are given by [9]

$$\nabla^{A\dot{C}}\varphi_{AB} = 0. \tag{3.1}$$

In Newmann-Penrose notation can be written

$$(\overline{\delta} - 2\alpha + \pi)\varphi_0 - (D - 2\rho)\varphi_1 - \kappa\varphi_2 = 0,$$

$$(\Delta - 2\gamma + \mu)\varphi_0 - (\delta - 2\tau)\varphi_1 - \sigma\varphi_2 = 0,$$

$$(\overline{\delta} - 2\pi)\varphi_1 - (D + 2\varepsilon - \rho)\varphi_2 - \lambda\varphi_0 = 0,$$

$$(\Delta + 2\mu)\varphi_1 - (\delta + 2\beta - \tau)\varphi_2 - \nu\varphi_0 = 0, \tag{3.2}$$

and the electromagnetic spinor given in terms of a vectorial potential $\Phi_{A\dot{A}}$ such that

$$\varphi_{AB} = \nabla_{\dot{A}(A}\Phi^{\dot{A}}_{B)} \tag{3.3}$$

Our interest is to find a symmetry operator for the Maxwell equations given in [3], that is consequence of the existence of decoupled equations for each component of φ_{AB}, then is necessary to determine a differential operator of first order in this form

$$\mathcal{S}^{B\dot{M}} = gH^{AB}\nabla_A{}^{\dot{M}}f, \tag{3.4}$$

Where f and g are scalar functions and H^{AB} is a spinor of two indexes, that when is applied to (3.1) is obtained a decoupled equation for each component of the Maxwell field , then defined

$$J_{B\dot{M}} \equiv \nabla_{\dot{M}}{}^{A}\varphi_{AB},$$

with φ_{AB} satisfy the Maxwell equations without sources, $J_{B\dot{M}}$ then the relation is such that

$$\mathcal{S}^{B\dot{M}}J_{B\dot{M}} = Scalarequationfor\varphi_0, \varphi_1 or \varphi_2. \tag{3.5}$$

Using the Newmann-Penrose notation this expression can be written like this

$$\begin{aligned}
\mathcal{S}^{B\dot{M}}J_{B\dot{M}} = fg\{H^{00}[(\delta - \overline{\alpha} - \beta + \overline{\pi} + \delta lnf)J_{0\dot{0}} \\
-(D - \varepsilon + \overline{\varepsilon} - \overline{\rho} + Dlnf)J_{0\dot{1}} + \sigma J_{1\dot{0}} - \kappa J_{1\dot{1}}] \\
H^{10}[(\delta + \beta - \overline{\alpha} + \overline{\pi} + \tau + \delta lnf)J_{1\dot{0}} \\
-(D + \varepsilon + \overline{\varepsilon} - \overline{\rho} + \rho + Dlnf)J_{1\dot{1}} \\
+(\Delta - \gamma + \overline{\gamma} - \mu + \overline{\mu} + \rho + \Delta lnf)J_{0\dot{0}} \\
-(\overline{\delta} - \alpha + \overline{\beta} - \overline{\tau} - \pi + \overline{\delta}lnf)J_{0\dot{1}}] \\
+H^{11}[(\Delta + \gamma - \overline{\gamma} + \overline{\mu} + \Delta lnf)J_{1\dot{0}} \\
-(\overline{\delta} + \alpha + \overline{\beta} - \overline{\tau} + \overline{\delta}lnf)J_{1\dot{1}}] - \nu J_{0\dot{0}} + \lambda J_{0\dot{1}}]\},
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
J_{0\dot{0}} &= [(\overline{\delta} - 2\alpha + \pi)\varphi_0 - (D - 2\rho)\varphi_1 - \kappa\varphi_2], \\
J_{0\dot{1}} &= [(\Delta - 2\gamma + \mu)\varphi_0 - (\delta - 2\tau)\varphi_1 - \sigma\varphi_2], \\
J_{1\dot{0}} &= [(\overline{\delta} + 2\pi)\varphi_1 - (D + 2\varepsilon - \rho)\varphi_2 - \lambda\varphi_0], \\
J_{1\dot{1}} &= [(\Delta + 2\mu)\varphi_1 - (\delta + 2\beta - \tau)\varphi_2 - \nu\varphi_0].
\end{aligned} \tag{3.7}$$

In the case that the component $H^{00} \neq 0$, then

$$\begin{aligned}
\mathcal{S}^{B\dot{M}}J_{B\dot{M}} = fgH^{00}\{[(\delta - \overline{\alpha} - \beta + \overline{\pi} + \delta lnf)(\overline{\delta} - 2\alpha + \pi) - \sigma\lambda \\
-(D - \varepsilon + \overline{\varepsilon} - \overline{\rho} + Dlnf)(\Delta - 2\gamma + \mu) + \kappa\nu)]\varphi_0 \\
+[(D - \varepsilon + \overline{\varepsilon} - \overline{\rho} + Dlnf)(\delta - 2\tau) - \kappa(\Delta + 2\mu) \\
-(\delta - \overline{\alpha} - \beta + \overline{\pi} + \delta lnf)(D - 2\rho) + \sigma(\overline{\delta} + 2\pi)]\varphi_1 \\
+[(D - \varepsilon + \overline{\varepsilon} - \overline{\rho} + Dlnf)\sigma + \kappa(\delta + 2\beta - \tau) \\
-(\delta - \overline{\alpha} - \beta + \overline{\pi} + \delta lnf)\kappa - \sigma(D + 2\varepsilon - \rho)]\varphi_2\}.
\end{aligned} \tag{3.8}$$

Using the commutator of D and δ, and the Ricci identities is obtained the differential operators are actuation over φ_1 and φ_2 and can be written as

$$\begin{aligned}
&\{(\Delta - 3\gamma - \overline{\gamma} + \overline{\mu})(-2\kappa) + (\overline{\delta} - 3\alpha + \overline{\beta} - \overline{\tau})(2\sigma) \\
&+2\rho\delta lnf - 2\tau Dlnf + (2\rho + Dlnf)\delta - (2\tau + \delta lnf)D - 4\Psi_1\} \\
&\{(2\rho + Dlnf)\sigma - (2\tau + \delta lnf)\kappa + \Psi_0\}, \qquad (3.9)
\end{aligned}$$

Under the condition that should be zero therefore

$$\begin{aligned}
&\kappa = \sigma = \Psi_0 = \Psi_1 = 0, \\
&2\tau = -\delta lnf, 2\rho = -Dlnf. \qquad (3.10)
\end{aligned}$$

Fu rthermore when φ_{AB} satisfy the Maxwell equations then the operator (3.4) give us a decoupled equation for $\varphi_0$under the condition that H^{00} is the unique component no equal to zero of the spinor H^{AB} and when satisfy the Eq. (3.10) the decoupled equation is

$$\begin{aligned}
\mathcal{S}^{B\dot{M}}J_{B\dot{M}} = fgH^{00}&[(\delta - \beta - \overline{\alpha} + \overline{\pi} - 2\tau)(\overline{\delta} - 2\alpha + \pi) \\
&-(D - \varepsilon + \overline{\varepsilon} - \overline{\rho} - 2\rho)(\Delta - 2\gamma + \mu)]\varphi_0 = 0.
\end{aligned} \qquad (3.11)$$

Now if the unique component that is not equal to zero is H^{11} , thus

$$\begin{aligned}
\mathcal{S}^{B\dot{M}}J_{B\dot{M}} = -fgH^{11}&\{[(\Delta + \gamma - \overline{\gamma} + \overline{\mu} + \Delta lnf)\lambda \\
&-(\overline{\delta} + \alpha + \overline{\beta} - \overline{\tau} + \overline{\delta}lnf)\nu \\
&+\nu(\overline{\delta} - 2\alpha + \pi) - \lambda(\Delta - 2\gamma + \mu)]\varphi_0 \\
&-[(\Delta + \gamma - \overline{\gamma} + \overline{\mu} + \Delta lnf)(\overline{\delta} + 2\pi) \\
&+(\overline{\delta} + \alpha + \overline{\beta} - \overline{\tau} + \overline{\delta}lnf)(\Delta + 2\mu) \\
&-\nu(D - 2\rho) + \lambda(\delta - 2\tau)]\varphi_1 \\
&+[(\Delta + \gamma - \overline{\gamma} + \overline{\mu} + \Delta lnf)(D + 2\varepsilon - \rho) \\
&-(\overline{\delta} + \alpha + \overline{\beta} - \overline{\tau} + \overline{\delta}lnf)(\delta + 2\beta - \tau) \\
&-\nu\kappa + \lambda\sigma]\varphi_2\}. \qquad (3.12)
\end{aligned}$$

Using the commutator of Δ and $\overline{\delta}$with the Ricci identities is obtained the differential operators over φ_0 and φ_1 that can be written of this form

$$\begin{aligned}
&\{(\Delta + 3\varepsilon + \overline{\varepsilon} - \overline{\rho})(2\nu) - (\delta + 3\beta - \overline{\alpha} + \overline{\pi})(2\lambda) - 2\mu\overline{\delta}lnf \\
&+2\pi\Delta lnf + (\Delta lnf - 2\mu)\overline{\delta} - (\overline{\delta}lnf - 2\pi)\Delta - 4\Psi_3\} \\
&\{(2\mu - \Delta lnf)\lambda - (2\pi - \overline{\delta}lnf)\nu + \Psi_4\}, \qquad (3.13)
\end{aligned}$$

The operator applied to the φ_0 and φ_1 in (3.12)conditions are zero if

$$\begin{aligned}
&\lambda = \nu = \Psi_3 = \Psi_4 = 0, \\
&2\mu = \Delta lnf, 2\pi = \overline{\delta} lnf.
\end{aligned} \tag{3.14}$$

When φ_{AB} satisfy the Maxwell equations, is obtained the decoupled equation for φ_2 with the condition that H^{11} is no equal to zero of the spinor H^{AB} and that are restricted to

$$\begin{aligned}
\mathcal{S}^{B\dot{M}} J_{B\dot{M}} = &-fgH^{11}[(\Delta + \gamma - \overline{\gamma} + 2\mu)(D + 2\varepsilon - \rho) \\
&-(\overline{\delta} + \alpha + \overline{\beta} - \overline{\tau} + 2\pi)(\delta + 2\beta - \tau)]\varphi_2 = 0.
\end{aligned} \tag{3.15}$$

Last one condition where $H^{10} \neq 0$, thus

$$\begin{aligned}
\mathcal{S}^{B\dot{M}} J_{B\dot{M}} = fgH^{10}[&(\delta + \beta - \overline{\alpha} + \overline{\pi} + \tau + \delta lnf)(\overline{\delta} + 2\pi) \\
&-(D + \varepsilon + \overline{\varepsilon} - \overline{\rho} + \rho + Dlnf)(\Delta + 2\mu) \\
&-(\Delta - \gamma - \overline{\gamma} + \overline{\mu} - \mu + \Delta lnf)(D - 2\rho) \\
&+(\overline{\delta} - \alpha + \overline{\beta} - \overline{\tau} - \pi + \overline{\delta} lnf)(\delta - 2\tau)]\varphi_1 \\
&-[(\delta + \beta - \overline{\alpha} + \overline{\pi} + \tau + \overline{\delta} lnf)(D + 2\varepsilon - \rho) \\
&-(D + \varepsilon + \overline{\varepsilon} - \overline{\rho} + \rho + Dlnf)(\delta + 2\beta - \tau) \\
&+(\Delta - \gamma - \overline{\gamma} + \overline{\mu} - \mu + \Delta lnf)\kappa \\
&-(\overline{\delta} - \alpha + \overline{\beta} - \overline{\tau} - \pi + \overline{\delta} lnf)\gamma]\varphi_2 \\
&+[-(\delta + \beta - \overline{\alpha} + \overline{\pi} + \tau + \delta lnf)\lambda \\
&+(D + \varepsilon + \overline{\varepsilon} - \overline{\rho} + \rho + Dlnf)\mu \\
&+(\Delta - \gamma - \overline{\gamma} + \overline{\mu} - \mu + \Delta lnf)(\overline{\delta} - 2\alpha + \pi) \\
&-(\overline{\delta} - \alpha + \overline{\beta} - \overline{\tau} - \pi + \overline{\delta} lnf)(\Delta - 2\gamma + \mu)]\varphi_0\}.
\end{aligned} \tag{3.16}$$

Using the commutator od D and δ, Δ and $\overline{\delta}$ and the Ricci identities is obtained a differential operator over φ_0 and φ_2 it can be written as

$$\begin{aligned}
&\{2(D + \varepsilon + \overline{\varepsilon} - \overline{\rho} + Dlnf^{1/2})\nu + (\Delta lnf)\pi \\
&-2(\delta + \beta - \overline{\alpha} + \overline{\pi} + \delta lnf^{1/2})\lambda - (\overline{\delta} lnf)\mu \\
&+(\overline{\delta} lnf - 2\pi)(\Delta + 2\gamma) + (\Delta lnf - 2\mu)(\overline{\delta} - 2\alpha)\},
\end{aligned}$$

and

$$\begin{aligned}
&\{2(\overline{\delta} - \alpha + \overline{\beta} - \overline{\tau} + \overline{\delta} lnf^{1/2})\sigma - (Dlnf)\tau \\
&-2(\Delta - \gamma - \overline{\gamma} + \overline{\mu} + \Delta lnf^{1/2})\kappa + (\delta lnf)\rho \\
&+(\delta lnf + 2\tau)(D - 2\varepsilon) + (Dlnf + 2\rho)(\delta + 2\beta)\},
\end{aligned} \tag{3.17}$$

That operators over φ_0 and φ_2 are equal to zero if

$$\begin{aligned} \kappa = \sigma = \lambda = \nu = 0, \\ 2\rho = -Dlnf, 2\tau = -\delta lnf \\ 2\mu = \Delta lnf, 2\pi = \overline{\delta} lnf. \end{aligned} \tag{3.18}$$

Furthermore when φ_{AB} satisfy the Maxwell equations then the operator (3.4) give us the decoupled equations for φ_1 under the condition that H^{10} is the component no equal to zero of the spinor H^{AB}, then by (3.18) is given as

$$\begin{aligned} \mathcal{S}^{B\dot{M}} J_{B\dot{M}} = fgH^{10}[(\delta + \beta - \overline{\alpha} + \overline{\pi} - \tau)(\overline{\delta} + 2\pi) \\ -(D + \varepsilon + \overline{\varepsilon} - \overline{\rho} - \rho)(\Delta + 2\mu) \\ (\Delta - \gamma - \overline{\gamma} + \overline{\mu} + \mu)(D - 2\rho) - \\ +(\overline{\delta} - \alpha + \overline{\beta} - \overline{\tau} + \pi)(\delta - 2\tau)]\varphi_1 = 0. \end{aligned} \tag{3.19}$$

For space-time type D is observed that exist a differential operator, that when is applied to the Maxwell equations without sources is obtained a decoupled equation for each component of the electromagnetis spinor given by

$$\mathcal{S}^{B\dot{M}} = \phi^2 H^{AB} \nabla_A{}^{\dot{M}} \phi^{-2}, \tag{3.20}$$

where ϕ is such that satisfy the eq.(2.25)and the decoupled equation for the electromagnetic spinor in covariant form is given by

$$\phi^2 \nabla_{(A}{}^{\dot{M}} \phi^{-2} \nabla_{\dot{M}}{}^{C} \varphi_{B)C} = 0. \tag{3.21}$$

Thus the eqs. (3.11),(3.15) and (3.19) are write in this form

$$\begin{aligned} [(\delta - \beta - \overline{\alpha} - 2\tau + \overline{\pi})(\overline{\delta} - 2\alpha + \pi) \\ -(D - \varepsilon + \overline{\varepsilon} - 2\rho - \overline{\rho})(\Delta - 2\gamma + \mu)]\varphi_0 = 0, \end{aligned}$$

$$\begin{aligned} [(\delta + \beta - \overline{\alpha} - \tau + \overline{\pi})(\overline{\delta} + 2\pi) \\ -(D + \varepsilon + \overline{\varepsilon} - \overline{\rho} - \rho)(\Delta + 2\mu) \\ +(\overline{\delta} + \overline{\beta} - \alpha + \pi - \overline{\tau})(\delta - 2\tau) - (\Delta - \gamma - \overline{\gamma} + \mu + \overline{\mu}(D - 2\rho)\varphi_1 = 0, \end{aligned}$$

$$[(\Delta + \gamma - \overline{\gamma} + 2\mu + \overline{\mu})(D + 2\varepsilon - \rho) - (\overline{\delta} + \alpha + \overline{\beta} + 2\pi - \overline{\tau})(\delta + 2\beta - \tau)]\varphi_2 = 0 \tag{3.22}$$

The equations for φ_0 and φ_2 were obtained by Teukolsky [12]and shown that when the background space-time is the Kerr metric, those equations have solution by variables separables. Thus the adjunt operator of (3.21)is given by

$$\nabla_{\dot{R}(A}\phi^{-2}\nabla^{S\dot{R}}\chi_{C)S} = 0. \tag{3.23}$$

If is defined

$$X^{\dot{R}}{}_{C} \equiv \phi^{-2}\nabla^{S\dot{R}}\phi^{2}\chi_{CS}. \tag{3.24}$$

Then is written the vectorial field $X^{\dot{R}}{}_{C}$ is the vectorial potential for the elctromagnetic field self-dual. Thus is possible writte a spinorial field that is solution of the Maxwell equations, let to define

$$W_{\dot{R}\dot{S}} \equiv \nabla^{B}{}_{(\dot{R}}\phi^{-2}\nabla^{S}{}_{\dot{S})}\phi^{2}\chi_{BS}, \tag{3.25}$$

Where

$$\chi_{AB} = L_{AC}L_{BD}\varphi^{CD}. \tag{3.26}$$

Thus if φ_{AB} is a solution to the Maxwell equations in a spce-time background type D solution to the Einstein equations in vacuum, with the possibility of the cosmological constant no equal to zero then

$$G_{\dot{R}\dot{S}} \equiv \nabla^{B}{}_{(\dot{R}}\phi^{-2}\nabla^{S}{}_{\dot{S})}\phi^{2}L_{BD}L_{SE}\varphi^{DE}, \tag{3.27}$$

Is another solution for the Maxwell equations.The complex conjugate of this is

$$G_{RS} = \nabla^{B}{}_{(R}\overline{\phi}^{-2}\nabla^{\dot{S}}{}_{S)}\overline{\phi}^{2}L_{\dot{B}\dot{D}}L_{\dot{S}\dot{E}}\varphi^{\dot{D}\dot{E}}, \tag{3.28}$$

Thus the complete operator is written as

$$W_{AB} \equiv \nabla^{\dot{B}}{}_{(A}\overline{\phi}^{-2}\nabla^{\dot{S}}{}_{B)}\overline{\phi}^{2}L_{\dot{B}\dot{D}}L_{\dot{S}\dot{E}}\nabla^{F(\dot{D}}\phi^{2}\nabla^{\dot{E})C}L_{FH}L_{CI}\varphi^{HI}, \tag{3.29}$$

Is another solution for the Maxwell equations.Now using the Newmann-Penrose notation can be written as

$$\begin{aligned}
W_0 &= 16[(D+\overline{\varepsilon}-\varepsilon-\overline{\rho})(D+2\overline{\varepsilon}+\overline{\rho})\\
&\overline{\phi}^{-2}(\delta+\beta+\overline{\alpha}-\tau)(\delta+2\beta+\tau)](\phi^{-2}\varphi_2),\\
W_1 &= 8[(D+\overline{\varepsilon}+\varepsilon-\overline{\rho}+\rho)(\overline{\delta}+2\overline{\beta}+\overline{\tau})\\
&+(\overline{\alpha}+\overline{\beta}-\alpha-\overline{\tau}-\pi)(D+2\overline{\varepsilon}+\overline{\rho})\\
&\overline{\phi}^{-2}(\delta+\beta+\overline{\alpha}-\tau)(\delta+2\beta+\tau)](\phi^{-2}\varphi_2),\\
W_2 &= 16[(\overline{\delta}+\overline{\beta}+\alpha-\overline{\tau})(\overline{\delta}+2\overline{\beta}+\overline{\tau})\\
&\overline{\phi}^{-2}(\delta+\beta+\overline{\alpha}-\tau)(\delta+2\beta+\tau)](\phi^{-2}\varphi_2).
\end{aligned} \tag{3.30}$$

Chapter 4

Solution for the Maxwell equations in the Carter A metric

4.1 The Carter A metric

The Carter A metric [14]is solution to the Einstein equations in vacuum with cosmological constant and is written as

$$ds^2 = \{\frac{\mathcal{Q}}{p^2+q^2}(du-p^2dv)^2 - \frac{p^2+q^2}{\mathcal{Q}}dq^2 - \frac{\mathcal{P}}{p^2+q^2}(du+q^2dv)^2 - \frac{p^2+q^2}{\mathcal{P}}dp^2\}, \tag{4.1}$$

where $\{p, q, u, v\}$ is a real system of coordinates and the functions $\mathcal{P} = \mathcal{P}(q)$, $\mathcal{Q} = \mathcal{Q}(q)$ are given by

$$\mathcal{P} = b + 2np - \varepsilon_0 p^2 - (\lambda_0/3)p^4,$$
$$\mathcal{Q} = b - 2mq + \varepsilon_0 q^2 - (\lambda_0/3)q^4,$$

The parameters m, n and λ_0 are representated by the mass, NUT parameter, and the cosmological constant each one, ε_0 and b are two additional parameters.The Kerr metric is obtained when $b = a^2, n = 0$ and $\varepsilon_0 = 1$.In Boyer-Lindquist coordinates $q = r, p = -acos\theta, u = -t + a\phi$ and $v = \phi/a$, where a represent the angular momentum.

The tangent vectors

$$D = \partial_q + \frac{1}{\mathcal{Q}}(\partial_v - q^2\partial_u),$$

$$
\begin{aligned}
\Delta &= 1/2\phi\overline{\phi}\mathcal{Q}(-\partial_q + \frac{1}{\mathcal{Q}}(\partial_v - q^2\partial_u)),\\
\delta &= (\mathcal{P}/2)^{1/2}\overline{\phi}(\partial_p + \frac{i}{\mathcal{P}}(\partial_v + p^2\partial_u)),\\
\overline{\delta} &= (\mathcal{P}/2)^{1/2}\phi(\partial_p - \frac{i}{\mathcal{P}}(\partial_v + p^2\partial_u)),
\end{aligned} \tag{4.2}
$$

with

$$
\phi \equiv \frac{1}{q+ip}.
$$

The spin coefficients are given by

$$
\begin{aligned}
\kappa = \gamma = \lambda = \nu = 0,\\
\varepsilon = 0, \beta = \delta ln\mathcal{P}^{1/4},\\
\alpha = -\overline{\delta} ln\frac{\mathcal{P}^{1/4}\mathcal{Q}^{1/2}}{q+ip}, \gamma = -\Delta ln\frac{\mathcal{P}^{1/4}\mathcal{Q}^{1/2}}{q+ip},\\
\rho = Dln\phi, \tau = \delta ln\phi, \pi = -\overline{\delta} ln\phi, \mu = -\Delta ln\phi.
\end{aligned} \tag{4.3}
$$

4.2 Solution of the equations for φ_0 and φ_2

The transformation of the differential operator with dependence over u and v are changing by

$$
\begin{aligned}
\mathcal{D} \longrightarrow \mathcal{D}_0 \quad \Delta \longrightarrow -1/2\phi\overline{\phi}\mathcal{Q}\mathcal{D}_0,\\
\delta \longrightarrow 1/\sqrt{2}\overline{\phi}\mathcal{L}_0^\dagger, \quad \overline{\delta} \longrightarrow 1/\sqrt{2}\phi\mathcal{L}_0,
\end{aligned} \tag{4.4}
$$

Where the solutions of the eq. (3.22)have dependence in the variables u and v given by [8] with k and l are separation constants, that defined the operator by

$$
\begin{aligned}
\mathcal{D}_n &\equiv \partial_q + \frac{i}{\mathcal{Q}}(l - kq^2) + n\frac{\dot{\mathcal{Q}}}{\mathcal{Q}},\\
\mathcal{D}_n^\dagger &\equiv \partial_q - \frac{i}{\mathcal{Q}}(l - kq^2) + n\frac{\dot{\mathcal{Q}}}{\mathcal{Q}},\\
\mathcal{L}_n &\equiv \mathcal{P}^{1/2}(\partial_p + \frac{1}{\mathcal{P}}(l + kp^2) + n/2\frac{\dot{\mathcal{P}}}{\mathcal{P}}),\\
\mathcal{L}_n^\dagger &\equiv \mathcal{P}^{1/2}(\partial_p - \frac{1}{\mathcal{P}}(l + kp^2) + n/2\frac{\dot{\mathcal{P}}}{\mathcal{P}}),
\end{aligned} \tag{4.5}
$$

Using the equations (4.3) and (4.4) the decoupled equation for φ_0 take the form

$$[\mathcal{Q}\mathcal{D}_{\prime}\mathcal{D}_0^\dagger + 2ikq + \mathcal{L}_0^\dagger\mathcal{L}_1 + 2kp]\varphi_0 = 0, \tag{4.6}$$

thus

$$\varphi_0 = e^{i(ku+lv)}R_{+1}(q)S_{+1}(p), \tag{4.7}$$

The functions $R_{+1}(q)$ and $S_{+1}(p)$ satisfy the differential equations

$$[\mathcal{Q}\mathcal{D}_0\mathcal{D}_0^\dagger + 2ikq]\mathcal{Q}R_{+1}(q) = A_1\mathcal{Q}R_{+1}(q),$$

$$[\mathcal{L}_0^\dagger\mathcal{L}_1 + 2kp]S_{+1}(p) = -A_1S_{+1}(p), \tag{4.8}$$

where A_1 is a separation constant.Of the same way for φ_2 thus

$$[\mathcal{Q}\mathcal{D}_0^\dagger\mathcal{D}_0 - 2ikq + \mathcal{L}_0\mathcal{L}_1^\dagger - 2kp](\phi^{-2}\varphi_2 = 0, \tag{4.9}$$

thus

$$\varphi_2 = \phi^2 e^{i(ku+lv)}R_{-1}(q)S_{-1}(p), \tag{4.10}$$

The functions $R_{-1}(q)$ and $S_{-1}(p)$ satisfy the differential equations

$$[\mathcal{Q}\mathcal{D}_0^\dagger\mathcal{D}_0 - 2ikq]R_{-1}(q) = A_2R_{-1}(q),$$

$$[\mathcal{L}_0\mathcal{L}_1^\dagger - 2kp]S_{-1}(p) = -A_2S_{-1}(p), \tag{4.11}$$

where A_2 is a separation constant.

The values of the constants should be equal and the functions $\mathcal{Q}R_{+1}$ and R_{-1} then satisfy the Teukolsky-Starobinsky identities [8]

$$\begin{aligned}\mathcal{D}_0\mathcal{D}_0R_{-1}(q) &= BR_{+1}(q),\\ \mathcal{Q}\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger\mathcal{Q}R_{+1}(q) &= BR_{-1}(q),\end{aligned} \tag{4.12}$$

The constant B is known how Starobinsky constant and is real, of the same way the functions $S_{\pm}(p)$ take the form

$$\begin{aligned}\mathcal{L}_0\mathcal{L}_1S_{+1}(p) &= BS_{-1}(p),\\ \mathcal{L}_0^\dagger\mathcal{L}_1^\dagger S_{-1}(p) &= BS_{+1}(p).\end{aligned} \tag{4.13}$$

From (4.12) and (4.13)the components of the electromagnetic field with maximal spin weight and the normalization right are given as

$$\varphi_0 = e^{i(ku+lv)} R_{+1}(q) S_{+1}(p),$$

$$\varphi_2 = 1/2\phi^2 e^{i(ku+lv)} R_{-1}(q) S_{-1}(p). \tag{4.14}$$

The relationship among the constants is

$$B^2 = A^2 + 4kl. \tag{4.15}$$

When was found the decoupled equation for φ_1 is not separable over p and q.Thus using (4.14) and with the use of the symmetry operator obtained in the previus chapter we can find the expression for φ_1.

4.3 The Full Solution

Using the equations (4.3) and (3.30) is obtained

$$\begin{aligned} W_0 &= 8\mathcal{D}_0\mathcal{D}_0\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger(\phi^{-2}\varphi_2), \\ W_1 &= 8/\sqrt{2}\phi^2[(q\mathcal{D}_0 - 1)\mathcal{L}_1) + i(p\mathcal{L}_1 - \sqrt{\mathcal{P}})\mathcal{D}_0]\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger(\phi^{-2}\varphi_2), \\ W_2 &= 4\phi^2\mathcal{L}_0\mathcal{L}_1\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger(\phi^{-2}\varphi_2). \end{aligned} \tag{4.16}$$

Now using the equations (4.12) and (4.14) we obtained

$$\begin{aligned} W_0 &= 4B^2 e^{i(ku+lv)} R_{+1}(q) S_{+1}(p), \\ W_1 &= 4B^2\phi^2/\sqrt{2} e^{i(ku+lv)}[(g_{+1}(q)\mathcal{L}_1 S_{+1}(p) + i f_{-1}(p)\mathcal{D}_0 R_{-1}(q)], \\ W_2 &= 4B^2\phi^2/2 e^{i(ku+lv)} R_{-1}(q) S_{-1}(p), \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} g_{+1}(q) &\equiv 1/B(q\mathcal{D}_0 R_{-1}(q) - R_{-1}(q)), \\ f_{-1}(p) &\equiv 1/B(p\mathcal{L}_1 S_{+1}(p) - \sqrt{\mathcal{P}} S_{+1}(p)). \end{aligned} \tag{4.18}$$

From (4.14) and (4.17)is obtained the expression for φ_1 with φ_0 and φ_2 , thus taking the right normalization can be written

$$\varphi_1 = 1/\sqrt{2}\phi^2 e^{i(ku+lv)}[g_{+1}(q)\mathcal{L}_1 S_{+1}(p) + i f_{-1}(p)\mathcal{D}_0 R_{-1}(q)]. \tag{4.19}$$

From (4.17)is observed that if φ_{AB} is a separable solution for the Maxwell equations without sources thus:

$$1/4\nabla^{\dot{B}}{}_{(A}\overline{\phi}^{2}\nabla^{\dot{S}}{}_{B)}\overline{\phi}^{2}L_{\dot{B}\dot{D}}L_{\dot{S}\dot{E}}G^{\dot{D}\dot{E}} = B^{2}\varphi_{AB}, \tag{4.20}$$

where

$$G^{\dot{D}\dot{E}} = \nabla^{F(\dot{D}}\phi^{-2}\nabla^{\dot{E})C}\phi^{2}L_{FH}L_{CI}\varphi^{HI}. \tag{4.21}$$

Chapter 5

Conclusions

According the result obtained in this report the symmetry operator (3.27) for the Maxwell equations there is a relationship with the Killing *spinor* of two index for all solutions type D of the Einstein equations in vacuum with or without cosmological constant, that can be resolved of (3.26). The equation for φ_0 and φ_2, but no for φ_1, with $\phi^{-1} = L_{10} = L_{AB}\iota^A \o^B$ expressed in term of an unique component differential of the Weyl *spinor* Ψ_2, obtained previously by Wald [13],whatsoever no show the importance of the Killing *spinor*. The equation (3.26) wa given by Torres del Castillo no only for electromagnetic field, furthermore for the case massless field spin $\frac{3}{2}$ with perturbation and gravitational space-time type D [15,16].The results of the chapter 3 can not apply in the case of perturbations in fields of spin $\frac{3}{2}$ and gravitational, due to the cases that no all the field components satisfy a decoupled equation . The result of the chapter 4 the symmetry operator given us the complete solution for the Maxwell equations.

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